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*Research Article*

# On Standing Wave Solutions for Discrete Nonlinear Schrödinger Equations

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The purpose of this paper is to study a class of discrete nonlinear Schrödinger equations. Under a weak superlinearity condition at infinity instead of the classical Ambrosetti-Rabinowitz condition, the existence of standing waves of the equations is obtained by using the Nehari manifold approach.

## 1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation was first derived in the context of nonlinear optics by Christodoulides and Joseph [1]; see also [2–5]. DNLS equation is one of the most important inherently discrete models, having a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology [6–10]. For example, Davydov [6] studied the equation in molecular biology and Su et al. [10] considered the equation in condensed matter physics. Eilbeck et al. [11] firstly pointed out the universal nature of the discrete nonlinear Schrödinger equation and reported a number of applications.

For the analytical study, many authors studied the existence results of standing wave solutions for DNLS equations. Much of the works concerns the periodic DNLS equations [12–14]. Recently, some authors considered the DNLS equations with infinitely growing potential. Zhang and Pankov [15, 16] devoted their efforts to the case of infinitely growing potential and power-like nonlinearity. In all these results, the nonlinearity is supposed to be either positive (self-focusing), or negative (defocusing). Pankov [17] studied the DNLS equatifvons with infinitely growing potential and sign-changing nonlinearity (a mixture of self-focusing and defocusing ones). Pankov and Zhang were concerned with the DNLS equations with infinitely growing potential and saturable nonlinearity in [18].

In this paper, we consider higher-dimensional generalizations of DNLS equation

$$\begin{gathered} i\dot{\phi}_n + (\mathscr{A}\phi)_n - v_n\phi_n + \sigma f(n,\phi_n) = 0, \\ n = (n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m, \end{gathered} \tag{1}$$

where

$$\begin{aligned} (\mathscr{A}\phi)_n = {} & \phi_{(n_1+1,n_2,\ldots,n_m)} + \phi_{(n_1,n_2+1,\ldots,n_m)} \\ & + \cdots + \phi_{(n_1,n_2,\ldots,n_m+1)} - 2m\phi_{(n_1,n_2,\ldots,n_m)} \\ & + \phi_{(n_1-1,n_2,\ldots,n_m)} + \phi_{(n_1,n_2-1,\ldots,n_m)} \\ & + \cdots + \phi_{(n_1,n_2,\ldots,n_m-1)}, \end{aligned} \tag{2}$$

and $\sigma = \pm 1$. The parameter $\sigma$ characterizes the focusing properties of the following equation: if $\sigma = 1$, the equation is self-focusing, while $\sigma = -1$ corresponds to the defocusing equation.

We assume that the nonlinearity $f(n,u)$ is gauge invariant, that is,

$$f\left(n, e^{i\theta}u\right) = e^{i\theta} f(n,u), \quad \theta \in \mathbb{R}. \tag{3}$$

Then we can consider the special solutions of the form $\phi_n = e^{-i\omega t}u_n$, for any $\omega \in \mathbb{R}$. These solutions are called breather solutions or standing waves, due to their periodic

time behavior. Inserting the ansatz of a breather solution into (1), it follows that $\phi_n$ satisfies the nonlinear system of algebraic equations

$$-(\mathscr{A}u)_n + v_n u_n - \omega u_n - \sigma f(n, u_n) = 0, \quad n \in \mathbb{Z}^m. \tag{4}$$

We need the following assumptions.

$(V_1)$ The discrete potential $V = \{v_n\}_{n\in\mathbb{Z}^m}$ satisfies

$$\lim_{|n|\to\infty} v_n = \infty, \tag{5}$$

where $|n| = |n_1| + |n_2| + \cdots + |n_m|$ is the length of multi-index $n$.

$(f_1)$ $f \in C(\mathbb{Z}^m \times \mathbb{R}, \mathbb{R})$, and there exists $a > 0$, $p \in (2, \infty)$ such that

$$|f(n, u)| \le a\left(1 + |u|^{p-1}\right), \quad \forall n \in \mathbb{Z}^m,\ u \in \mathbb{R}. \tag{6}$$

$(f_2)$ $\lim_{|u|\to 0} f(n, u)/u = 0$ uniformly for $n \in \mathbb{Z}^m$.

$(f_3)$ $\lim_{|u|\to\infty} F(n, u)/u^2 = +\infty$ uniformly for $n \in \mathbb{Z}^m$, where $F(n, u)$ is the primitive function of $f(n, u)$, that is,

$$F(n, u) = \int_0^u f(n, s)\, ds. \tag{7}$$

$(f_4)$ $u \mapsto f(n, u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

We are concerned with the existence of ground state solutions, that is, solutions corresponding to the least positive critical value of the variational functional. To obtain the existence of ground states, usually besides the growth condition on the nonlinearity and a Nehari type condition, the following classical Ambrosetti-Rabinowitz superlinear condition (see, e.g., [19]) is assumed:

$$0 < \mu F(n, u) \le f(n, u)\, u, \quad \text{for some } \mu > 2,\ u \ne 0. \tag{8}$$

It is easy to see that (8) implies that $F(n, u) \ge C|u|^\mu$, for some constant $C > 0$ and $|u| \ge 1$.

In this paper, instead of (8) we assume the superquadratic condition $(f_3)$. It is easy to see that (8) implies $(f_3)$. It is well known that many nonlinearities such as

$$f(n, u) = u \ln(1 + |u|), \tag{9}$$

do not satisfy (8). A crucial role that (8) plays is to ensure the boundedness of Palais-Smale sequences.

This paper is organized as follows. In Section 2, we establish the variational framework associated with (4). We then present the main results of this paper and compare them with the existing ones. Section 3 is devoted to prove some useful lemmas, and the proof of the main results is completed in Section 4.

## 2. Preliminaries

In order to apply the critical point theory, we will establish the corresponding variational framework associated with (4).

For some positive integer $m$, we consider the real sequence spaces

$$l^p \equiv l^p(\mathbb{Z}^m) = \left\{ u = \{u_n\}_{n\in\mathbb{Z}^m} : \forall n \in \mathbb{Z}^m,\ u_u \in \mathbb{R}, \right.$$
$$\left. \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}^m} |u_n|^p\right)^{1/p} < \infty \right\}. \tag{10}$$

Then the following embedding between $l^p$ spaces holds:

$$l^q \subset l^p, \quad \|u\|_{l^p} \le \|u\|_{l^q}, \quad 1 \le q \le p \le \infty. \tag{11}$$

Let

$$L = -\mathscr{A} + V, \tag{12}$$

which is a self-adjoint operator defined on $l^p(\mathbb{Z}^m)$ (see [20]).

Define the space

$$E := \left\{ u \in l^2(\mathbb{Z}^m) : L^{1/2} u \in l^2(\mathbb{Z}^m) \right\}. \tag{13}$$

Then $E$ is a Hilbert space equipped with the norm

$$\|u\| = \left\| L^{1/2} u \right\|_{l^2(\mathbb{Z}^m)}. \tag{14}$$

Now we consider the variational functional $J$ defined on $E$ by

$$J(u) = \frac{1}{2}((L - \omega)u, u) - \sigma \sum_{n\in\mathbb{Z}^m} F(n, u_n), \tag{15}$$

where $(\cdot, \cdot)$ is the inner product in $l^2$. Then $J \in C^1(E, \mathbb{R})$. And for the derivative of $J$, we have the following formula:

$$\left(J'(u), v\right) = ((L - \omega)u, v) - \sigma \sum_{n\in\mathbb{Z}^m} f(n, u_n)\, v_n, \quad \forall v \in E. \tag{16}$$

Equation (16) implies that (4) is the corresponding Euler-Lagrange equation for $J$. Thus, we have reduced the problem of finding a nontrivial solution of (4) to that of seeking a nonzero critical point of the functional $J$ on $E$.

The following lemma plays an important role in this paper; it was established in [20].

**Lemma 1.** *If $V$ satisfies the condition $(V_1)$, then*

(1) *for any $2 \le p \le \infty$, the embedding map from $E$ into $l^p(\mathbb{Z}^m)$ is compact,*

(2) *the spectrum $\sigma(L)$ is discrete and consists of simple eigenvalues accumulating to $+\infty$.*

By Lemma 1, we may assume that $\lambda_1$ is the smallest eigenvalue of $L$, that is

$$\lambda_1 = \inf \sigma(L). \tag{17}$$

Now we are ready to state the main results.

**Theorem 2.** *Suppose that conditions $(V_1)$ and $(f_1)$–$(f_4)$ are satisfied. Then one has the following conclusions.*

(1) *If $\sigma = -1$, $\omega \le \lambda_1$, (4) has no nontrivial solution.*

(2) *If $\sigma = 1$, $\omega < \lambda_1$, (4) has a nontrivial ground state solution.*

(3) *If $\sigma = 1$, $\omega < \lambda_1$, and $f(n,u)$ is odd in $u$ for each $n \in \mathbb{Z}^m$, (4) has infinitely many pairs of solutions $\pm u^{(k)}$ in $E$.*

*Remark 3.* In [20], the author considered the following DNLS equation:

$$Lu_n - \omega u_n - \sigma \gamma_n f(u_n) = 0, \tag{18}$$

where there exists a positive constant $\overline{\gamma}$, such that for any $n \in \mathbb{Z}^m$, $0 < \gamma_n \le \overline{\gamma}$. Clearly, (18) corresponds (4) if we let $f(n,u) = \gamma_n f(u)$. Therefore, (18) is a special case of (4).

In [20], the nonlinearity $f \in C^1(\mathbb{R})$ satisfies the following condition:

$$0 < (q-1) f(u) u \le f'(u) u^2, \quad \forall u \ne 0,\ 2 < q < \infty, \tag{19}$$

which implies (8). So it is a stronger condition than $(f_3)$. Therefore, our results generalize the corresponding ones.

*Remark 4.* In [16], the authors also considered (18) and assumed that the nonlinearity $f \in C^1(\mathbb{R})$ satisfies the classical Ambrosetti-Rabinowitz superlinear condition (8). Clearly, it is a stronger condition than $(f_3)$.

Since $\omega < \lambda_1$, we may introduce an equivalent norm in $E$ by setting

$$\|u\|^2 := ((L-\omega)u, u), \tag{20}$$

and then the functional $J$ can be rewritten as

$$J(u) = \frac{1}{2}\|u\|^2 - \sigma \sum_{n\in\mathbb{Z}^m} F(n, u_n). \tag{21}$$

To prove the multiplicity results, we need the following lemma.

**Lemma 5** (see [21]). *Let $S = \{w \in E : \|w\| = 1\}$. If $E$ is a infinite-dimensional Hilbert space, $\Phi \in C^1(S, \mathbb{R})$ is even and bounded below and satisfies the Palais-Smale condition. Then $\Phi$ has infinitely many pairs of critical points.*

## 3. Some Lemmas

In this section, we always assume that $\sigma = 1$.

We define the Nehari manifold

$$\mathcal{N} = \left\{u \in E \setminus \{0\} : J'(u) u = 0\right\}. \tag{22}$$

To prove the main results, we need some lemmas.

**Lemma 6.** *Suppose that conditions $(V_1)$ and $(f_1)$–$(f_4)$ are satisfied. Then one has*

(1) *$F(n,u) > 0$ and $(1/2) f(n,u)u > F(n,u)$ for all $u \ne 0$,*

(2) *$J(u) > 0$, for all $u \in \mathcal{N}$.*

*Proof.* (1) From $(f_2)$ and $(f_4)$, it is easy to get that

$$F(n,u) > 0, \quad \forall u \ne 0. \tag{23}$$

By $(f_4)$, we have

$$\frac{u}{2} f(n,u) - \int_0^u f(n,s)\, ds > \frac{u}{2} f(n,u) - \frac{f(n,u)}{u} \int_0^u s\, ds = 0. \tag{24}$$

So $(1/2) f(n,u)u > F(n,u)$ for all $u \ne 0$.

(2) For all $u \in \mathcal{N}$, by (1), we have

$$\begin{aligned} J(u) &= J(u) - \frac{1}{2} J'(u) u \\ &= \sigma \sum_{n\in\mathbb{Z}^m} \left(\frac{1}{2} f(n,u) u - F(n,u)\right) > 0. \end{aligned} \tag{25}$$

□

**Lemma 7.** *Suppose that conditions $(V_1)$ and $(f_1)$–$(f_4)$ are satisfied, and let $I(u) = \sum_{n\in\mathbb{Z}^m} F(n, u_n)$. Then one has the following.*

(1) *$I'(u) = o(\|u\|)$ as $u \to 0$.*

(2) *$s \mapsto I'(su)u/s$ is strictly increasing for all $u \ne 0$ and $s > 0$.*

(3) *$I(su)/s^2 \to \infty$ uniformly for $u$ on the weakly compact subsets of $E \setminus \{0\}$, as $s \to \infty$.*

*Proof.* (1) and (2) are easy to be shown from $(f_2)$ and $(f_4)$, respectively. Next, we verify (3). Let $W \subset E \setminus \{0\}$ be weakly compact and let $\{u^{(k)}\} \subset W$. It suffices to show that if $s^{(k)} \to \infty$ as $k \to \infty$, then so does a subsequence of $I(s^{(k)}u^{(k)})/(s^{(k)})^2$. Passing to a subsequence if necessary, $u^{(k)} \rightharpoonup u \in E \setminus \{0\}$ and $u_n^{(k)} \to u_n$ for every $n$, as $k \to \infty$.

Since $|s^{(k)} u_n^{(k)}| \to \infty$ and $u^{(k)} \ne 0$, by $(f_3)$ and (23), we have

$$\frac{I\left(s^{(k)}u^{(k)}\right)}{\left(s^{(k)}\right)^2} = \sum_{n\in\mathbb{Z}^m} \frac{F\left(n, s^{(k)}u_n^k\right)}{\left(s^{(k)}u_n^{(k)}\right)^2} \left(u_n^{(k)}\right)^2 \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{26}$$

□

**Lemma 8.** *Under the assumptions $(V_1)$ and $(f_1)$–$(f_4)$, for each $w \in E \setminus \{0\}$, there exists a unique $s_w > 0$ such that $s_w w \in \mathcal{N}$.*

*Proof.* Let $g(s) := J(sw)$, $s > 0$. Note that

$$g'(s) = J'(sw) w = s\left(\|w\|^2 - s^{-1} I'(sw) w\right), \tag{27}$$

and from (2) of Lemma 7, then there exists a unique $s_w$, such that $g'(s) > 0$ whenever $0 < s < s_w$, $g'(s) < 0$ whenever $s > s_w$, and $g'(s_w) = J'(s_w w) w = 0$. So $s_w w \in \mathcal{N}$. □

*Remark 9.* By (1) and (3) of Lemma 7, $g(s) > 0$ for $s > 0$ small and $g(s) < 0$ for $s > 0$ large. Together with Lemma 8, we have that $s_w$ is a unique maximum of $g(s)$ and $s_w w$ is the unique point on the ray $s \mapsto sw$ $(s > 0)$ which intersects with $\mathcal{N}$. That is, $u \in \mathcal{N}$ is the unique maximum of $J$ on the ray. Therefore, we may define the mapping $\widehat{m} : E \setminus \{0\} \to \mathcal{N}$ and $m : S \to \mathcal{N}$ by setting

$$\widehat{m}(w) := s_w w, \qquad m := \widehat{m}|_S, \tag{28}$$

where $S = \{u \in E : \|u\| = 1\}$.

**Lemma 10.** *For each compact subset $\mathcal{V} \subset S$, there exists a constant $C_{\mathcal{V}}$ such that $s_w \le C_{\mathcal{V}}$ for all $w \in \mathcal{V}$.*

*Proof.* Suppose that, by contradiction, $s_w^{(k)} \to \infty$ as $k \to \infty$. By Lemma 6 and $(f_3)$, we have

$$\begin{aligned} 0 < \frac{J\left(s_w^{(k)} w\right)}{\left(s_w^{(k)}\right)^2} &= \frac{1}{2}\|w\|^2 \\ &- \sum_{n\in\mathbb{Z}^m} \frac{F\left(n, s_w^{(k)} w_n\right)}{\left(s_w^{(k)}\right)^2 w_n^2} w_n^2 \longrightarrow -\infty, \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{29}$$

This is a contradiction. □

**Lemma 11.** *(1) The mapping $\widehat{m}$ is continuous.*

*(2) The mapping $m$ is a homeomorphism between $S$ and $\mathcal{N}$, and the inverse of $m$ is given by $m^{-1}(u) = u/\|u\|$.*

*Proof.* (1) Suppose that $w_n \to w \neq 0$. Since $\widehat{m}(tu) = \widehat{m}(u)$ for each $t > 0$, we may assume that $w_n \in S$ for all $n$. Write $\widehat{m}(w_n) = s_{w_n} w_n$. By Lemmas 8 and 10, $\{s_{w_n}\}$ is bounded, and hence $s_{w_n} \to s > 0$ after passing to a subsequence if needed. Since $\mathcal{N}$ is closed and $\widehat{m}(w_n) = s_{w_n} w_n \to sw$, $sw \in \mathcal{N}$. Hence $sw = s_w w = \widehat{m}(w)$ by the uniqueness of $s_w$ of Lemma 8. (2) This is an immediate consequence of (1). □

**Lemma 12.** *$J$ satisfies the Palais-Smale condition on $\mathcal{N}$.*

*Proof.* Let $\{u^{(k)}\} \subset \mathcal{N}$ be a sequence such that $J(u^{(k)}) \le d$ for some $d > 0$ and $J'(u^{(k)}) \to 0$ as $k \to \infty$.

Firstly, we prove that $\{u^{(k)}\}$ is bounded. In fact, if not, we may assume by contradiction that $\|u^{(k)}\| \to \infty$ as $k \to \infty$. Let $v^{(k)} = u^{(k)}/\|u^{(k)}\|$. Then there exists a subsequence, still denoted by the same notation, such that $v^{(k)} \rightharpoonup v$ in $E$ as $k \to \infty$.

Suppose that $v = 0$. For every $s > 0$, from Remark 9, we have

$$d \ge J\left(u^{(k)}\right) \ge J\left(sv^{(k)}\right) = \frac{1}{2}s^2\left\|v^{(k)}\right\|^2 - I\left(sv^{(k)}\right) \longrightarrow \frac{1}{2}s^2. \tag{30}$$

This is a contradiction if $s \ge \sqrt{2d}$. Therefore, $v \neq 0$.

According to Lemma 7(3), we have

$$\begin{aligned} 0 &\le \frac{J\left(u^{(k)}\right)}{\left\|u^{(k)}\right\|^2} \\ &= \frac{1}{2} - \frac{I\left(\left\|u^{(k)}\right\| v^{(k)}\right)}{\left\|u^{(k)}\right\|^2} \longrightarrow -\infty, \quad k \longrightarrow \infty, \end{aligned} \tag{31}$$

a contradiction again. Thus, $\{u^{(k)}\}$ is bounded.

Finally, we show that there exists a convergent subsequence of $\{u^{(k)}\}$. Actually, there exists a subsequence, still denoted by the same notation, such that $u^{(k)} \rightharpoonup u$. By Lemma 1, for any $2 \le q \le \infty$, then

$$u^{(k)} \longrightarrow u \quad \text{in } l^q\left(\mathbb{Z}^m\right). \tag{32}$$

Note that

$$\begin{aligned} &\left\|u^{(k)} - u\right\|^2 - \omega\left\|u^{(k)} - u\right\|_2^2 \\ &\quad = \left(J'\left(u^{(k)}\right) - J'(u), \left(u^{(k)} - u\right)\right) \\ &\qquad + \sum_{n\in\mathbb{Z}^m} \left(f\left(n, u_n^{(k)}\right) - f(n, u_n)\right)\left(u_n^{(k)} - u_n\right). \end{aligned} \tag{33}$$

The first term $(J'(u^{(k)}) - J'(u), (u^{(k)} - u)) \to 0$ as $k \to \infty$ because of the weak convergence.

By $(f_1)$ and $(f_2)$, it is easy to show that for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$, such that

$$\left|f(n,u)\right| \le \varepsilon|u| + c_\varepsilon|u|^{p-1}, \qquad |F(n,u)| \le \varepsilon|u|^2 + c_\varepsilon|u|^p. \tag{34}$$

Then,

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^m} \left(f\left(n, u_n^{(k)}\right) - f(n, u_n)\right)\left(u_n^{(k)} - u_n\right) \\ &\quad \le \sum_{n\in\mathbb{Z}^m} \left[\varepsilon\left(\left|u_n^{(k)}\right| + |u_n|\right)\right. \\ &\qquad \left. + c_\varepsilon\left(\left|u_n^{(k)}\right|^{p-1} + |u_n|^{p-1}\right)\right]\left|u_n^{(k)} - u_n\right| \\ &\quad \le \varepsilon\left(\left\|u^{(k)}\right\|_{l^2} + \|u\|_{l^2}\right)\left\|u^{(k)} - u\right\|_{l^2} \\ &\qquad + c_\varepsilon\left(\left\|u^{(k)}\right\|_{l^p}^{p-1} + \|u\|_{l^p}^{p-1}\right)\left\|u^{(k)} - u\right\|_{l^p}. \end{aligned} \tag{35}$$

Combining (32) and the boundedness of $\{u^{(k)}\}$, the above inequality implies

$$\begin{aligned} &\sum_{n\in\mathbb{Z}^m} \left(f\left(n, u_n^{(k)}\right) - f(n, u_n)\right) \\ &\qquad \times \left(u_n^{(k)} - u_n\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{36}$$

It follows from (33) that $u^{(k)} \to u$ in $E$; that is, $J$ satisfies Palais-Smale condition.

The proof is complete. □

Now we define the functional $\widehat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ by

$$\widehat{\Psi}(w) := J(\widehat{m}(w)), \qquad \Psi(w) := \widehat{\Psi}|_S. \tag{37}$$

**Lemma 13.** *(1) $\widehat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$, and*

$$\widehat{\Psi}'(w) z = \frac{\|\widehat{m}(w)\|}{\|w\|} J'(\widehat{m}(w)) z \quad \forall w, z \in E,\ w \neq 0. \tag{38}$$

*(2) $\Psi \in C^1(S, \mathbb{R})$, and*

$$\begin{gathered} \Psi'(w) = \| m(w) \| J'(m(w)) z \\ \forall z \in T_w(S) = \{v \in E : (w, v) = 0\}. \end{gathered} \tag{39}$$

*(3) $\{w_n\}$ is a Palais-Smale sequence for $\Psi$ if and only if $\{m(w_n)\}$ is a Palais-Smale sequence for $J$.*

*(4) $w$ is a critical point of $\Psi$ if and only if $m(w)$ is a nontrivial critical point of $J$. Moreover, the corresponding values of $\Psi$ and $J$ coincide and $\inf_S \Psi = \inf_{\mathcal{N}} J$.*

*Proof.* (1) Let $w \in E \setminus \{0\}$ and $z \in E$. By Remark 9 and the mean value theorem, we obtain

$$\begin{aligned} \widehat{\Psi}(w + tz) - \widehat{\Psi}(w) &= J(s_{w+tz}(w + tz)) - J(s_w w) \\ &\leq J(s_{w+tz}(w + tz)) - J(s_{w+tz}(w)) \\ &= J'(s_{w+tz}(w + \tau_t tz)) s_{w+tz} tz, \end{aligned} \tag{40}$$

where $|t|$ is small enough and $\tau_t \in (0, 1)$. Similarly,

$$\begin{aligned} \widehat{\Psi}(w + tz) - \widehat{\Psi}(w) &= J(s_{w+tz}(w + tz)) - J(s_w w) \\ &\geq J(s_w(w + tz)) - J(s_w(w)) \\ &= J'(s_w(w + \eta_t tz)) s_w tz, \end{aligned} \tag{41}$$

where $\eta_t \in (0, 1)$. From the proof of Lemma 11, the function $w \mapsto s_w$ is continuous, combining these two inequalities that

$$\begin{aligned} &\lim_{t \to 0} \frac{\widehat{\Psi}(w + tz) - \widehat{\Psi}(w)}{t} \\ &\quad = s_w J'(s_w w) z \\ &\quad = \frac{\|\widehat{m}(w)\|}{\|w\|} J'(\widehat{m}(w)) z. \end{aligned} \tag{42}$$

Hence the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in $z$ and continuous in $w$. It follows that $\widehat{\Psi}$ is a class of $C^1$ (see [19, Proposition 1.3]).

(2) follows from (1). Note only that since $w \in S$, $m(w) = \widehat{m}(w)$.

(3) Let $\{w_n\}$ be a Palais-Smale sequence for $\Psi$, and let $u_n = m(w_n) \in \mathcal{N}$. Since for every $w_n \in S$ we have an orthogonal splitting $E = T_{w_n} S \oplus \mathbb{R} w_n$, using (2) we have

$$\begin{aligned} \|\Psi'(w_n)\| &= \sup_{\substack{z \in T_{w_n} S \\ \|z\| = 1}} \Psi'(w_n) z \\ &= \|m(w_n)\| \sup_{\substack{z \in T_{w_n} S \\ \|z\| = 1}} J'(m(w_n)) z \\ &= \|u_n\| \sup_{\substack{z \in T_{w_n} S \\ \|z\| = 1}} J'(u_n) z. \end{aligned} \tag{43}$$

Using (2) again, then

$$\begin{aligned} \|\Psi'(w_n)\| &\leq \|u_n\| \|J'(u_n)\| \\ &= \|u_n\| \sup_{\substack{z \in T_{w_n} S, t \in \mathbb{R} \\ z + tw \neq 0}} \frac{J'(u_n)(z + tw)}{\|z + tw\|} \\ &\leq \|u_n\| \sup_{z \in T_{w_n} S \setminus \{0\}} \frac{J'(u_n)(z)}{\|z\|} = \|\Psi'(w_n)\|. \end{aligned} \tag{44}$$

Therefore,

$$\|\Psi'(w_n)\| = \|u_n\| \|J'(u_n)\|. \tag{45}$$

According to Lemma 6, for $u_n \in \mathcal{N}$, $J(u_n) > 0$, so there exists a constant $c > 0$ such that $J(u_n) > c$. And since $c \leq J(u_n) = (1/2)\|u_n\|^2 - I(u_n) \leq (1/2)\|u_n\|^2$, $\|u_n\| \geq \sqrt{2c}$. Together with Lemma 12, $\sqrt{2c} \leq \|u_n\| \leq \sup_n \|u_n\| < \infty$. Hence $\{w_n\}$ is a Palais-Smale sequence for $\Psi$ if and only if $\{u_n\}$ is a Palais-Smale sequence for $J$.

(4) By (45), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. The other part is clear. □

## 4. Proof of Main Results

*Proof of Theorem 2.* (1) If $\sigma = -1$, $\omega \leq \lambda_1$, we suppose that (4) has a nontrivial solution $u \in E$. Then $u$ is a nonzero critical point of $J$ in $E$ and $J'(u) = 0$. But

$$\begin{aligned} (J'(u), u) &= ((L - \omega) u, u) - \sigma \sum_{n \in \mathbb{Z}^m} f(n, u_n) u_n \\ &\geq \sum_{n \in \mathbb{Z}^m} f(n, u_n) u_n > 0. \end{aligned} \tag{46}$$

This is a contradiction.

(2) If $\sigma = 1$, $\omega < \lambda_1$. We firstly show that $\Psi$ satisfies the Palais-Smale condition.

Let $\{w^{(k)}\}$ be a Palais-Smale sequence for $\Psi$; then $\{u^{(k)}\}$ is a Palais-Smale sequence for $J$ by Lemma 13(3), where $u^{(k)} := m(w^{(k)}) \in \mathcal{N}$. From Lemma 12, $u^{(k)} \to u$ after passing to a subsequence and $w^{(k)} \to m^{-1}(u)$, so $\Psi$ satisfies the Palais-Smale condition.

Let $\{w^{(k)}\} \subset S$ be a minimizing sequence for $\Psi$. By Ekeland's variational principle, we may assume that $\Psi'(w^{(k)}) \to 0$ as $k \to \infty$, so $\{w^{(k)}\}$ is a Palais-Smale sequence for $\Psi$.

By Palais-Smale condition, $w^{(k)} \to w$ after passing to a subsequence if needed. Hence $w$ is a minimizer for $\Psi$ and therefore a critical point of $\Psi$, and then $u = m(w)$ is a critical point of $J$ and is also a minimizer for $J$ by Lemma 13. Therefore, $u$ is a ground state solution of (4).

(3) If $\sigma = 1$, $\omega < \lambda_1$, and $f(n,u)$ is odd in $u$ for each $n \in \mathbb{Z}^m$, then $J$ is even and so is $\Psi$. Since $\inf_S \Psi = \inf_{\mathcal{N}} J > 0$ and $\Psi$ satisfies the Palais-Smale condition, $\Psi$ has infinitely many pairs of critical points by Lemma 5. It follows that (4) has infinitely many pairs of solutions $\pm u^{(k)}$ in $E$ from Lemma 13.

This completes Theorem 2. □

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