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Research Article

Stability for the Kirchhoff Plates Equations with Viscoelastic Boundary Conditions in Noncylindrical Domains

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We study Kirchhoff plates equations with viscoelastic boundary conditions in a noncylindrical domain. This work is devoted to proving the global existence, uniqueness of solutions, and decay of the energy of solutions for Kirchhoff plates equations in a noncylindrical domain.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^2$ containing the origin and having C^2 boundary. Let $\gamma : [0,\infty[\to \mathbb{R}$ be a continuously differentiable function. Consider the family of subdomains $\{\Omega_t\}_{0\le t<\infty}$ of $\mathbb{R}^2$ given by

$$\Omega_t = T(\Omega), \quad T : y \in \Omega \longrightarrow x = \gamma(t)\, y, \tag{1}$$

whose boundaries are denoted by Γ_t, and let $\widehat{Q}$ be the noncylindrical domain of $\mathbb{R}^3$ given by

$$\widehat{Q} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\} \tag{2}$$

with boundary

$$\widehat{\Sigma} = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}. \tag{3}$$

In this paper, we consider the following Kirchhoff plates equations with viscoelastic boundary conditions:

$$u'' + \Delta^2 u = 0 \quad \text{in } \Omega_t \times (0,\infty), \tag{4}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_{0,t} \times (0,\infty), \tag{5}$$

$$-u + \int_0^t g_1(t-s)\,\mathscr{B}_2 u(s)\, ds = 0 \quad \text{on } \Gamma_{1,t} \times (0,\infty), \tag{6}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\,\mathscr{B}_1 u(s)\, ds = 0 \quad \text{on } \Gamma_{1,t} \times (0,\infty), \tag{7}$$

$$u(0,x) = u_0(x), \quad u'(0,x) = u_1(x) \quad \text{in } \Omega_0, \tag{8}$$

where $\nu = (\nu_1, \nu_2)$ is the unit normal at $(\sigma, t) \in \widehat{\Sigma}$ directed towards the exterior of $\widehat{Q}$. We divide the boundary into two parts:

$$\Gamma_t = \Gamma_{0,t} \cup \Gamma_{1,t} \quad \text{with } \overline{\Gamma}_{0,t} \cap \overline{\Gamma}_{1,t} = \emptyset,\ \Gamma_{0,t} \neq \emptyset. \tag{9}$$

We are denoting by $\mathscr{B}_1$ and $\mathscr{B}_2$ the following differential operators:

$$\mathscr{B}_1 u = \Delta u + (1-\mu) B_1 u, \qquad \mathscr{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}, \tag{10}$$

where B_1 and B_2 are given by

$$\begin{gathered} B_1 u = 2\nu_1\nu_2 u_{x_1x_2} - \nu_1^2 u_{x_2x_2} - \nu_2^2 u_{x_1x_1}, \\ B_2 u = \left(\nu_1^2 - \nu_2^2\right) u_{x_1x_2} + \nu_1\nu_2\left(u_{x_2x_2} - u_{x_1x_1}\right), \end{gathered} \tag{11}$$

and the constant μ, $0 < \mu < 1/2$, represents Poisson's ratio. From the physics point of view, system (4) describes the small transversal vibrations of a thin plate with a moving boundary device. The integral equations (6) and (7) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. The relaxation functions $g_1, g_2 \in C^1(0,\infty)$ are positive and nondecreasing.

The uniform stabilization of plates equations with linear or nonlinear boundary feedback in cylindrical domain was investigated by several authors; see for example [1–3] among others. The uniform decay for viscoelastic plates with memory was studied by [4, 5] and the references therein. Santos et al. [6] studied the asymptotic behavior of the solutions of a nonlinear wave equation of Kirchhoff type with boundary condition of memory type. Santos and Junior [7] investigated the stability of solutions for Kirchhoff plate equations with boundary memory condition. Park and Kang [8] studied the exponential decay for the Kirchhoff plate equations with nonlinear dissipation and boundary memory condition. They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation functions. But the existence of solutions and decay of energy for the Kirchhoff plate equations with viscoelastic boundary conditions in noncylindrical domain are not studied yet. In a moving domain, the transverse deflection $u(x,t)$ of the thin plate which changes its configuration at each instant of time increases its deformation and hence increases its tension. Moreover, the horizontal movement of the boundary yields nonlinear terms involving derivatives in the space variables. To control these nonlinearities, we add in the boundary a memory type. This term will play an important role in the dissipative nature of the problem.

In [9–17], the authors considered the global existence and the uniform decay of solution in noncylindrical domains. Dal Passo and Ughi [15] investigated a certain class of parabolic equations in noncylindrical domains. Benabidallah and Ferreira [9] proved the existence of solutions for the nonlinear beam equation in noncylindrical domains. Santos et al. [17] studied the global solvability and asymptotic behavior for the nonlinear coupled system of viscoelastic waves with memory in noncylindrical domains. Park and Kang [14] investigated the global existence and stability for von Karman equations with memory in noncylindrical domains. Motivated by these results, we prove the exponential decay of the energy to the Kirchhoff plate equations with viscoelastic boundary conditions in noncylindrical domains.

This paper is organized as follows. In Section 2, we recall notations and hypotheses. In Section 3, we prove the existence and uniqueness of solutions by employing Faedo-Galerkin's method. In Section 4, we establish the exponential decay rate of the solution.

2. Notations and Hypotheses

We begin this section introducing notations and some hypotheses. Throughout this paper we use standard functional spaces and denote that $\|\cdot\|_p$, $\|\cdot\|_{p,t}$ are $L^p(\Omega)$ norm and $L^p(\Omega_t)$ norm. We define the inner product

$$(u,v)=\int_\Omega u(x)\,v(x)\,dx,\qquad (u,v)_t=\int_{\Omega_t} u(x)\,v(x)\,dx. \tag{12}$$

Also, let us assume that there exists $x_0\in\mathbb{R}^2$ such that

$$\begin{aligned}\Gamma_{0,t}&=\{x\in\Gamma_t:\nu(x)\cdot(x-x_0)\le 0\},\\ \Gamma_{1,t}&=\{x\in\Gamma:\nu(x)\cdot(x-x_0)>0\}.\end{aligned} \tag{13}$$

The method used to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time dependent. This is done using a suitable change of variable. Then we show the existence and uniqueness for this new problem. Our existence result on noncylindrical domains will follow by using the inverse transformation. That is, by using the diffeomorphism

$$\tau:\widehat{Q}\longrightarrow Q,\quad (x,t)\in\Omega_t\times\{t\}\longrightarrow(y,t)=\left(\frac{x}{\gamma(t)},t\right) \tag{14}$$

and $\tau^{-1}:Q\to\widehat{Q}$ defined by

$$\tau^{-1}(y,t)=(x,t)=(\gamma(t)\,y,t). \tag{15}$$

For each function u we denote by v the function

$$v(y,t)=u\circ\tau^{-1}(y,t)=u(x,t), \tag{16}$$

the initial boundary value problem (4)–(8) becomes

$$v''+\gamma^{-4}\Delta^2 v+A(t)\,v+b(y,t)\cdot\nabla v+c(y,t)\cdot\nabla v'=0\quad\text{in }\Omega\times(0,\infty), \tag{17}$$

$$v=\frac{\partial v}{\partial\nu}=0\quad\text{on }\Gamma_0\times(0,\infty), \tag{18}$$

$$-v+\int_0^t g_1(t-s)\,\gamma^{-2}(s)\,\mathscr{B}_2 v(s)\,ds=0\quad\text{on }\Gamma_1\times(0,\infty), \tag{19}$$

$$\frac{\partial v}{\partial\nu}+\int_0^t g_2(t-s)\,\gamma^{-2}(s)\,\mathscr{B}_1 v(s)\,ds=0\quad\text{on }\Gamma_1\times(0,\infty), \tag{20}$$

$$v(y,0)=v_0(y),\quad v'(y,0)=v_1(y)\quad\text{in }\Omega, \tag{21}$$

where

$$\begin{aligned}A(t)\,v&=\sum_{i,j=1}^{2}\partial_{y_i}\left(a_{ij}\partial_{y_j}v\right),\\ a_{ij}&=\left(\gamma'\gamma^{-1}\right)^2 y_i y_j\qquad(i,j=1,2),\\ b(y,t)&=-\gamma^{-2}\left(\gamma''\gamma+\left(\gamma'\right)^2\right)y,\\ c(y,t)&=-2\gamma'\gamma^{-1}y.\end{aligned} \tag{22}$$

The above method was introduced by Dal Passo and Ughi [15] for studying a certain class of parabolic equations in noncylindrical domains. This idea was used in [11, 13, 14, 16, 17].

We will use (19) and (20) to estimate the values $\mathscr{B}_1$ and $\mathscr{B}_2$ on Γ_1. Denoting by

$$\left(g * v\right)(t) = \int_0^t g(t-s)\,v(s)\,ds \tag{23}$$

the convolution product operator and differentiating (19) and (20) we arrive at the following Volterra equations:

$$\begin{aligned} \frac{\mathscr{B}_2 v}{\gamma^2} + \frac{1}{g_1(0)} g_1' * \frac{\mathscr{B}_2 v}{\gamma^2} &= \frac{1}{g_1(0)} v', \\ \frac{\mathscr{B}_1 v}{\gamma^2} + \frac{1}{g_2(0)} g_2' * \frac{\mathscr{B}_1 v}{\gamma^2} &= -\frac{1}{g_2(0)} \frac{\partial v'}{\partial \nu}. \end{aligned} \tag{24}$$

Applying Volterra's inverse operator, we get

$$\begin{aligned} \frac{\mathscr{B}_2 v}{\gamma^2} &= \frac{1}{g_1(0)} \left\{ v' + k_1 * v' \right\}, \\ \frac{\mathscr{B}_1 v}{\gamma^2} &= -\frac{1}{g_2(0)} \left\{ \frac{\partial v'}{\partial \nu} + k_2 * \frac{\partial v'}{\partial \nu} \right\}, \end{aligned} \tag{25}$$

where the resolvent kernels of $-g_i'/g_i(0)$ satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i', \quad \forall i = 1, 2. \tag{26}$$

Denoting by $\tau_1 = 1/g_1(0)$ and $\tau_2 = 1/g_2(0)$, we obtain

$$\frac{\mathscr{B}_2 v}{\gamma^2} = \tau_1 \left\{ v' + k_1(0)\,v - k_1(t)\,v_0 + k_1' * v \right\}, \tag{27}$$

$$\begin{aligned} \frac{\mathscr{B}_1 v}{\gamma^2} = -\tau_2 \Bigg\{ & \frac{\partial v'}{\partial \nu} + k_2(0) \frac{\partial v}{\partial \nu} \\ & - k_2(t) \frac{\partial v_0}{\partial \nu} + k_2' * \frac{\partial v}{\partial \nu} \Bigg\}. \end{aligned} \tag{28}$$

Therefore, we use (27) and (28) instead of the boundary conditions (19) and (20).

Let us define the bilinear form $a(\cdot,\cdot)$ as follows:

$$\begin{aligned} a(w, v) = {} & w_{x_1x_1} v_{x_1x_1} + w_{x_2x_2} v_{x_2x_2} \\ & + \mu \left(w_{x_1x_1} v_{x_2x_2} + w_{x_2x_2} v_{x_1x_1} \right) \\ & + 2(1-\mu)\, w_{x_1x_2} v_{x_1x_2}. \end{aligned} \tag{29}$$

Since $\Gamma_0 \neq \emptyset$ we know that $\int_\Omega a(v, v)dy$ is equivalent to the $H^2(\Omega)$ norm, that is,

$$c_0 \|v\|^2_{H^2(\Omega)} \le \int_\Omega a(v, v)\,dy \le C_0 \|v\|^2_{H^2(\Omega)}, \tag{30}$$

where c_0 and C_0 are generic positive constants.

Let us denote that

$$\begin{aligned} (g \circ v)(t) &:= \int_0^t g(t-s)\,(v(t) - v(s))\,ds, \\ (g \square v)(t) &:= \int_0^t g(t-s)\,|v(t) - v(s)|^2 ds. \end{aligned} \tag{31}$$

The following lemma states an important property of the convolution operator.

Lemma 1. *For $g, v \in C^1([0,\infty) : \mathbb{R})$ one has*

$$\begin{aligned} (g * v)\,v' = {} & -\frac{1}{2} g(t)\,|v(t)|^2 + \frac{1}{2} g' \square v \\ & - \frac{1}{2}\frac{d}{dt} \left[g \square v - \left(\int_0^t g(s)\,ds \right) |v|^2 \right]. \end{aligned} \tag{32}$$

The proof of this lemma follows by differentiating the term $g \square v$.

We state the following lemma which will be useful in what follows.

Lemma 2 (see [7]). *Let w and v be functions in $H^4(\Omega) \cap H_0^2(\Omega)$. Then one has*

$$\begin{aligned} \int_\Omega \left(\Delta^2 w\right) v\,dy = {} & \int_\Omega a(w, v)\,dy \\ & + \int_{\Gamma_1} \left\{ (\mathscr{B}_2 w)\,v - (\mathscr{B}_1 w) \frac{\partial v}{\partial \nu} \right\} d\Gamma. \end{aligned} \tag{33}$$

Lemma 3 (see [18]). *Suppose that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_1)$, and $h \in H^{3/2}(\Gamma_1)$; then, any solution of*

$$\begin{aligned} \int_\Omega a(v, w)\,dy = {} & \int_\Omega f w\,dy + \int_{\Gamma_1} g w\,d\Gamma \\ & + \int_{\Gamma_1} h \frac{\partial w}{\partial \nu} d\Gamma, \quad \forall w \in H_0^2(\Omega) \end{aligned} \tag{34}$$

satisfies $v \in H^4(\Omega)$ and also

$$\begin{aligned} \Delta^2 v = f, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad & \text{on } \Gamma_0, \\ \mathscr{B}_1 v = h, \quad \mathscr{B}_2 v = g \quad & \text{on } \Gamma_1. \end{aligned} \tag{35}$$

To show the existence of solution, we will use the following hypotheses:

$$\gamma' \le 0, \qquad \gamma \in L^\infty(0,\infty), \qquad \inf_{0 \le t < \infty} \gamma(t) = \gamma_0 > 0, \tag{36}$$

$$\gamma' \in W^{2,\infty}(0,\infty) \cap W^{2,1}(0,\infty), \tag{37}$$

$$0 < \max_{0 \le t < \infty} \left|\gamma'(t)\right| \gamma(t) \le \frac{1}{\sqrt{2 c_1 c_0^{-1} M d}}, \tag{38}$$

where $d = \operatorname{diam}(\Omega)$, $M = \operatorname{meas}(\Omega)$, and c_0 is a positive imbedding constant such that $\|\nabla v\|^2 \le c_1 \|\Delta v\|^2$, for all $v \in H_0^2(\Omega)$.

3. Existence and Regularity

In this section we will study the existence and regularity of solutions for system (4)–(8).

The well posedness of system (17)–(21) is given by the following theorem.

Theorem 4. *Let $k_i \in C^2(\mathbb{R}^+)$ be such that*

$$k_i, -k_i', k_i'' \geq 0. \tag{39}$$

The function γ satisfies that

$$\left|\gamma'(t)\right| \gamma^{-1}(t) < \min\left\{1, -\frac{k_i'(t)}{2}\right\}. \tag{40}$$

If $(v_0, v_1) \in (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$ satisfy the compatibility condition

$$\mathscr{B}_2 v_0 - \tau_1 \gamma^2(0) v_1 = 0, \qquad \mathscr{B}_1 v_0 + \tau_2 \gamma^2(0) \frac{\partial v_1}{\partial \nu} \quad \text{on } \Gamma_1 \tag{41}$$

then there exists only one solution for system (17)–(21) *satisfying*

$$\begin{gathered} v \in L^\infty\left(0, T; H^4(\Omega) \cap H_0^2(\Omega)\right), \\ v' \in L^\infty\left(0, T; H_0^2(\Omega)\right), \quad v'' \in L^\infty\left(0, T; L^2(\Omega)\right). \end{gathered} \tag{42}$$

Proof. The main idea is to use the Galerkin method. To do this let us denote by B the operator

$$Bw = \Delta^2 w, \qquad D(B) = H_0^2(\Omega) \cap H^4(\Omega). \tag{43}$$

It is well known that B is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of eigenfunctions and eigenvalues of B such that the set of linear combinations of $\{w_n\}_{n\in\mathbb{N}}$ is dense in $D(B)$ and $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$. Let us define

$$v_{0m} = \sum_{j=1}^m \left(v_0, w_j\right) w_j, \qquad v_{1m} = \sum_{j=1}^m \left(v_1, w_j\right) w_j. \tag{44}$$

Note that for any $(v_0, v_1) \in D(B) \times H_0^2(\Omega)$, we have $v_{0m} \to v_0$ strong in $D(B)$ and $v_{1m} \to v_1$ strong in $H_0^2(\Omega)$.

Let us denote by V_m the space generated by $w_1, w_2, \ldots, w_m$. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$v_m(t) = \sum_{j=1}^m g_{jm}(t) w_j, \tag{45}$$

of the approximate system

$$\begin{aligned} &\int_\Omega v_m'' w_j \, dy + \gamma^{-4} \int_\Omega a\left(v_m, w_j\right) dy + \int_\Omega A(t) v_m w_j \, dy \\ &\quad + \int_\Omega c(y,t) \cdot \nabla v_m' w_j \, dy + \int_\Omega b(y,t) \cdot \nabla v_m w_j \, dy \\ &= -\tau_1 \gamma^{-2} \int_{\Gamma_1} \left\{ v_m' + k_1(0) v_m - k_1(t) v_{0m} + k_1' * v_m \right\} w_j \, d\Gamma \\ &\quad - \tau_2 \gamma^{-2} \int_{\Gamma_1} \left\{ \frac{\partial v_m'}{\partial \nu} + k_2(0) \frac{\partial v_m}{\partial \nu} - k_2(t) \frac{\partial v_{0m}}{\partial \nu} \right. \\ &\qquad \left. + k_2' * \frac{\partial v_m}{\partial \nu} \right\} \frac{w_j}{\partial \nu} d\Gamma \quad (j = 1, 2, \ldots, m), \end{aligned} \tag{46}$$

$$v_m(x, 0) = v_{0m}, \quad v_m'(x, 0) = v_{1m}. \tag{47}$$

By standard methods for differential equations, we prove the existence of solutions to the approximate equation (46) on some interval $[0, t_m)$. Then, this solution can be extended to the whole interval $[0, T]$, for all $T > 0$, by using the following first estimate.

The First Estimate. Multiplying (46) by $g_{jm}'(t)$, summing up the product result $j = 1, 2, \ldots, m$, and making some calculations using Lemma 1, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_\Omega \left|v_m'\right|^2 dy + \gamma^{-4} \int_\Omega a\left(v_m, v_m\right) dy \right. \\ &\qquad + \tau_1 \gamma^{-2} \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\ &\qquad \left. + \tau_2 \gamma^{-2} \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \right] \\ &\quad + 2\gamma^{-5}\gamma' \int_\Omega a\left(v_m, v_m\right) dy \\ &\quad + \tau_1 \gamma^{-3}\gamma' \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\ &\quad + \tau_2 \gamma^{-3}\gamma' \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \\ &= -\int_\Omega A(t) v_m v_m' \, dy - \int_\Omega c(y,t) \cdot \nabla v_m' v_m' \, dy \\ &\quad - \int_\Omega b(y,t) \cdot \nabla v_m v_m' \, dy \\ &\quad - \tau_1 \gamma^{-2} \int_{\Gamma_1} \left(\left|v_m'\right|^2 - k_1(t) v_{0m} v_m' - \frac{1}{2} k_1'(t) \left|v_m\right|^2 \right. \\ &\qquad \left. + \frac{1}{2} k_1'' \square v_m \right) d\Gamma \\ &\quad - \tau_2 \gamma^{-2} \int_{\Gamma_1} \left(\left|\frac{\partial v_m'}{\partial \nu}\right|^2 - k_2(t) \frac{\partial v_{0m}}{\partial \nu} \frac{\partial v_m'}{\partial \nu} \right. \\ &\qquad \left. - \frac{1}{2} k_2'(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 + \frac{1}{2} k_2'' \square \frac{\partial v_m}{\partial \nu} \right) d\Gamma. \end{aligned} \tag{48}$$

Now we will estimate terms of the right-hand side of (48). From the hypotheses on γ and Green's formula, we get

$$\begin{aligned} &-\int_\Omega A(t) v_m v_m' \, dy \\ &\quad = -\int_\Omega \sum_{i,j=1}^2 \partial_{y_i} \left(a_{ij} \partial_{y_j} v_m\right) v_m' \, dy \\ &\quad = \int_\Omega \sum_{i,j=1}^2 \left(a_{ij} \partial_{y_j} v_m\right) \partial_{y_i} v_m' \, dy \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{i,j=1}^{2} \left(\gamma'\gamma^{-1}\right)^2 y_i y_j \partial_{y_j} v_m \partial_{y_i} v_m' \, dy \\
&= \frac{d}{dt} \int_\Omega \frac{1}{2} \left(\gamma'\gamma^{-1}\right)^2 \left|\nabla v_m \cdot y\right|^2 dy \\
&\quad - \left(\gamma'\gamma^{-1}\right) \left[\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right] \left\|\nabla v_m \cdot y\right\|_2^2, \\
&\int_\Omega c(y,t) \cdot \nabla v_m' v_m' \, dy \\
&\quad = -\int_\Omega 2\gamma'\gamma^{-1} y \cdot \nabla v_m' v_m' \, dy \\
&\quad = -\int_\Omega \gamma'\gamma^{-1} y \cdot \nabla \left|v_m'\right|^2 dy = 2\gamma'\gamma^{-1} \left\|v_m'\right\|_2^2, \\
&- \int_\Omega b(y,t) \cdot \nabla v_m v_m' \, dy \\
&\quad = \int_\Omega \gamma^{-2} \left(\gamma''\gamma + \left(\gamma'\right)^2\right) y \cdot \nabla v_m v_m' \, dy \\
&\quad \le \left(\frac{\left|\gamma''\gamma^{-1}\right| + \left|\gamma'\gamma^{-1}\right|^2}{2}\right) \left(\left\|y \cdot \nabla v_m\right\|_2^2 + \left\|v_m'\right\|_2^2\right) \\
&\quad \le C_1 \left(\left\|\nabla v_m\right\|_2^2 + \left\|v_m'\right\|_2^2\right).
\end{aligned} \tag{49}
$$

Young's inequality yields

$$
\begin{aligned}
&\int_{\Gamma_1} k_1(t) v_{0m} v_m' \, d\Gamma \le \frac{1}{2} \int_{\Gamma_1} \left|v_m'\right|^2 d\Gamma + \frac{k_1^2(t)}{2} \int_{\Gamma_1} \left|v_{0m}\right|^2 d\Gamma, \\
&\quad \int_{\Gamma_1} k_2(t) \frac{\partial v_{0m}}{\partial \nu} \frac{\partial v_m'}{\partial \nu} d\Gamma \\
&\quad \le \frac{1}{2} \int_{\Gamma_1} \left|\frac{\partial v_m'}{\partial \nu}\right|^2 d\Gamma + \frac{k_2^2(t)}{2} \int_{\Gamma_1} \left|\frac{\partial v_{0m}}{\partial \nu}\right|^2 d\Gamma.
\end{aligned} \tag{50}
$$

Replacing the above calculations in (48) and using our assumptions $k_i, -k_i', k_i'' \ge 0$ and (30), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left[\left\|v_m'\right\|_2^2 + \gamma^{-4} \int_\Omega a(v_m, v_m) \, dy - \left(\gamma'\gamma^{-1}\right)^2 \left\|\nabla v_m \cdot y\right\|_2^2 \right. \\
&\quad + \tau_1 \gamma^{-2} \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\
&\quad \left. + \tau_2 \gamma^{-2} \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \right] \\
&\le C_2 \left[\left\|v_m'\right\|_2^2 + \int_\Omega a(v_m, v_m) \, dy \right. \\
&\quad + \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\
&\quad \left. + \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \right] \\
&\quad + \frac{\tau_1 \gamma^{-2}}{2} k_1^2(t) \int_{\Gamma_1} \left|v_{0m}\right|^2 d\Gamma + \frac{\tau_2 \gamma^{-2}}{2} k_2^2(t) \int_{\Gamma_1} \left|\frac{\partial v_{0m}}{\partial \nu}\right|^2 d\Gamma.
\end{aligned} \tag{51}
$$

From our choice of v_{0m} and v_{1m} and integrating (51) over $(0, t)$ with $t \in (0, t_m)$, we obtain

$$
\begin{aligned}
&\left\|v_m'\right\|_2^2 + \gamma^{-4} \int_\Omega a(v_m, v_m) \, dy - \left(\gamma'\gamma^{-1}\right)^2 \left\|\nabla v_m \cdot y\right\|_2^2 \\
&\quad + \tau_1 \gamma^{-2} \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\
&\quad + \tau_2 \gamma^{-2} \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \\
&\le C_3 \int_0^t \left[\left\|v_m'(s)\right\|_2^2 + \int_\Omega a(v_m(s), v_m(s)) \, dy \right. \\
&\quad + \int_{\Gamma_1} \left(k_1(s) \left|v_m(s)\right|^2 - \left(k_1' \square v_m\right)(s)\right) d\Gamma \\
&\quad \left. + \int_{\Gamma_1} \left(k_2(s) \left|\frac{\partial v_m(s)}{\partial \nu}\right|^2 - \left(k_2' \square \frac{\partial v_m}{\partial \nu}\right)(s)\right) d\Gamma \right] ds \\
&\quad + C_4.
\end{aligned} \tag{52}
$$

We observe that, from (30) and (38),

$$
\begin{aligned}
\left(\gamma'\gamma^{-1}\right)^2 \left\|\nabla v_m \cdot y\right\|_2^2 &\le \left(\gamma'\gamma^{-1}\right)^2 M d^2 \left\|\nabla v_m\right\|_2^2 \\
&\le \left(\gamma'\gamma^{-1}\right)^2 c_1 c_0^{-1} M d^2 \int_\Omega a(v_m, v_m) \, dy \\
&\le \frac{\gamma^{-4}}{2} \int_\Omega a(v_m, v_m) \, dy,
\end{aligned} \tag{53}
$$

for all $t \ge 0$. Hence, by Gronwall's lemma we get

$$
\begin{aligned}
&\left\|v_m'\right\|_2^2 + \int_\Omega a(v_m, v_m) \, dy + \int_{\Gamma_1} \left(k_1(t) \left|v_m\right|^2 - k_1' \square v_m\right) d\Gamma \\
&\quad + \int_{\Gamma_1} \left(k_2(t) \left|\frac{\partial v_m}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m}{\partial \nu}\right) d\Gamma \le C_5,
\end{aligned} \tag{54}
$$

where C_5 is a positive constant which is independent of m and t.

The Second Estimate. First of all, we are going to estimate $v_m''(0)$ in $L^2(\Omega)$-norm. Letting $t \to 0^+$ in (46), multiplying the result by $g''_{jm}(0)$, and using the compatibility condition (41), we have

$$\left\|v''_m(0)\right\|_2^2 \le C_6. \tag{55}$$

Now, differentiating (46) with respect to t, we obtain

$$\begin{aligned}
&\int_\Omega v'''_m w_j\,dy + \gamma^{-4}\int_\Omega a\left(v'_m, w_j\right)dy \\
&- 4\gamma^{-5}\gamma'\int_\Omega a\left(v_m, w_j\right)dy \\
=& -\int_\Omega \frac{d}{dt}\left[A(t)v_m\right]w_j\,dy - \int_\Omega \frac{d}{dt}\left[c(y,t)\cdot\nabla v'_m\right]w_j\,dy \\
&- \int_\Omega \frac{d}{dt}\left[b(y,t)\cdot\nabla v_m\right]w_j\,dy \\
&- \tau_1\gamma^{-2}\int_{\Gamma_1}\left\{v''_m + k_1(0)v'_m + k'_1 * v'_m\right\}w_j\,d\Gamma \\
&- \tau_2\gamma^{-2}\int_{\Gamma_1}\left\{\frac{\partial v''_m}{\partial \nu} + k_2(0)\frac{\partial v'_m}{\partial\nu} + k'_2 * \frac{\partial v'_m}{\partial\nu}\right\}\frac{w_j}{\partial\nu}d\Gamma \\
&+ 2\tau_1\gamma^{-3}\gamma'\int_{\Gamma_1}\left\{v'_m + k_1(0)v_m - k_1(t)v_{0m}\right. \\
&\qquad\qquad \left. + k'_1 * v_m\right\}w_j\,d\Gamma \\
&+ 2\tau_2\gamma^{-3}\gamma'\int_{\Gamma_1}\left\{\frac{\partial v'_m}{\partial\nu} + k_2(0)\frac{\partial v_m}{\partial\nu} - k_2(t)\frac{\partial v_{0m}}{\partial\nu}\right. \\
&\qquad\qquad \left. + k'_2 * \frac{\partial v_m}{\partial\nu}\right\}\frac{w_j}{\partial\nu}d\Gamma.
\end{aligned} \tag{56}$$

Multiplying (56) by $g''_{jm}(t)$, summing up the product result in j, and using Lemma 1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|v''_m\right\|_2^2 + \gamma^{-4}\int_\Omega a\left(v'_m, v'_m\right)dy\right. \\
&\qquad - 8\gamma^{-5}\gamma'\int_\Omega a\left(v_m, v'_m\right)dy \\
&\qquad + \tau_1\gamma^{-2}\int_{\Gamma_1}\left(k_1(t)\left|v'_m\right|^2 - k'_1\Box v'_m\right)d\Gamma \\
&\qquad \left. + \tau_2\gamma^{-2}\int_{\Gamma_1}\left(k_2(t)\left|\frac{\partial v'_m}{\partial\nu}\right|^2 - k'_2\Box\frac{\partial v'_m}{\partial\nu}\right)d\Gamma\right] \\
&+ 6\gamma^{-5}\gamma'\int_\Omega a\left(v'_m, v'_m\right)dy \\
&+ 4\gamma^{-4}\left(\gamma''\gamma^{-1} - 5\left(\gamma'\gamma^{-1}\right)^2\right)\int_\Omega a\left(v_m, v'_m\right)dy \\
&+ \tau_1\gamma^{-3}\gamma'\int_{\Gamma_1}\left(k_1(t)\left|v'_m\right|^2 - k'_1\Box v'_m\right)d\Gamma \\
&+ \tau_2\gamma^{-3}\gamma'\int_{\Gamma_1}\left(k_2(t)\left|\frac{\partial v'_m}{\partial\nu}\right|^2 - k'_2\Box\frac{\partial v'_m}{\partial\nu}\right)d\Gamma \\
=& -\int_\Omega \frac{d}{dt}\left[A(t)v_m\right]v''_m\,dy - \int_\Omega\frac{d}{dt}\left[c(y,t)\cdot\nabla v'_m\right]v''_m\,dy \\
&- \int_\Omega\frac{d}{dt}\left[b(y,t)\cdot\nabla v_m\right]v''_m\,dy \\
&- \tau_1\gamma^{-2}\int_{\Gamma_1}\left(\left|v''_m\right|^2 + \frac{1}{2}k''_1\Box v'_m - \frac{1}{2}k'_1(t)\left|v'_m\right|^2\right)d\Gamma \\
&- \tau_2\gamma^{-2}\int_{\Gamma_1}\left(\left|\frac{\partial v''_m}{\partial\nu}\right|^2 + \frac{1}{2}k''_2\Box\frac{\partial v'_m}{\partial\nu} - \frac{1}{2}k'_2(t)\left|\frac{\partial v'_m}{\partial\nu}\right|^2\right)d\Gamma \\
&+ 2\tau_1\gamma^{-3}\gamma'\int_{\Gamma_1}\left\{v'_m + k_1(0)v_m - k_1(t)v_{0m}\right. \\
&\qquad\qquad \left. + k'_1 * v_m\right\}v''_m\,d\Gamma \\
&+ 2\tau_2\gamma^{-3}\gamma'\int_{\Gamma_1}\left\{\frac{\partial v'_m}{\partial\nu} + k_2(0)\frac{\partial v_m}{\partial\nu} - k_2(t)\frac{\partial v_{0m}}{\partial\nu}\right. \\
&\qquad\qquad \left. + k'_2 * \frac{\partial v_m}{\partial\nu}\right\}\frac{v''_m}{\partial\nu}d\Gamma.
\end{aligned} \tag{57}$$

Now we will estimate terms of the right-hand side of (57). From the hypotheses on γ and Green's formula, we get

$$\begin{aligned}
&-\int_\Omega\frac{d}{dt}\left[A(t)v_m\right]v''_m\,dy \\
&= -\int_\Omega\frac{d}{dt}\left[\sum_{i,j=1}^2\partial_{y_i}\left(\left(\gamma'\gamma^{-1}\right)^2 y_i y_j \partial_{y_j} v_m\right)\right]v''_m\,dy \\
&= -\int_\Omega\left[\sum_{i,j=1}^2\partial_{y_i}\left(2\gamma'\gamma^{-1}\left(\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right)y_i y_j\partial_{y_j}v_m\right.\right. \\
&\qquad\qquad \left.\left. + \left(\gamma'\gamma^{-1}\right)^2 y_i y_j \partial_{y_j} v'_m\right)\right]v''_m\,dy \\
&= -\int_\Omega\left[\sum_{i,j=1}^2\partial_{y_i}\left(2\gamma'\gamma^{-1}\left(\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right)\right.\right. \\
&\qquad\qquad \left.\left. \times y_i y_j \partial_{y_j} v_m\right)\right]v''_m\,dy \\
&\quad + \int_\Omega\sum_{i,j=1}^2\left(\gamma'\gamma^{-1}\right)^2 y_i y_j \partial_{y_j} v'_m \partial_{y_i} v''_m\,dy
\end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \left[\sum_{i,j=1}^{2} \partial_{y_i}\left(2\gamma'\gamma^{-1}\left(\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right) y_i y_j \partial_{y_j} v_m\right)\right] v''_m \, dy \\
&\quad + \frac{d}{dt}\int_\Omega \frac{1}{2}\left(\gamma'\gamma^{-1}\right)^2 \left|\nabla v'_m \cdot y\right|^2 dy \\
&\quad - \left(\gamma'\gamma^{-1}\right)\left[\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right] \left\|\nabla v'_m \cdot y\right\|_2^2,
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
&- \int_\Omega \frac{d}{dt}\left[c(y,t)\cdot \nabla v'_m\right] v''_m \, dy \\
&\quad = \int_\Omega \frac{d}{dt}\left[2\gamma'\gamma^{-1} y \cdot \nabla v'_m\right] v''_m \, dy \\
&\quad = \int_\Omega \left[2\left(\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right) y \cdot \nabla v'_m \right. \\
&\qquad \left. + 2\gamma'\gamma^{-1} y \cdot \nabla v''_m\right] v''_m \, dy \\
&\quad = \int_\Omega \left[2\left(\gamma''\gamma^{-1} - \left(\gamma'\gamma^{-1}\right)^2\right) y \cdot \nabla v'_m\right] v''_m \, dy \\
&\qquad + \int_\Omega \gamma'\gamma^{-1} y \cdot \nabla |v''_m|^2 \, dy \\
&\quad \le \left(\left|\gamma''\gamma^{-1}\right| + \left|\gamma'\gamma^{-1}\right|^2\right)\left(\left\|y \cdot \nabla v'_m\right\|_2^2 + \left\|v''_m\right\|_2^2\right) \\
&\qquad - 2\gamma'\gamma^{-1}\left\|v''_m\right\|_2^2,
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
&- \int_\Omega \frac{d}{dt}\left[b(y,t)\cdot \nabla v_m\right] v''_m \, dy \\
&\quad = \int_\Omega \frac{d}{dt}\left[\left(\gamma''\gamma^{-1} + \left(\gamma'\gamma^{-1}\right)^2\right) y \cdot \nabla v_m\right] v''_m \, dy \\
&= \int_\Omega \frac{d}{dt}\left[\gamma''\gamma^{-1} + \left(\gamma'\gamma^{-1}\right)^2\right] y \cdot \nabla v_m v''_m \, dy \\
&\quad + \int_\Omega \left(\gamma''\gamma^{-1} + \left(\gamma'\gamma^{-1}\right)^2\right) y \cdot \nabla v'_m v''_m \, dy \\
&\le C_7 \left(\left\|y \cdot \nabla v_m\right\|_2^2 + \left\|y \cdot \nabla v'_m\right\|_2^2 + \left\|v''_m\right\|_2^2\right).
\end{aligned} \tag{60}
$$

We know that

$$
\begin{aligned}
\left(k'_1 * v_m\right)(t) = {} & \int_0^t k'_1(t-s)\left(v_m(s) - v_m(t)\right) ds \\
& + k_1(t) v_m(t) - k_1(0) v_m(t).
\end{aligned} \tag{61}
$$

By using Hölder's inequality and our assumption $k'_1 \le 0$, we note that

$$
\begin{aligned}
&\left\|\int_0^t k'_1(t-s)\left(v(t) - v(s)\right) ds\right\|_{\Gamma_1}^2 \\
&\quad \le \left(\int_0^t k'_1(s)\, ds\right) \int_{\Gamma_1} \int_0^t k'_1(t-s)\left(v(t) - v(s)\right)^2 ds\, d\Gamma \\
&\quad \le \int_{\Gamma_1} k_1(0)\left|k'_1\right| \square v \, d\Gamma
\end{aligned} \tag{62}
$$

and, hence, by applying Young's inequality, we obtain

$$
\begin{aligned}
&\left|2\tau_1\gamma^{-3}\gamma' \int_{\Gamma_1} \left\{v'_m + k_1(0) v_m - k_1(t) v_{0m} + k'_1 * v_m\right\} v''_m \, d\Gamma\right| \\
&\quad \le \tau_1\gamma^{-3}\left|\gamma'\right| \int_{\Gamma_1} \left|v''_m\right|^2 d\Gamma + \tau_1\gamma^{-3}\left|\gamma'\right| \\
&\qquad \times \int_{\Gamma_1} \left(\left|v'_m\right|^2 + k_1^2(t)\left|v_{0m}\right|^2 + k_1(0)\left|k'_1\right| \square v_m \right. \\
&\qquad \left. + k_1^2(t)\left|v_m\right|^2\right) d\Gamma.
\end{aligned} \tag{63}
$$

By the same argument of (63), we can obtain the similar estimate

$$
\begin{aligned}
&\left|2\tau_2\gamma^{-3}\gamma' \int_{\Gamma_1} \left\{\frac{\partial v'_m}{\partial \nu} + k_2(0)\frac{\partial v_m}{\partial \nu} \right.\right. \\
&\qquad\qquad \left.\left. - k_2(t)\frac{\partial v_{0m}}{\partial \nu} + k'_2 * \frac{\partial v_m}{\partial \nu}\right\} \frac{v''_m}{\partial \nu} d\Gamma\right| \\
&\quad \le \tau_2\gamma^{-3}\left|\gamma'\right| \int_{\Gamma_1} \left|\frac{v''_m}{\partial \nu}\right|^2 d\Gamma + \tau_2\gamma^{-3}\left|\gamma'\right| \\
&\qquad \times \int_{\Gamma_1} \left(\left|\frac{v'_m}{\partial \nu}\right|^2 + k_2^2(t)\left|\frac{\partial v_{0m}}{\partial \nu}\right|^2 \right. \\
&\qquad\qquad \left. + k_2(0)\left|k'_2\right| \square \frac{\partial v_m}{\partial \nu} + k_2^2(t)\left|\frac{\partial v_m}{\partial \nu}\right|^2\right) d\Gamma.
\end{aligned} \tag{64}
$$

Applying (58)–(64) to (57) and using the first estimate (54) and our assumptions $k_i, -k'_i, k''_i \ge 0$ and $|\gamma'|\gamma^{-1} < \min\{1, -(k'_i/2)\}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|v''_m\right\|_2^2 + \gamma^{-4}\int_\Omega a\left(v'_m, v'_m\right) dy - \left(\gamma'\gamma^{-1}\right)^2\left\|\nabla v'_m \cdot y\right\|_2^2 \right. \\
&\qquad - 8\gamma^{-5}\gamma' \int_\Omega a\left(v_m, v'_m\right) dy \\
&\qquad + \tau_1\gamma^{-2}\int_{\Gamma_1} \left(k_1(t)\left|v'_m\right|^2 - k'_1 \square v'_m\right) d\Gamma \\
&\qquad \left. + \tau_2\gamma^{-2}\int_{\Gamma_1} \left(k_2(t)\left|\frac{\partial v'_m}{\partial \nu}\right|^2 - k'_2 \square \frac{\partial v'_m}{\partial \nu}\right) d\Gamma\right]
\end{aligned}
$$

$$\le C_8 \left[\|v_m''\|_2^2 + \int_\Omega a(v_m', v_m')\,dy + \int_\Omega a(v_m, v_m')\,dy \right.$$
$$+ \int_{\Gamma_1} \left(k_1(t)\left|v_m'\right|^2 - k_1' \square v_m'\right) d\Gamma$$
$$\left. + \int_{\Gamma_1} \left(k_2(t)\left|\frac{\partial v_m'}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m'}{\partial \nu}\right) d\Gamma \right]$$
$$+ \tau_1 \gamma^{-3} \left|\gamma'\right| \int_{\Gamma_1} k_1^2(t) \left|v_{0m}\right|^2 d\Gamma$$
$$+ \tau_2 \gamma^{-3} \left|\gamma'\right| \int_{\Gamma_1} k_2^2(t) \left|\frac{\partial v_{0m}}{\partial \nu}\right|^2 d\Gamma + C_9. \tag{65}$$

From (55) and our choice of v_{0m} and v_{1m} and integrating (65) over $(0,t)$ with $t \in (0,t_m)$, we obtain

$$\|v_m''\|_2^2 + \gamma^{-4} \int_\Omega a(v_m', v_m')\,dy - \left(\gamma'\gamma^{-1}\right)^2 \left\|\nabla v_m' \cdot y\right\|_2^2$$
$$- 8\gamma^{-5}\gamma' \int_\Omega a(v_m, v_m')\,dy$$
$$+ \tau_1 \gamma^{-2} \int_{\Gamma_1} \left(k_1(t)\left|v_m'\right|^2 - k_1' \square v_m'\right) d\Gamma$$
$$+ \tau_2 \gamma^{-2} \int_{\Gamma_1} \left(k_2(t)\left|\frac{\partial v_m'}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m'}{\partial \nu}\right) d\Gamma$$
$$\le 2C_8 \int_0^t \left[\|v_m''(s)\|_2^2 + \int_\Omega a(v_m'(s), v_m'(s))\,dy \right.$$
$$+ \int_\Omega a(v_m(s), v_m'(s))\,dy$$
$$+ \int_{\Gamma_1} \left(k_1(s)\left|v_m'(s)\right|^2 - \left(k_1' \square v_m'\right)(s)\right) d\Gamma$$
$$\left. + \int_{\Gamma_1} \left(k_2(s)\left|\frac{\partial v_m'(s)}{\partial \nu}\right|^2 - \left(k_2' \square \frac{\partial v_m'}{\partial \nu}\right)(s)\right) d\Gamma \right] ds$$
$$+ C_{10}. \tag{66}$$

Using the same arguments as for (53), we get

$$\left(\gamma'\gamma^{-1}\right)^2 \left\|\nabla v_m' \cdot y\right\|_2^2 < \frac{\gamma^{-4}}{2} \int_\Omega a(v_m', v_m')\,dy, \tag{67}$$

for all $t \ge 0$. Therefore, by Gronwall's lemma, we obtain

$$\|v_m''\|_2^2 + \int_\Omega a(v_m', v_m')\,dy + \int_{\Gamma_1} \left(k_1(t)\left|v_m'\right|^2 - k_1' \square v_m'\right) d\Gamma$$
$$+ \int_{\Gamma_1} \left(k_2(t)\left|\frac{\partial v_m'}{\partial \nu}\right|^2 - k_2' \square \frac{\partial v_m'}{\partial \nu}\right) d\Gamma \le C_{11}, \tag{68}$$

where C_{11} is a positive constant which is independent of m and t.

According to (54) and (68), we get

$$\{v_m\} \text{ is bounded in } L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{69}$$
$$\{v_m'\} \text{ is bounded in } L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{70}$$
$$\{v_m''\} \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right). \tag{71}$$

From (69) to (71), there exists a subsequence of $\{v_m\}$, which we still denote by $\{v_m\}$, such that

$$v_m \longrightarrow v \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{72}$$
$$v_m' \longrightarrow v' \text{ weak star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \tag{73}$$
$$v_m'' \longrightarrow v'' \text{ weak star in } L^\infty\left(0,T;L^2(\Omega)\right). \tag{74}$$

Letting $m \to \infty$ in (46) and using (72)–(74), we obtain

$$\int_\Omega a(v,w)\,dy$$
$$= -\gamma^4 \int_\Omega v'' w\,dy - \gamma^4 \int_\Omega A(t) v w\,dy$$
$$- \gamma^4 \int_\Omega c(y,t) \cdot \nabla v' w\,dy$$
$$- \gamma^4 \int_\Omega b(y,t) \cdot \nabla v w\,dy \tag{75}$$
$$- \tau_1 \gamma^2 \int_{\Gamma_1} \left\{v' + k_1(0) v - k_1(t) v_0 + k_1' * v\right\} w\,d\Gamma$$
$$- \tau_2 \gamma^2 \int_{\Gamma_1} \left\{\frac{\partial v'}{\partial \nu} + k_2(0)\frac{\partial v}{\partial \nu} - k_2(t)\frac{\partial v_0}{\partial \nu}\right.$$
$$\left. + k_2' * \frac{\partial v}{\partial \nu}\right\} \frac{\partial w}{\partial \nu} d\Gamma$$

for any $w \in H_0^2(\Omega)$. From Lemma 3 we obtain that $v \in L^\infty(0,T;H^4(\Omega))$. The uniqueness of solutions follows by using standard arguments. □

Theorem 5. *Under the hypotheses of Theorem 4, let $u_0 \in H_0^2(\Omega_0) \cap H^4(\Omega_0)$, $u_1 \in H_0^2(\Omega_0)$. Then there exists a unique solution u of the problem* (4)–(8) *satisfying*

$$u \in L^\infty\left(0,\infty; H_0^2(\Omega_t) \cap H^4(\Omega_t)\right),$$
$$u' \in L^\infty\left(0,\infty; H_0^2(\Omega_t)\right), \tag{76}$$
$$u'' \in L^\infty\left(0,\infty; L^2(\Omega_t)\right).$$

Proof. This idea was used in [11, 13, 14, 16, 17]. To show the existence in noncylindrical domains, we return to our original problem in the noncylindrical domains by using the change variable given in (14) by $(y,t) = \tau(x,t)$, $(x,t) \in \widehat{Q}$.

Let v be the solution obtained from Theorem 4 and u defined by (16); then u belongs to the class

$$\begin{aligned} u &\in L^{\infty}\left(0,\infty;H_0^2\left(\Omega_t\right)\cap H^4\left(\Omega_t\right)\right),\\ u' &\in L^{\infty}\left(0,\infty;H_0^2\left(\Omega_t\right)\right),\\ u'' &\in L^{\infty}\left(0,\infty;L^2\left(\Omega_t\right)\right). \end{aligned} \tag{77}$$

Denoting by

$$u(x,t)=v(y,t)=(v\circ\tau)(x,t), \tag{78}$$

then from (15) it is easy to see that u satisfies (4)–(8) in the sense of $L^{\infty}(0,\infty;L^2(\Omega_t))$. If u_1,u_2 are two solutions obtained through the diffeomorphism τ given by (14), then $v_1=v_2$, so $u_1=u_2$. Thus the proof of Theorem 5 is completed. □

4. Exponential Decay

In this section, we show that the solution of system (4)–(8) decays exponentially. First of all, we introduce the useful lemma for a noncylindrical domain.

Lemma 6 (see [11, 12]). *Let $G(\cdot,\cdot)$ be the smooth function defined in $\Omega_t\times[0,\infty[$. Then*

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_t}G(x,t)\,dx=&\int_{\Omega_t}\frac{d}{dt}G(x,t)\,dx\\ &+\gamma'\gamma^{-1}\int_{\Gamma_t}G(x,t)(x\cdot\overline{\nu})\,d\Gamma, \end{aligned} \tag{79}$$

where $\overline{\nu}$ is the x-component of the unit normal exterior ν.

By the same argument of (27) and (28), it can be written as

$$\mathscr{B}_2u=\tau_1\left\{u'+k_1(0)u-k_1(t)u_0+k_1'*u\right\}, \tag{80}$$

$$\mathscr{B}_1u=-\tau_2\left\{\frac{\partial u'}{\partial\nu}+k_2(0)\frac{\partial u}{\partial\nu}-k_2(t)\frac{\partial u_0}{\partial\nu}+k_2'*\frac{\partial u}{\partial\nu}\right\}. \tag{81}$$

We use (80) and (81) instead of the boundary conditions (6) and (7).

We will use the following lemma.

Lemma 7 (see [4]). *For every $u\in H^4(\Omega)$ and for every $\mu\in\mathbb{R}$, one has*

$$\begin{aligned} &\int_{\Omega_t}(m\cdot\nabla u)\Delta^2u\,dx\\ &=\int_{\Omega_t}a(u,u)\,dx+\frac{1}{2}\int_{\Gamma_t}(m\cdot\nu)a(u,u)\,d\Gamma\\ &\quad+\int_{\Gamma_t}\left[(\mathscr{B}_2u)(m\cdot\nabla u)-(\mathscr{B}_1u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right]d\Gamma. \end{aligned} \tag{82}$$

Now, we define the energy of problem (4)–(8) by

$$\begin{aligned} E(t)=\frac{1}{2}\Bigg[&\left\|u'\right\|_{2,t}^2+\int_{\Omega_t}a(u,u)\,dx\\ &+\tau_1\int_{\Gamma_{1,t}}\left(k_1(t)|u|^2-k_1'\Box u\right)d\Gamma\\ &+\tau_2\int_{\Gamma_{1,t}}\left(k_2(t)\left|\frac{\partial u}{\partial\nu}\right|^2-k_2'\Box\frac{\partial u}{\partial\nu}\right)d\Gamma\Bigg]. \end{aligned} \tag{83}$$

We observe that $E(t)$ is a positive function. Using Lemmas 6 and 1, we have

$$\begin{aligned} E'(t)\le&\frac{\gamma'\gamma^{-1}}{2}\int_{\Gamma_{1,t}}\left[\left|u'\right|^2+a(u,u)\right](x\cdot\overline{\nu})\,d\Gamma\\ &-\frac{\tau_1}{2}\int_{\Gamma_{1,t}}\left|u'\right|^2d\Gamma+\frac{\tau_1}{2}k_1^2(t)\int_{\Gamma_{1,t}}\left|u_0\right|^2d\Gamma\\ &+\frac{\tau_1}{2}k_1'(t)\int_{\Gamma_{1,t}}|u|^2d\Gamma-\frac{\tau_1}{2}\int_{\Gamma_{1,t}}k_1''\Box u\,d\Gamma\\ &-\frac{\tau_2}{2}\int_{\Gamma_{1,t}}\left|\frac{\partial u'}{\partial\nu}\right|^2d\Gamma+\frac{\tau_2}{2}k_2^2(t)\int_{\Gamma_{1,t}}\left|\frac{\partial u_0}{\partial\nu}\right|^2d\Gamma\\ &+\frac{\tau_2}{2}k_2'(t)\int_{\Gamma_{1,t}}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma-\frac{\tau_2}{2}\int_{\Gamma_{1,t}}k_2''\Box\frac{\partial u}{\partial\nu}d\Gamma. \end{aligned} \tag{84}$$

Let us consider the following functional:

$$\psi(t)=\int_{\Omega_t}(m\cdot\nabla u)u'\,dx. \tag{85}$$

The following lemma plays an important role for the construction of the Lyapunov functional.

Lemma 8. *Let one suppose that the initial data $(u_0,u_1)\in(H^4(\Omega_0)\cap H_0^2(\Omega_0))\times H_0^2(\Omega_0)$ and satisfies the compatibility condition (41). Then the solution of system (4)–(8) satisfies*

$$\begin{aligned} \psi'(t)\le&\frac{1}{2}\int_{\Gamma_{1,t}}(m\cdot\nu)\left|u'\right|^2d\Gamma-\int_{\Omega_t}\left|u'\right|^2dx\\ &-\int_{\Omega_t}a(u,u)\,dx-\frac{1}{2}\int_{\Gamma_{1,t}}(m\cdot\nu)a(u,u)\,d\Gamma\\ &-\int_{\Gamma_{1,t}}\left[(\mathscr{B}_2u)(m\cdot\nabla u)-(\mathscr{B}_1u)\frac{\partial}{\partial\nu}(m\cdot\nabla u)\right]d\Gamma\\ &+\gamma'\gamma^{-1}\int_{\Gamma_{1,t}}(m\cdot\nabla u)u'(x\cdot\overline{\nu})\,d\Gamma. \end{aligned} \tag{86}$$

Proof. Differentiating ψ and using (4) and Lemmas 6 and 7, we get

$$\begin{aligned}\psi'(t) = {} & \int_{\Omega_t} \left(m \cdot \nabla u'\right) u' dx + \int_{\Omega_t} (m \cdot \nabla u) u'' dx \\ & + \gamma' \gamma^{-1} \int_{\Gamma_{1,t}} (m \cdot \nabla u) u' (x \cdot \overline{\nu}) d\Gamma \\ = {} & \frac{1}{2} \int_{\Gamma_{1,t}} (m \cdot \nu) \left|u'\right|^2 d\Gamma - \int_{\Omega_t} \left|u'\right|^2 dx \\ & - \int_{\Omega_t} a(u,u) dx - \frac{1}{2} \int_{\Gamma_t} (m \cdot \nu) a(u,u) d\Gamma \\ & - \int_{\Gamma_t} \left[(\mathscr{B}_2 u)(m \cdot \nabla u) - (\mathscr{B}_1 u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) \right] d\Gamma \\ & + \gamma' \gamma^{-1} \int_{\Gamma_t} (m \cdot \nabla u) u' (x \cdot \overline{\nu}) d\Gamma. \end{aligned} \tag{87}$$

Let us next examine the integrals over $\Gamma_{0,t}$ in (87). Since $u = \partial u / \partial \nu = 0$ on $\Gamma_{0,t}$, we have

$$\begin{gathered} B_1 u = B_2 u = \nabla u = 0 \quad \text{on } \Gamma_{0,t}, \\ u_{x_1} = \frac{\partial u}{\partial \nu} \nu_1, \qquad u_{x_2} = \frac{\partial u}{\partial \nu} \nu_2, \end{gathered} \tag{88}$$

and hence

$$\begin{aligned} \int_{\Gamma_{0,t}} (\mathscr{B}_1 u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma &= \int_{\Gamma_{0,t}} \Delta u (m \cdot \nu) \frac{\partial^2 u}{\partial \nu^2} d\Gamma \\ &= \int_{\Gamma_{0,t}} (m \cdot \nu) |\Delta u|^2 d\Gamma, \end{aligned} \tag{89}$$

$$\int_{\Gamma_{0,t}} (m \cdot \nu) a(u,u) d\Gamma = \int_{\Gamma_{0,t}} (m \cdot \nu) |\Delta u|^2 d\Gamma. \tag{90}$$

Therefore, from (87)–(90) we have

$$\begin{aligned} \psi'(t) = {} & \frac{1}{2} \int_{\Gamma_{1,t}} (m \cdot \nu) \left|u'\right|^2 d\Gamma - \int_{\Omega_t} \left|u'\right|^2 dx - \int_{\Omega_t} a(u,u) dx \\ & + \frac{1}{2} \int_{\Gamma_{0,t}} (m \cdot \nu) |\Delta u|^2 d\Gamma \\ & - \frac{1}{2} \int_{\Gamma_{1,t}} (m \cdot \nu) a(u,u) d\Gamma - \int_{\Gamma_{1,t}} (\mathscr{B}_2 u)(m \cdot \nabla u) d\Gamma \\ & + \int_{\Gamma_{1,t}} (\mathscr{B}_1 u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma \\ & + \gamma' \gamma^{-1} \int_{\Gamma_{1,t}} (m \cdot \nabla u) u' (x \cdot \overline{\nu}) d\Gamma. \end{aligned} \tag{91}$$

Noting that $m \cdot \nu \le 0$ on $\Gamma_{0,t}$ follows from (91), we have the conclusion of the lemma. □

Let us introduce the Lyapunov functional

$$\mathscr{L}(t) = NE(t) + \psi(t), \tag{92}$$

with $N > 0$. Using Young's inequality and choosing $N > 0$ sufficiently large, we see that

$$q_0 E(t) \le \mathscr{L}(t) \le q_1 E(t) \tag{93}$$

for q_0 and q_1 are positive constants. We will show later that the functional $\mathscr{L}$ satisfies the inequality of the following result.

Lemma 9 (see [7]). *Let f be a real positive function of class C^1. If there exist positive constants p_0, p_1, and p_2 such that*

$$f'(t) \le -p_0 f(t) + p_1 e^{-p_2 t} \tag{94}$$

then there exist positive constants p and c such that

$$f(t) \le \left(f(0) + c\right) e^{-pt}. \tag{95}$$

Finally, we will show the main result of this section.

Theorem 10. *Assume that there exist positive constants β_1 and β_2 such that*

$$\begin{gathered} k_i(0) > 0, \quad k_i'(t) \le -\beta_1 k_i(t), \\ k_i''(t) \ge -\beta_2 k_i'(t), \quad i = 1,2. \end{gathered} \tag{96}$$

If $(u_0, u_1) \in H_0^2(\Omega_0) \times L^2(\Omega_0)$ then there exist constants $\omega, C > 0$ such that

$$E(t) \le CE(0) e^{-\omega t}, \quad \forall t \ge 0. \tag{97}$$

Proof. From (84) and Lemma 8 we have

$$\begin{aligned} \mathscr{L}'(t) \le {} & \frac{\gamma' \gamma^{-1} N}{2} \int_{\Gamma_{1,t}} \left[\left|u'\right|^2 + a(u,u) \right] (x \cdot \overline{\nu}) d\Gamma \\ & - \frac{\tau_1 N}{2} \int_{\Gamma_{1,t}} \left|u'\right|^2 d\Gamma + \frac{\tau_1 N}{2} k_1^2(t) \int_{\Gamma_{1,t}} \left|u_0\right|^2 d\Gamma \\ & + \frac{\tau_1 N}{2} k_1'(t) \int_{\Gamma_{1,t}} |u|^2 d\Gamma - \frac{\tau_1 N}{2} \int_{\Gamma_{1,t}} k_1'' \Box u d\Gamma \\ & - \frac{\tau_2 N}{2} \int_{\Gamma_{1,t}} \left|\frac{\partial u'}{\partial \nu}\right|^2 d\Gamma \\ & + \frac{\tau_2 N}{2} k_2^2(t) \int_{\Gamma_{1,t}} \left|\frac{\partial u_0}{\partial \nu}\right|^2 d\Gamma \\ & + \frac{\tau_2 N}{2} k_2'(t) \int_{\Gamma_{1,t}} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma \\ & - \frac{\tau_2 N}{2} \int_{\Gamma_{1,t}} k_2'' \Box \frac{\partial u}{\partial \nu} d\Gamma \\ & + \frac{1}{2} \int_{\Gamma_{1,t}} (m \cdot \nu) \left|u'\right|^2 d\Gamma \end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega_t}\left|u'\right|^2dx-\int_{\Omega_t}a\left(u,u\right)dx\\
&-\frac{1}{2}\int_{\Gamma_{1,t}}\left(m\cdot\nu\right)a\left(u,u\right)d\Gamma\\
&-\int_{\Gamma_{1,t}}\left[\left(\mathscr{B}_2u\right)\left(m\cdot\nabla u\right)-\left(\mathscr{B}_1u\right)\frac{\partial}{\partial\nu}\left(m\cdot\nabla u\right)\right]d\Gamma\\
&+\gamma'\gamma^{-1}\int_{\Gamma_{1,t}}\left(m\cdot\nabla u\right)u'\left(x\cdot\overline{\nu}\right)d\Gamma.
\end{aligned}\tag{98}
$$

Since the boundary conditions (80) and (81) can be written as

$$
\begin{aligned}
\mathscr{B}_2u&=\tau_1\left\{u'+k_1\left(t\right)u-k_1\left(t\right)u_0-k_1'\circ u\right\},\\
\mathscr{B}_1u&=-\tau_2\left\{\frac{\partial u'}{\partial\nu}+k_2\left(t\right)\frac{\partial u}{\partial\nu}-k_2\left(t\right)\frac{\partial u_0}{\partial\nu}-k_2'\circ\frac{\partial u}{\partial\nu}\right\},
\end{aligned}\tag{99}
$$

by using Young's inequality we obtain

$$
\begin{aligned}
\left|-\int_{\Gamma_{1,t}}\left(\mathscr{B}_2u\right)\left(m\cdot\nabla u\right)d\Gamma\right|\le{}&\frac{\tau_1}{2\epsilon}\int_{\Gamma_{1,t}}\left|u'\right|^2d\Gamma\\
&+\frac{\tau_1}{2\epsilon}k_1^2\left(t\right)\int_{\Gamma_{1,t}}|u|^2d\Gamma\\
&+\frac{\tau_1}{2\epsilon}k_1^2\left(t\right)\int_{\Gamma_{1,t}}\left|u_0\right|^2d\Gamma\\
&+\frac{\tau_1}{2\epsilon}\int_{\Gamma_{1,t}}k_1\left(0\right)\left|k_1'\right|\Box u\,d\Gamma\\
&+\frac{\epsilon}{2}\int_{\Gamma_{1,t}}|m\cdot\nabla u|^2d\Gamma,
\end{aligned}\tag{100}
$$

$$
\begin{aligned}
\left|\int_{\Gamma_{1,t}}\left(\mathscr{B}_1u\right)\frac{\partial}{\partial\nu}\left(m\cdot\nabla u\right)d\Gamma\right|\le{}&\frac{\tau_2}{2\epsilon}\int_{\Gamma_{1,t}}\left|\frac{\partial u'}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2}{2\epsilon}k_2^2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2}{2\epsilon}k_2^2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u_0}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2}{2\epsilon}\int_{\Gamma_{1,t}}k_2\left(0\right)\left|k_2'\right|\Box\frac{\partial u}{\partial\nu}d\Gamma\\
&+\frac{\epsilon}{2}\int_{\Gamma_{1,t}}\left|\frac{\partial}{\partial\nu}\left(m\cdot\nabla u\right)\right|^2d\Gamma,
\end{aligned}\tag{101}
$$

where ϵ is a positive constant. Since the bilinear form $a(u,u)$ is strictly coercive, using the trace theory and the fact $m\cdot\nu\ge\delta_0$ on $\Gamma_{1,t}$, we get

$$
\begin{aligned}
&\int_{\Gamma_{1,t}}|m\cdot\nabla u|^2d\Gamma+\int_{\Gamma_{1,t}}\left|\frac{\partial}{\partial\nu}\left(m\cdot\nabla u\right)\right|^2d\Gamma\\
&\quad\le\lambda_0\int_{\Omega_t}a\left(u,u\right)dx+\frac{\lambda_0}{\delta_0}\int_{\Gamma_{1,t}}\left(m\cdot\nu\right)a\left(u,u\right)d\Gamma,
\end{aligned}\tag{102}
$$

where λ_0 is a constant depending on Ω and μ. Substituting inequalities (100)–(102) into (98) we have

$$
\begin{aligned}
\mathscr{L}'\left(t\right)\le{}&\frac{\gamma'\gamma^{-1}N}{2}\int_{\Gamma_{1,t}}\left[\left|u'\right|^2+a\left(u,u\right)\right]\left(x\cdot\overline{\nu}\right)d\Gamma\\
&-\frac{\tau_1N}{2}\int_{\Gamma_{1,t}}\left|u'\right|^2d\Gamma+\frac{\tau_1N}{2}k_1^2\left(t\right)\int_{\Gamma_{1,t}}\left|u_0\right|^2d\Gamma\\
&-\frac{\tau_1\beta_1N}{2}k_1\left(t\right)\int_{\Gamma_{1,t}}|u|^2d\Gamma+\frac{\tau_1\beta_2N}{2}\int_{\Gamma_{1,t}}k_1'\Box u d\Gamma\\
&-\frac{\tau_2N}{2}\int_{\Gamma_{1,t}}\left|\frac{\partial u'}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2N}{2}k_2^2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u_0}{\partial\nu}\right|^2d\Gamma\\
&-\frac{\tau_2\beta_1N}{2}k_2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2\beta_2N}{2}\int_{\Gamma_{1,t}}k_2'\Box\frac{\partial u}{\partial\nu}d\Gamma\\
&-\int_{\Omega_t}\left|u'\right|^2dx-\left(1-\frac{\epsilon\lambda_0}{2}\right)\int_{\Omega_t}a\left(u,u\right)dx\\
&-\left(\frac{1}{2}-\frac{\epsilon\lambda_0}{2\delta_0}\right)\int_{\Gamma_{1,t}}\left(m\cdot\nu\right)a\left(u,u\right)d\Gamma\\
&+\frac{1}{2}\int_{\Gamma_{1,t}}\left(m\cdot\nu\right)\left|u'\right|^2d\Gamma+\frac{\tau_1}{2\epsilon}\int_{\Gamma_{1,t}}\left|u'\right|^2d\Gamma\\
&+\frac{\tau_1}{2\epsilon}k_1^2\left(t\right)\int_{\Gamma_{1,t}}|u|^2d\Gamma+\frac{\tau_1}{2\epsilon}k_1^2\left(t\right)\int_{\Gamma_{1,t}}\left|u_0\right|^2d\Gamma\\
&+\frac{\tau_1}{2\epsilon}\int_{\Gamma_{1,t}}k_1\left(0\right)\left|k_1'\right|\Box u d\Gamma+\frac{\tau_2}{2\epsilon}\int_{\Gamma_{1,t}}\left|\frac{\partial u'}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2}{2\epsilon}k_2^2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma+\frac{\tau_2}{2\epsilon}k_2^2\left(t\right)\int_{\Gamma_{1,t}}\left|\frac{\partial u_0}{\partial\nu}\right|^2d\Gamma\\
&+\frac{\tau_2}{2\epsilon}\int_{\Gamma_{1,t}}k_2\left(0\right)\left|k_2'\right|\Box\frac{\partial u}{\partial\nu}d\Gamma\\
&+\gamma'\gamma^{-1}\int_{\Gamma_{1,t}}\left(m\cdot\nabla u\right)u'\left(x\cdot\overline{\nu}\right)d\Gamma.
\end{aligned}\tag{103}
$$

First, choose $\epsilon > 0$ sufficiently small such that

$$1 - \frac{\epsilon\lambda_0}{2} > 0, \quad \frac{1}{2} - \frac{\epsilon\lambda_0}{2\delta_0} > 0. \tag{104}$$

Then, choosing N large enough, we have

$$\mathscr{L}'(t) \le -c_2 E(t) + c_3 K^2(t) E(0), \tag{105}$$

where $c_2, c_3 > 0$ and $K(t) = k_1(t) + k_2(t)$. From (93), (96), and (105), we obtain

$$\mathscr{L}'(t) \le -\frac{c_2}{q_1}\mathscr{L}(t) + c_4 c_3 E(0) e^{-2\beta_1 t} \text{ for some } c_4 > 0. \tag{106}$$

By Lemma 9, there exist positive constants c_5 and c_6 such that

$$\mathscr{L}(t) \le \left(\mathscr{L}(0) + c_5 E(0)\right) e^{-c_6 t}, \quad \forall t \ge 0. \tag{107}$$

Using (93), we conclude that

$$E(t) \le CE(0) e^{-\omega t}, \quad \forall t \ge 0 \tag{108}$$

for some positive constants C and ω. □

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