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Research Article

Stability and Bifurcation Analysis for a Predator-Prey Model with Discrete and Distributed Delay

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We propose a two-dimensional predatory-prey model with discrete and distributed delay. By the use of a new variable, the original two-dimensional system transforms into an equivalent three-dimensional system. Firstly, we study the existence and local stability of equilibria of the new system. And, by choosing the time delay τ as a bifurcation parameter, we show that Hopf bifurcation can occur as the time delay τ passes through some critical values. Secondly, by the use of normal form theory and central manifold argument, we establish the direction and stability of Hopf bifurcation. At last, an example with numerical simulations is provided to verify the theoretical results. In addition, some simple discussion is also presented.

1. Introduction

Since the pioneering theoretical works by Lotka [1] and Volterra [2], there were a lot of authors who studied all kinds of predator-prey models modeled by ordinary differential equations (ODEs). To reflect that the dynamical behavior of the models depends on the past history of the system, it is often necessary to incorporate time delays into the models. Therefore, a more realistic predator-prey model should be described by delayed differential equations (DDEs) [3–11]. In general, delay differential equations exhibit more complicated dynamics on stability, periodic structure, bifurcation, and so on [12–26]. In [27, 28], the authors investigated the effect of the discrete delay on the stability of the model. In [29], the effect of the distributed delay on the stability of the model was investigated. In [11], the authors proposed a Logistic model with discrete and distributed delays:

$$x'(t) = rx(t)\left[1 - a_1 x(t-\tau) - a_2 \int_{-\infty}^{t} f(t-s)\, x(s)\, \mathrm{d}s\right], \tag{1}$$

where the parameters r, τ, a_1, a_2 are positive constants. The function f in (1) is called the delayed kernel, which is the weight given to the population t time units ago. And it was assumed that $f(t) \geq 0$ for all $t \geq 0$, together with the normalization condition

$$\int_0^{\infty} f(t)\, \mathrm{d}t = 1, \tag{2}$$

which ensures that the steady states of the model (1) are unaffected by the delay. They studied the stability of the positive equilibrium and existence of Hopf bifurcations, and direction and stability of the Hopf bifurcation were also analyzed. In [7], the authors proposed and investigated the following predator-prey model with time delay:

$$\begin{aligned} x'(t) &= x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)\right], \\ y'(t) &= y(t)\left[-r_2 + a_{21}x(t-\tau) - a_{22}y(t-\tau)\right], \end{aligned} \tag{3}$$

where $x(t)$ and $y(t)$ can be interpreted as the population densities of the prey and the predator at time t, respectively. $r_1 > 0$ denotes the intrinsic growth rate of the prey, and $r_2 > 0$ denotes the death rate of the predator. For the convenience of computation, they chose the same $\tau > 0$ as delays; the delay τ represents the feedback time delay of the prey species to the growth of itself in term $a_{11}x(t-\tau)$, represents the feedback time delay of the predator species to the growth of itself in term $a_{22}y(t-\tau)$, represents the hunting delay in term $a_{12}y(t-\tau)$, and represents the time of the predator maturation in term $a_{21}x(t-\tau)$. The parameters a_{ij} $(i,j=1,2)$ are all positive constants. They studied the stability of the positive equilibrium and existence of Hopf bifurcations.

Motivated by [7, 11, 27–29] and the references cited therein, in the present paper, we will consider the following predator-prey model with discrete and distributed delay:

$$\begin{aligned} N'(t) &= N(t)\left[\alpha_1 - a_{11}N(t-\tau) - a_{12}P(t-\tau)\right], \\ P'(t) & \\ &= P(t)\left[-\alpha_2 + a_{21}\int_{-\infty}^{t} G(t-s)N(s)\,\mathrm{d}s - a_{22}P(t-\tau)\right], \end{aligned} \tag{4}$$

where $N(t)$ and $P(t)$ can be interpreted as the population densities of the prey and the predator at time t, respectively. $\alpha_1>0$ denotes the intrinsic growth rate of the prey and $\alpha_2>0$ denotes the death rate of the predator; $\tau>0$ represents the feedback time delay of the prey species to the growth of itself in term $a_{11}N(t-\tau)$, represents the feedback time delay of the predator species to the growth of itself in term $a_{22}P(t-\tau)$, and represents the hunting delay in term $a_{12}P(t-\tau)$; $a_{ij}>0\,(i,j=1,2)$. The function $G(t)$ is the same as the function $f(s)$ in system (1). Following the ideas of Cushing [30], we define $G(t)$ as the following weak kernel function:

$$G(t) = \alpha e^{-\alpha t}, \quad \alpha > 0. \tag{5}$$

Next, we define a new variable:

$$u(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} N(s)\,\mathrm{d}s, \tag{6}$$

then using the linear chain trick technique, system (4) can be transformed into the following equivalent system:

$$\begin{aligned} N'(t) &= N(t)\left[\alpha_1 - a_{11}N(t-\tau) - a_{12}P(t-\tau)\right], \\ P'(t) &= P(t)\left[-\alpha_2 + a_{21}u(t) - a_{22}P(t-\tau)\right], \\ u'(t) &= \alpha N(t) - \alpha u(t). \end{aligned} \tag{7}$$

The organization of this paper is as follows. In Section 2, we will consider the existence and stability of equilibria of system (7). The existence of Hopf bifurcation is also discussed. In Section 3, by use of normal form theory and central manifold argument, we illustrate the direction and stability of Hopf bifurcation. In Section 4, we provide an example with some numerical simulations to verify the theoretical results, and we also give some brief discussion.

2. Local Stability of Equilibria and the Existence of Hopf Bifurcations

In this section, we will finish two tasks: (a) investigating the existence and stability of equilibriums of system (7) and (b) studying the effect of time delay on the system (7); that is, we will choose τ as bifurcating parameter to analyze Hopf bifurcation.

Let the right equations of system (7) equal zero; we get the following algebraic equations:

$$\begin{aligned} N &= N(\alpha_1 - a_{11}N - a_{12}P), \\ P &= P(-\alpha_2 + a_{21}u - a_{22}P), \\ u &= \alpha N - \alpha u. \end{aligned} \tag{8}$$

By simple computation, we know that the trivial equilibrium and boundary equilibrium of system (7) always exist with values $E_0 = (0,0,0)$ and $E_1 = (\alpha_1/a_{11}, 0, \alpha_1/a_{11})$, respectively. In addition, we have the following results.

(i) The eigenvalues of characteristic equations at the trivial equilibrium E_0 are $\lambda_1=\alpha_1>0$, $\lambda_2=-\alpha_2<0$, and $\lambda_3=-\alpha<0$, which means that this equilibrium is always unstable.

(ii) The eigenvalues of characteristic equations at the boundary equilibrium E_1 are $\lambda_1=0$ and $\lambda_2=-\alpha<0$, and other eigenvalues are determined by $\lambda+\alpha_1 e^{-\lambda\tau}=0$. When $\tau=0$, it is easy to see that $\lambda_3=-\alpha_1<0$, which means that this equilibrium is locally stable; whereas when $\tau>0$, the sign of the real part of the eigenvalues can not be determined, which means that this equilibrium may be locally stable or unstable.

In fact, there exists a unique positive equilibrium $E_2=(N^*,u^*,P^*)$ for system (7) provided that $\alpha_1 a_{21}-\alpha_2 a_{11}>0$, holds. Here

$$N^* = u^* = \frac{\alpha_1 a_{22} + \alpha_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \qquad P^* = \frac{\alpha_1 a_{21} - \alpha_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}. \tag{9}$$

Next, we always assume that

(H1) $\alpha_1 a_{21} - \alpha_2 a_{11} > 0$ holds.

The characteristic equation for system (7) at the equilibrium E_2 takes the form

$$\lambda^3 + d_1\lambda^2 + \left(d_2\lambda^2 + d_3\lambda + d_4\right)e^{-\lambda\tau} + \left(d_5\lambda + d_6\right)e^{-2\lambda\tau} = 0, \tag{10}$$

where

$$\begin{aligned} &d_1 = \alpha, \qquad d_2 = a_{11}N^* + a_{22}P^*, \\ &d_3 = \alpha a_{11}N^* + \alpha a_{22}P^*, \\ &d_4 = \alpha a_{21}a_{12}N^*P^*, \qquad d_5 = a_{11}a_{22}N^*P^*, \\ &d_6 = \alpha a_{11}a_{22}N^*P^*. \end{aligned} \tag{11}$$

Multiplying $e^{\lambda\tau}$ on both sides of (10), we obtain equivalent characteristic equation as

$$\left(\lambda^3 + d_1\lambda^2\right)e^{\lambda\tau} + \left(d_2\lambda^2 + d_3\lambda + d_4\right) + \left(d_5\lambda + d_6\right)e^{-\lambda\tau} = 0. \tag{12}$$

When $\tau = 0$, characteristic equation (10) or (12) becomes

$$\lambda^3 + (d_1 + d_2)\lambda^2 + (d_3 + d_5)\lambda + d_4 + d_6 = 0. \quad (13)$$

It is easy to confirm that $d_1 + d_2 > 0$, $d_4 + d_6 > 0$ and $(d_1 + d_2)(d_3 + d_5) > d_4 + d_6$. By the Routh-Hurwitz criterion we know that all the roots of (13) have negative real parts. Thus, the positive equilibrium E_2 is locally asymptotically stable for $\tau = 0$.

Next, we will consider the eigenvalues of (12) for $\tau > 0$. Suppose that there is a pure imaginary root $\lambda = i\omega$, $\omega > 0$, then we get

$$\begin{aligned}&\left(-i\omega^3 - d_1\omega^2\right)(\cos\omega\tau + i\sin\omega\tau) - d_2\omega^2 + d_3 i\omega \\ &\quad + d_4 + (d_5 i\omega + d_6)(\cos\omega\tau - i\sin\omega\tau) = 0.\end{aligned} \quad (14)$$

Separating the real and imaginary parts, we have

$$\begin{aligned}&\left(-d_1\omega^2 + d_6\right)\cos\omega\tau + \left(\omega^3 + d_5\omega\right)\sin\omega\tau = d_2\omega^2 - d_4, \\ &\left(d_5\omega - \omega^3\right)\cos\omega\tau - \left(d_1\omega^2 + d_6\right)\sin\omega\tau = -d_3\omega.\end{aligned} \quad (15)$$

By simple calculation, we can obtain the following equations:

$$\begin{aligned}\sin\omega\tau &= \frac{d_2\omega^5 + (d_1d_3 - d_4 - d_2d_5)\omega^3 + (d_4d_5 - d_3d_6)\omega}{\omega^6 + d_1^2\omega^4 - d_5^2\omega^2 - d_6^2}, \\ \cos\omega\tau &= \frac{(d_3 - d_1d_2)\omega^4 + (d_1d_4 - d_2d_6 + d_3d_5)\omega^2 + d_4d_6}{\omega^6 + d_1^2\omega^4 - d_5^2\omega^2 - d_6^2}.\end{aligned} \quad (16)$$

Let

$$\begin{aligned}&e_1 = d_1^2, \quad e_2 = -d_5^2, \quad e_3 = -d_6^2, \\ &e_4 = d_2, \quad e_5 = d_1d_3 - d_4 - d_2d_5, \\ &e_6 = d_4d_5 - d_3d_6, \quad e_7 = d_3 - d_1d_2, \\ &e_8 = d_1d_4 - d_2d_6 + d_3d_5, \quad e_9 = d_4d_6,\end{aligned} \quad (17)$$

then $\sin\omega\tau$, $\cos\omega\tau$ can be written as

$$\sin\omega\tau = \frac{e_4\omega^5 + e_5\omega^3 + e_6\omega}{\omega^6 + e_1\omega^4 + e_2\omega^2 + e_3}, \quad (18)$$

$$\cos\omega\tau = \frac{e_7\omega^4 + e_8\omega^2 + e_9}{\omega^6 + e_1\omega^4 + e_2\omega^2 + e_3}. \quad (19)$$

By adding the square of (18) and (19), we obtain

$$\omega^{12} + f_1\omega^{10} + f_2\omega^8 + f_3\omega^6 + f_4\omega^4 + f_5\omega^2 + f_6 = 0, \quad (20)$$

where

$$\begin{aligned}&f_1 = 2e_1 - e_4^2, \quad f_2 = e_1^2 + 2e_2 - 2e_4e_5 - e_7^2, \\ &f_3 = 2e_3 + 2e_1e_2 - 2e_4e_6 - 2e_7e_8 - e_5^2, \\ &f_4 = 2e_1e_3 + e_2^2 - 2e_5e_6 - 2e_7e_9 - e_8^2, \\ &f_5 = 2e_2e_3 - e_6^2 - 2e_8e_9, \quad f_6 = e_3^2 - e_9^2.\end{aligned} \quad (21)$$

Denote $z = \omega^2$, then (20) becomes

$$z^6 + f_1z^5 + f_2z^4 + f_3z^3 + f_4z^2 + f_5z + f_6 = 0. \quad (22)$$

Let

$$G(z) = z^6 + f_1z^5 + f_2z^4 + f_3z^3 + f_4z^2 + f_5z + f_6, \quad (23)$$

then the following assumption holds true.

(H2) Equation (22) has at least one positive real root.

In fact, if all the parameters of system (7) are given, it is easy to calculate the root of (22) by using a computer. Since $\lim_{z\to\infty} G(z) = +\infty$, we conclude that if $f_6 < 0$, then (22) has at least one positive real root. Without loss of generality, we assume that (22) has six positive roots, defined by z_1, z_2, z_3, z_4, z_5, and z_6, respectively. Then (20) has six positive roots:

$$\begin{aligned}&\omega_1 = \sqrt{z_1}, \quad \omega_2 = \sqrt{z_2}, \quad \omega_3 = \sqrt{z_3}, \\ &\omega_4 = \sqrt{z_4}, \quad \omega_5 = \sqrt{z_5}, \quad \omega_6 = \sqrt{z_6}.\end{aligned} \quad (24)$$

By (19), we get

$$\cos\omega_k\tau = \frac{e_7\omega_k^4 + e_8\omega_k^2 + e_9}{\omega_k^6 + e_1\omega_k^4 + e_2\omega_k^2 + e_3}. \quad (25)$$

If we denote

$$\tau_k^{(j)} = \frac{1}{\omega_k}\left\{\arccos\left(\frac{e_7\omega_k^4 + e_8\omega_k^2 + e_9}{\omega_k^6 + e_1\omega_k^4 + e_2\omega_k^2 + e_3}\right) + 2j\pi\right\}, \quad (26)$$

where $k = 1, 2, \ldots, 6$ and $j = 0, 1, 2, \ldots$, then $\pm i\omega_k$ is a pair of purely imaginary roots of (12). Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,\ldots,6\}}\left\{\tau_k^{(0)}\right\}, \qquad \omega_0 = \omega_{k_0}. \quad (27)$$

Here, we use the method by [25–27], which is different from [31].

In order to obtain the main result, it is necessary to make the following assumption.

(H3) $\mathrm{Re}(d\lambda/d\tau)|_{\tau=\tau_0} \neq 0$.

Taking the derivative of λ with respect to τ in (12), it is easy to obtain

$$\begin{aligned}&\left(3\lambda^2 + 2d_1\lambda\right)e^{\lambda\tau}\frac{d\lambda}{d\tau} + \left(\lambda^3 + d_1\lambda^2\right)e^{\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) \\ &\quad + (2d_2\lambda + d_3)\frac{d\lambda}{d\tau} + d_5e^{-\lambda\tau}\frac{d\lambda}{d\tau} \\ &\quad + (d_5\lambda + d_6)e^{-\lambda\tau}\left(-\lambda - \tau\frac{d\lambda}{d\tau}\right) = 0,\end{aligned} \quad (28)$$

which is equivalent to

$$\begin{aligned}\frac{d\lambda}{d\tau} &= \left(\lambda(d_5\lambda + d_6)e^{-\lambda\tau} - \lambda\left(\lambda^3 + d_1\lambda^2\right)e^{\lambda\tau}\right) \\ &\quad \times \left(\left(3\lambda^2 + 2d_1\lambda\right)e^{\lambda\tau} + \left(\lambda^3 + d_1\lambda^2\right)\tau e^{\lambda\tau}\right. \\ &\quad \left. + d_5e^{-\lambda\tau} - (d_5\lambda + d_6)\tau e^{-\lambda\tau} + 2d_2\lambda + d_3\right)^{-1}.\end{aligned} \quad (29)$$

By (12), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{\left(3\lambda^2 + 2d_1\lambda\right)e^{\lambda\tau} + d_5e^{-\lambda\tau} + 2d_2\lambda + d_3}{\lambda\left(d_5\lambda + d_6\right)e^{-\lambda\tau} - \lambda\left(\lambda^3 + d_1\lambda^2\right)e^{\lambda\tau}} - \frac{\tau}{\lambda} \tag{30}$$

$$= \frac{\left(3\lambda^2 + 2d_1\lambda\right)e^{\lambda\tau} + d_5e^{-\lambda\tau} + 2d_2\lambda + d_3}{d_2\lambda^3 + d_3\lambda^2 + d_4\lambda + 2\lambda\left(d_5\lambda + d_6\right)e^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{31}$$

Take $\lambda = i\omega$ into the above equation, we get

$$\begin{aligned}
&\Big(\left(-3\omega^2 + 2d_1 i\omega\right)(\cos\omega\tau + i\sin\omega\tau) \\
&\quad + d_5(\cos\omega\tau - i\sin\omega\tau) + 2d_2 i\omega + d_3\Big) \\
&\quad \times \Big(-d_2 i\omega^3 - d_3\omega^2 + d_4 i\omega + 2i\omega\left(d_5 i\omega + d_6\right) \\
&\quad \times (\cos\omega\tau - i\sin\omega\tau)\Big)^{-1} - \frac{\tau}{i\omega} \\
&= \Big(\left(-3\omega^2 + d_5\right)\cos\omega\tau - 2d_1\omega\sin\omega\tau + d_3 \\
&\quad + i\left[\left(-3\omega^2 - d_5\right)\sin\omega\tau + 2d_1\omega\cos\omega\tau + 2d_2\omega\right]\Big) \\
&\quad \times \Big(\left(-3d_3\omega^2 - 2d_5\omega^2\cos\omega\tau + 2d_6\omega\sin\omega\tau\right) \\
&\quad + i\Big(-d_2\omega^3 + d_4\omega + 2d_6\omega\cos\omega\tau \\
&\quad + 2d_5\omega^2\sin\omega\tau\Big)^{-1}\Big) - \frac{\tau}{i\omega}.
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
Q &= \left(-3d_3\omega^2 - 2d_5\omega^2\cos\omega\tau + 2d_6\omega\sin\omega\tau\right)^2 \\
&\quad + \left(-d_2\omega^3 + d_4\omega + 2d_6\omega\cos\omega\tau + 2d_5\omega^2\sin\omega\tau\right)^2 > 0.
\end{aligned} \tag{33}$$

Then we get

$$\begin{aligned}
&Q\,\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&= \left[\left(-3d_3\omega^2 + d_5\right)\cos\omega\tau - 2d_1\omega\sin\omega\tau + d_3\right] \\
&\quad \times \left[-3d_3\omega^2 - 2d_5\omega^2\cos\omega\tau + 2d_6\omega\sin\omega\tau\right] \\
&\quad + \left[\left(-3\omega^2 - d_5\right)\sin\omega\tau + 2d_1\omega\cos\omega\tau + 2d_2\omega\right] \\
&\quad \times \left[-d_2\omega^3 + d_4\omega + 2d_6\omega\cos\omega\tau + 2d_5\omega^2\sin\omega\tau\right].
\end{aligned} \tag{34}$$

Note that

$$\mathrm{Sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau=\tau_0}\right\} = \mathrm{Sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_0}\right\}. \tag{35}$$

Now, we can use the following lemma to get our result.

Lemma 1 (see [32]). *Consider the exponential polynomial*

$$\begin{aligned}
&P\left(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}\right) \\
&\quad = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\
&\qquad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} \\
&\qquad + \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m},
\end{aligned} \tag{36}$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ and $p_j^{(i)}$ $(i = 0, 1, \ldots, m;\ j = 1, 2, \ldots, n)$ are constants. As $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the order of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Theorem 2. *Suppose that (H1), (H2), and (H3) hold; then the following results hold.*

(i) *The positive equilibrium E_2 of system* (7) *(or the positive equilibrium (N^*, P^*) of system* (4)*) is asymptotically stable for $\tau \in [0, \tau_0)$.*

(ii) *The positive equilibrium E_2 of system* (7) *or system* (4) *undergoes a Hopf bifurcation when $\tau = \tau_0$. That is, system* (7) *has a branch of periodic of solutions bifurcating from the positive equilibrium E_2 near $\tau = \tau_0$.*

3. Direction and Stability of the Hopf Bifurcation

In this section, following the ideas of [33], we derive the explicit formulae for determining the properties of the Hopf bifurcation at critical value of τ_0 by using the normal form and the center manifold theory. Throughout this section, we always assume that system (7) undergoes Hopf bifurcation at the positive equilibrium E_2 for $\tau = \tau_0$, and then $\pm i\omega_0$ is the corresponding purely imaginary roots of the characteristic equation at the positive equilibrium E_2.

Let $x_1 = N - N^*$, $x_2 = u - u^*$, $x_3 = P - P^*$, $\overline{x}_i(t) = x_i(\tau t)$, $\tau = \tau_0 + \mu$, dropping the bars for simplification of notations; then system (7) is transformed into functional differential equations in $C = C([-1, 0], R^3)$ as

$$x'(t) = L_\mu\left(x_t\right) + f\left(\mu, x_t\right), \tag{37}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R^3$, $L_\mu : C \to R$, $f : R \times C \to R^3$, and

$$L_\mu(\phi) = (\tau_0+\mu)\begin{bmatrix} 0 & 0 & 0 \\ \alpha & -\alpha & 0 \\ 0 & a_{21}P^* & 0 \end{bmatrix}\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{bmatrix} + (\tau_0+\mu)\begin{bmatrix} -a_{11}N^* & 0 & -a_{12}N^* \\ 0 & 0 & 0 \\ 0 & 0 & -a_{22}P^* \end{bmatrix}\begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{bmatrix}, \tag{38}$$

$$f(\mu,\phi) = (\tau_0+\mu)\begin{bmatrix} -a_{11}\phi_1(0)\phi_1(-1) - a_{12}\phi_1(0)\phi_3(-1) \\ 0 \\ a_{21}\phi_2(0)\phi_3(0) - a_{22}\phi_3(0)\phi_3(-1) \end{bmatrix}, \tag{39}$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$. By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\,\phi(\theta), \quad \text{for } \phi \in C. \tag{40}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{bmatrix} 0 & 0 & 0 \\ \alpha & -\alpha & 0 \\ 0 & a_{21}P^* & 0 \end{bmatrix}\delta(\theta) - (\tau_0+\mu)\begin{bmatrix} -a_{11}N^* & 0 & -a_{12}N^* \\ 0 & 0 & 0 \\ 0 & 0 & -a_{22}P^* \end{bmatrix}\delta(\theta+1), \tag{41}$$

where δ is the Dirac delta function. For $\phi \in C^1([-1,0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\,\phi(s), & \theta = 0, \end{cases} \qquad R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{42}$$

Then system (37) is equivalent to

$$x'(t) = A(\mu)x_t + R(\mu)x_t, \tag{43}$$

where $x_t = x(t+\theta)$, $\theta \in [-1,0]$. For $\psi \in C^1([0,1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\,\psi(-t), & s = 0, \end{cases} \tag{44}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \tag{45}$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 2, we know $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively.

Suppose that $q(\theta) = (1, q_1, q_2)^T e^{i\theta\omega_0\tau_0}$ is the eigenvalues of $A(0)$ corresponding to $i\omega_0\tau_0$: then $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$. It follows from the definition of $A(0)$ and $\eta(\theta,\mu)$ that

$$\tau_0\begin{bmatrix} 0 & 0 & 0 \\ \alpha & -\alpha & 0 \\ 0 & a_{21}P^* & 0 \end{bmatrix} q(0) + \tau_0\begin{bmatrix} -a_{11}N^* & 0 & -a_{12}N^* \\ 0 & 0 & 0 \\ 0 & 0 & -a_{22}P^* \end{bmatrix} q(-1) = i\omega_0\tau_0 q(0), \tag{46}$$

because $q(-1) = q(0)e^{-i\omega_0\tau_0}$; then we get

$$\begin{bmatrix} i\omega_0 + a_{11}N^* e^{-i\omega_0\tau_0} & 0 & a_{12}N^* e^{-i\omega_0\tau_0} \\ -\alpha & i\omega_0 + \alpha & 0 \\ 0 & -a_{21}P^* & i\omega_0 + a_{22}P^* e^{-i\omega_0\tau_0} \end{bmatrix}\begin{bmatrix} 1 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{47}$$

where

$$q_1 = \frac{\alpha i\omega_0 + \alpha(a_{12} + a_{11})N^* e^{-i\omega_0\tau_0}}{i\omega_0 a_{12}N^* \exp^{-i\omega_0\tau_0}}, \qquad q_2 = -\frac{i\omega_0 + a_{11}N^* e^{-i\omega_0\tau_0}}{a_{12}N^* \exp^{-i\omega_0\tau_0}}. \tag{48}$$

Similarly, let $q^*(\theta) = D(1, q_1^*, q_2^*)e^{i\theta\omega_0\tau_0}$ be the eigenvalues of A^* corresponding to $-i\omega_0\tau_0$; according to the definition of A^* we get

$$\tau_0\begin{bmatrix} 0 & \alpha & 0 \\ 0 & -\alpha & a_{21}P^* \\ 0 & 0 & 0 \end{bmatrix} q^*(0) + \tau_0\begin{bmatrix} -a_{11}N^* & 0 & 0 \\ 0 & 0 & 0 \\ -a_{12}N^* & 0 & -a_{22}P^* \end{bmatrix} q^*(-1) = -i\omega_0\tau_0 q^*(0), \tag{49}$$

where $q^*(-1) = q^*(0)e^{-i\omega_0\tau_0}$, from which we obtain

$$\begin{bmatrix} -i\omega_0 + a_{11}N^* e^{-i\omega_0\tau_0} & -\alpha & 0 \\ 0 & -i\omega_0 + \alpha & -a_{21}P^* \\ a_{12}N^* e^{i\omega_0\tau_0} & 0 & -i\omega_0 + a_{22}P^* e^{-i\omega_0\tau_0} \end{bmatrix} \times \begin{bmatrix} 1 \\ q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$q_1^* = \frac{-i\omega_0 + a_{11}N^* e^{i\omega_0\tau_0}}{\alpha},$$

$$q_2^* = -\frac{-i\omega_0 + a_{22}P^* e^{i\omega_0\tau_0}}{a_{12}N^* e^{i\omega_0\tau_0}}. \quad (50)$$

By (45) we get

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \overline{D}\left(1, \overline{q_1^*}, \overline{q_2^*}\right)(1, q_1, q_2)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}\left(1, \overline{q_1^*}, \overline{q_2^*}\right) e^{-i\omega_0\tau_0(\xi-\theta)} \\ &\quad \times d\eta(\theta)(1, q_1, q_2)^T e^{i\omega_0\tau_0\xi} d\xi \\ &= \overline{D}\left\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* \right. \\ &\quad - \int_{-1}^{0}\left(1, \overline{q_1^*}, \overline{q_2^*}\right)\theta e^{i\omega_0\tau_0\theta} d\eta(\theta) \\ &\quad \left. \times (1, q_1, q_2)^T \right\} \\ &= \overline{D}\left\{1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* \right. \\ &\quad + \tau\left(-a_{11}N^* - a_{12}N^* q_2 \right. \\ &\quad \left.\left. - a_{22}P^* q_2\overline{q_2^*}\right) e^{-i\omega_0\tau_0}\right\}, \end{aligned} \quad (51)$$

then we choose

$$\begin{aligned} \overline{D} = 1 \times &\left(1 + q_1\overline{q}_1^* + q_2\overline{q}_2^* \right. \\ &\left. + \tau\left(-a_{11}N^* - a_{12}N^* q_2 - a_{22}P^* q_2\overline{q}_2^*\right) e^{-i\omega_0\tau_0}\right)^{-1} \end{aligned} \quad (52)$$

such that $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Next, we will use the ideas in [33] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$z(t) = \langle q^*, x_t\rangle, \qquad W(t,\theta) = x_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}. \quad (53)$$

On the center manifold C_0, we have

$$W(z(t), \overline{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \quad (54)$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of $q^*(s)$ and $\overline{q}(\theta)$. Note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \in C_0$ of (43), since $\mu = 0$, we have

$$\begin{aligned} z'(t) &= i\omega_0\tau_0 z + \overline{q}^*(0) f(0, W(z, \overline{z}, 0)) + 2\operatorname{Re}\{zq(\theta)\} \\ &= i\omega_0\tau_0 z + \overline{q}^*(0) f_0(z, \overline{z}) \\ &= i\omega_0\tau_0 z(t) + g(z, \overline{z},), \end{aligned} \quad (55)$$

where

$$\begin{aligned} g(z, \overline{z},) = \overline{q}^*(0) f_0(z, \overline{z}) &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} \\ &\quad + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \quad (56)$$

From (53) and (54), we have

$$\begin{aligned} x_t &= (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta)) = W(t,\theta) + 2\operatorname{Re}\{z(t)q(\theta)\} \\ &= W(t,\theta) + zq(\theta) + \overline{zq(\theta)} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}\frac{\overline{z}^2}{2} \\ &\quad + (1, q_1, q_2)^T e^{i\theta\omega_0\tau_0} z + (1, \overline{q}_1, \overline{q}_2)^T e^{i\theta\omega_0\tau_0}\overline{z} + \cdots, \end{aligned} \quad (57)$$

then we can obtain

$$\begin{aligned} x_{1t}(0) &= W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} \\ &\quad + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + z + \overline{z} + O\left(|(z,\overline{z})|^3\right), \\ x_{2t}(0) &= W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} \\ &\quad + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + q_1 z + \overline{q_1 z} + O\left(|(z,\overline{z})|^3\right), \\ x_{3t}(0) &= W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} \\ &\quad + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + q_2 z + \overline{q_2 z} + O\left(|(z,\overline{z})|^3\right), \\ x_{1t}(-1) &= W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z} \\ &\quad + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + ze^{-i\omega_0\tau_0} + \overline{z}e^{i\omega_0\tau_0} + O\left(|(z,\overline{z})|^3\right), \end{aligned}$$

$$
\begin{aligned}
x_{3t}(-1) = {} & W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1) z\overline{z} \\
& + W_{02}^{(3)}(-1)\frac{\overline{z}^2}{2} + z q_2 e^{-i\omega_0\tau_0} + \overline{z q_2} e^{i\omega_0\tau_0} \\
& + O\left(|(z,\overline{z})|^3\right).
\end{aligned}
\tag{58}
$$

From the definition of $f(\mu, x_t)$, we have

$$
\begin{aligned}
g(z,\overline{z}) & = \overline{q^*}(0) f_0(z,\overline{z}) \\
& = \tau_0 \overline{D}\left(1, \overline{q_1^*}, \overline{q_2^*}\right) \\
& \quad \times \begin{bmatrix} -a_{11} x_{1t}(0) x_{1t}(-1) - a_{12} x_{1t}(0) x_{3t}(-1) \\ 0 \\ a_{21} x_{3t}(0) x_{2t}(0) - a_{22} x_{3t}(0) x_{3t}(-1) \end{bmatrix} \\
& = \tau_0 \overline{D}\left\{ z^2 \left[-a_{11} e^{-i\omega_0\tau_0} - a_{12} q_2 e^{-i\omega_0\tau_0} \right.\right. \\
& \qquad \left. + a_{21} q_1 q_2 \overline{q_2^*} - a_{22} q_2^2 \overline{q_2^*} e^{-i\omega_0\tau_0} \right] \\
& \quad + 2z\overline{z} \left[-a_{11} \operatorname{Re}\left\{ e^{i\omega_0\tau_0} \right\} - a_{12} \operatorname{Re}\left\{ q_2 e^{-i\omega_0\tau_0} \right\} \right. \\
& \quad \left. + a_{21} \overline{q_2^*} \operatorname{Re}\left\{ q_1 \overline{q_2} \right\} + a_{22} q_2^2 q_2^* \operatorname{Re}\left\{ e^{i\omega_0\tau_0} \right\} \right] \\
& \quad + \overline{z^2} \left[-a_{11} e^{i\omega_0\tau_0} - a_{12} \overline{q_2} e^{i\omega_0\tau_0} \right. \\
& \qquad \left. + a_{21} \overline{q_1 q_2} \overline{q_2^*} - a_{22} \overline{q_2^2} \overline{q_2^*} e^{i\omega_0\tau_0} \right] \\
& \quad + \frac{1}{2} z^2 \overline{z} \left[-a_{11} \left(W_{20}^{(1)}(0) e^{i\omega_0\tau_0} \right. \right. \\
& \qquad + 2W_{11}^{(1)}(0) e^{-i\omega_0\tau_0} \\
& \qquad \left. + 2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1) \right) \\
& \qquad - a_{12} \left(W_{20}^{(1)}(0) \overline{q_2} e^{i\omega_0\tau_0} \right. \\
& \qquad + 2W_{11}^{(1)}(0) q_2 e^{-i\omega_0\tau_0} \\
& \qquad \left. + 2W_{11}^{(3)}(-1) + W_{20}^{(3)}(-1) \right) \\
& \qquad + a_{21} \overline{q_2^*} \left(W_{20}^{(2)}(0) \overline{q_2} + 2W_{11}^{(2)}(0) q_2 \right. \\
& \qquad + 2W_{11}^{(3)}(0) q_1 \\
& \qquad \left. + W_{20}^{(3)}(0) \overline{q_1} \right) \\
& \qquad - a_{22} \overline{q_2^*} \left(W_{20}^{(3)}(0) \overline{q_2} e^{i\omega_0\tau_0} \right. \\
& \qquad + 2W_{11}^{(3)}(0) q_2 e^{-i\omega_0\tau_0} \\
& \qquad + 2W_{11}^{(3)}(-1) q_2 \\
& \qquad \left.\left. \left. + W_{20}^{(3)}(-1) \overline{q_2} \right) \right] + \cdots \right\}.
\end{aligned}
\tag{59}
$$

Comparing the coefficients with (56), we obtain

$$
\begin{aligned}
g_{20} = {} & 2\tau_0 \overline{D} \left[-a_{11} e^{-i\omega_0\tau_0} - a_{12} q_2 e^{-i\omega_0\tau_0} \right. \\
& \left. + a_{21} q_1 q_2 \overline{q_2^*} - a_{22} q_2^2 \overline{q_2^*} e^{-i\omega_0\tau_0} \right], \\
g_{11} = {} & 2\tau_0 \overline{D} \left[-a_{11} \operatorname{Re}\left\{ e^{i\omega_0\tau_0} \right\} - a_{12} \operatorname{Re}\left\{ q_2 e^{-i\omega_0\tau_0} \right\} \right. \\
& \left. + a_{21} \overline{q_2^*} \operatorname{Re}\left\{ q_1 \overline{q_2} \right\} - a_{22} q_2 \overline{q_2} \overline{q_2^*} \operatorname{Re}\left\{ e^{i\omega_0\tau_0} \right\} \right], \\
g_{02} = {} & 2\tau_0 \overline{D} \left[-a_{11} e^{i\omega_0\tau_0} - a_{12} \overline{q_2} e^{i\omega_0\tau_0} \right. \\
& \left. + a_{21} \overline{q_1 q_2} \overline{q_2^*} - a_{22} \overline{q_2^2} \overline{q_2^*} e^{i\omega_0\tau_0} \right], \\
g_{21} = {} & \tau_0 \overline{D} \left[-a_{11} \left(W_{20}^{(1)}(0) e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(0) e^{-i\omega_0\tau_0} \right. \right. \\
& \left. + 2W_{11}^{(1)}(-1) + W_{20}^{(3)}(-1) \right) \\
& - a_{12} \left(W_{20}^{(1)}(0) \overline{q_2} e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(0) q_2 e^{-i\omega_0\tau_0} \right. \\
& \left. + 2W_{11}^{(3)}(-1) + W_{20}^{(3)}(-1) \right) \\
& + a_{21} \overline{q_2^*} \left(W_{20}^{(2)}(0) \overline{q_2} + 2W_{11}^{(2)}(0) q_2 \right. \\
& \left. + 2W_{11}^{(3)}(0) q_1 + W_{20}^{(3)}(0) \right) \\
& - a_{22} \overline{q_2} \left(W_{20}^{(3)}(0) \overline{q_2^*} e^{i\omega_0\tau_0} + 2W_{11}^{(3)}(0) q_2 e^{-i\omega_0\tau_0} \right. \\
& \left.\left. + 2W_{11}^{(3)}(-1) q_2 + W_{20}^{(3)}(-1) q_2 \right) \right].
\end{aligned}
\tag{60}
$$

In order to determine g_{21} we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (43) and (53), we have

$$
\begin{aligned}
\dot{W} & = \dot{x}_t - \dot{z} q - \dot{\overline{z}} \overline{q} \\
& = \begin{cases} A(0) W - 2\operatorname{Re}\left\{ \overline{q}^*(0) f_0 q(\theta) \right\}, & \theta \in [-1, 0), \\ A(0) W - 2\operatorname{Re}\left\{ \overline{q}^*(0) f_0 q(\theta) \right\} + f_0, & \theta = 0, \end{cases} \\
& \doteq A(0) W - H(z, \overline{z}, \theta),
\end{aligned}
\tag{61}
$$

where

$$
H(z, \overline{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02} \frac{\overline{z^2}}{2} + \cdots. \tag{62}
$$

Note that on the center manifold C_0 near to the origin,

$$
\dot{W} = W_z \dot{z} + W_{\overline{z}} \dot{\overline{z}}, \tag{63}
$$

thus we obtain

$$
\begin{aligned}
\left(A(0) - 2i\omega_0\tau_0 \right) W_{20}(\theta) & = -H_{20}(\theta), \\
A(0) W_{11}(\theta) & = -H_{11}(\theta).
\end{aligned}
\tag{64}
$$

By (62) we know that for $\theta \in [-1, 0)$,

$$
\begin{aligned}
H(z, \overline{z}, \theta) & = -\overline{q^*}(0) f_0 q(\theta) - q_0^* \overline{f_0} \overline{q}(\theta) \\
& = -g(z, \overline{z}) q(\theta) - \overline{g(z, \overline{z}) q(\theta)}.
\end{aligned}
\tag{65}
$$

Comparing the coefficients with (62), we get that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \quad H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{66}$$

From (64), (66), and the definition of A, we have

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{67}$$

Noting $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}, \tag{68}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ is a constant vector. Similarly, we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2, \tag{69}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ is a constant vector. In the following, we will find out E_1 and E_2. From the definition of A and (64), we can obtain

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) - H_{20}(0), \tag{70}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(\theta), \tag{71}$$

where $\eta(\theta) = \eta(\theta, 0)$. From (61) and (62) we have

$$\begin{aligned} H(z,\overline{z},0) &= -2\operatorname{Re}\{\overline{q}^*(0) f_0 q(0)\} + f_0 \\ &= -\overline{q}^*(0) f_0 q(0) - q^*(0)\overline{f_0}\overline{q}(0) + f_0 \\ &= -g(z,\overline{z})q(0) - \overline{g}(z,\overline{z})\overline{q}(0) + f_0. \end{aligned} \tag{72}$$

That is,

$$\begin{aligned} &H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}\frac{\overline{z^2}}{2} + \cdots \\ &= -q(0)\left(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z^2}}{2} + \cdots\right) \\ &\quad - \overline{q}(0)\left(\overline{g}_{20}\frac{\overline{z}^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g_{02}}\frac{z^2}{2} + \cdots\right) + f_0. \end{aligned} \tag{73}$$

By (39) and (53) we have

$$\begin{aligned} f_0 &= \tau_0\begin{bmatrix} -a_{11}x_{1t}(0)x_{1t}(-1) - a_{12}x_{1t}(0)x_{3t}(-1) \\ 0 \\ a_{21}x_{2t}(0)x_{3t}(0) - a_{22}x_{3t}(0)x_{3t}(-1) \end{bmatrix}, \\ x_t(\theta) &= W(t,\theta) + 2\operatorname{Re}\{z(t)q(\theta)\} \\ &= W(t,\theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + z(t)q(\theta) \\ &\quad + \overline{z}(t)\overline{q}(\theta) + \cdots. \end{aligned} \tag{74}$$

Thus,

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_0 \times \begin{bmatrix} -a_{11}e^{-i\omega_0\tau_0} - a_{12}q_2e^{-i\omega_0\tau_0} \\ 0 \\ a_{21}q_1q_2 - a_{22}q_2^2e^{-i\omega_0\tau_0} \end{bmatrix}, \tag{75}$$

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau_0 \times \begin{bmatrix} -a_{11}\operatorname{Re}\{e^{i\omega_0\tau_0}\} - a_{12}\operatorname{Re}\{q_2e^{-i\omega_0\tau_0}\} \\ 0 \\ a_{21}\operatorname{Re}\{q_1\overline{q}_2\} - a_{22}q_2^2\operatorname{Re}\{e^{i\omega_0\tau_0}\} \end{bmatrix}. \tag{76}$$

Since $i\omega_0\tau_0$ is the eigenvalues of $A(0)$ and $q(0)$ is the corresponding eigenvector, we obtain

$$\begin{aligned} \left(i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right)q(0) &= 0, \\ \left(-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right)\overline{q}(0) &= 0. \end{aligned} \tag{77}$$

Thus, substituting (68) and (75) into (70), we have

$$\begin{aligned} &\left(2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right)E_1 \\ &= 2\tau_0\begin{bmatrix} -a_{11}e^{-i\omega_0\tau_0} - a_{12}q_2e^{-i\omega_0\tau_0} \\ 0 \\ a_{21}q_1q_2 - a_{22}q_2^2e^{-i\omega_0\tau_0} \end{bmatrix}, \end{aligned} \tag{78}$$

or

$$\begin{aligned} &\begin{bmatrix} 2i\omega_0 + a_{11}N^*e^{-2i\omega_0\tau_0} & 0 & a_{12}N^*e^{-2i\omega_0\tau_0} \\ -\alpha & 2i\omega_0 + \alpha & 0 \\ 0 & -a_{21}P^* & 2i\omega_0 + a_{22}P^*e^{-2i\omega_0\tau_0} \end{bmatrix}E_1 \\ &= 2\begin{bmatrix} -a_{11}e^{-i\omega_0\tau_0} - a_{12}q_2e^{-i\omega_0\tau_0} \\ 0 \\ a_{21}q_1q_2 - a_{22}q_2^2e^{-i\omega_0\tau_0} \end{bmatrix}. \end{aligned} \tag{79}$$

From which we can get

$$\begin{aligned} E_1 &= 2\begin{bmatrix} 2i\omega_0 + a_{11}N^*e^{-2i\omega_0\tau_0} & 0 & a_{12}N^*e^{-2i\omega_0\tau_0} \\ -\alpha & 2i\omega_0 + \alpha & 0 \\ 0 & -a_{21}P^* & 2i\omega_0 + a_{22}P^*e^{-2i\omega_0\tau_0} \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} -a_{11}e^{-i\omega_0\tau_0} - a_{12}q_2e^{-i\omega_0\tau_0} \\ 0 \\ a_{21}q_1q_2 - a_{22}q_2^2e^{-i\omega_0\tau_0} \end{bmatrix}. \end{aligned} \tag{80}$$

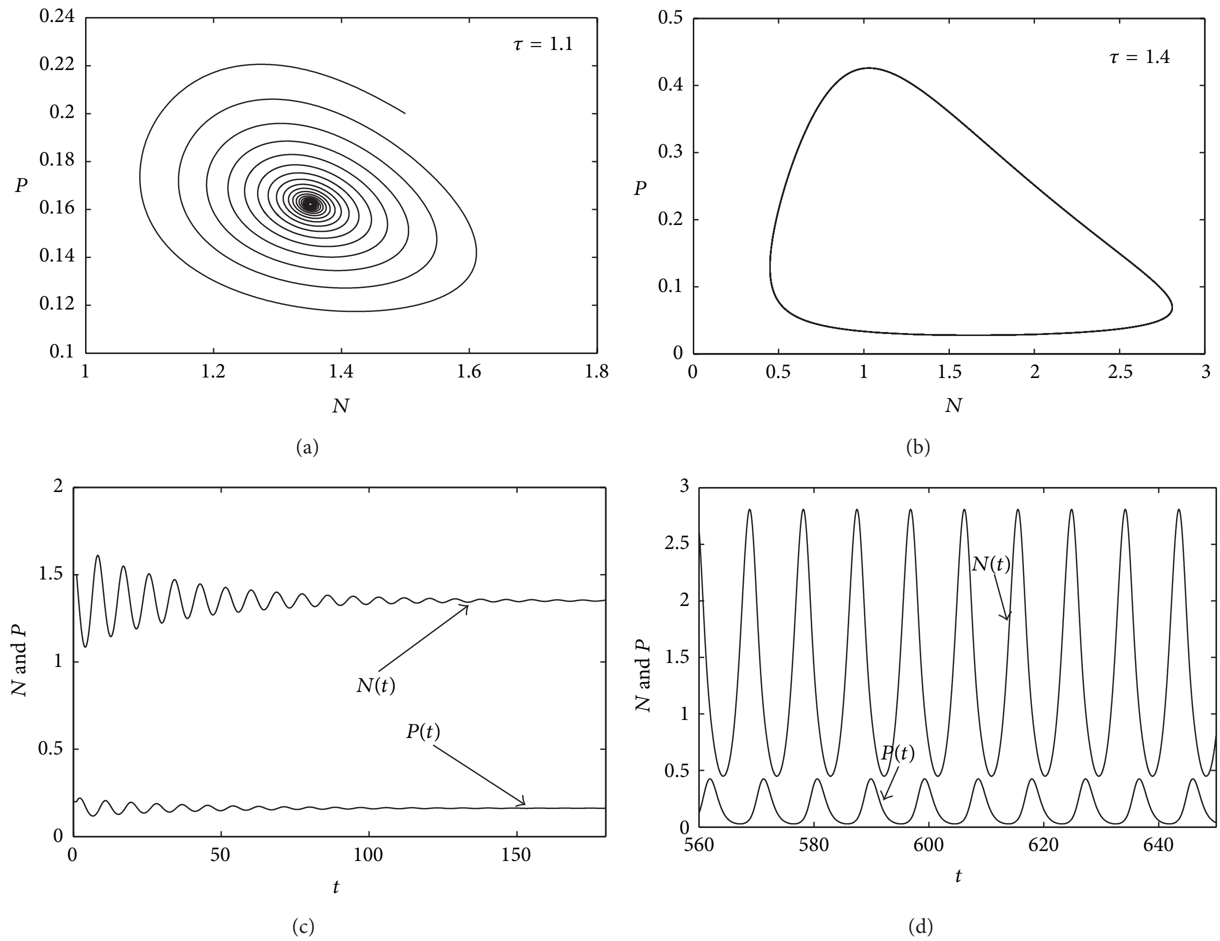

FIGURE 1: The stability of unique positive equilibrium E_2. (a), (c) The equilibrium E_2 is stable for $\tau = 1.1$. (b), (d) The equilibrium E_2 is unstable and a stable periodic solution appears for $\tau = 1.4$.

Similarly, substituting (69) and (76) into (71), we can get

$$\begin{bmatrix} a_{11}N^* & 0 & a_{12}N^* \\ -\alpha & \alpha & 0 \\ 0 & -a_{21}P^* & a_{22}P^* \end{bmatrix} E_2 = 2\begin{bmatrix} -a_{11}\operatorname{Re}\left\{e^{i\omega_0\tau_0}\right\} - a_{12}\operatorname{Re}\left\{q_2 e^{-i\omega_0\tau_0}\right\} \\ 0 \\ a_{21}\operatorname{Re}\left\{q_1\bar{q}_2\right\} - a_{22}q_2^2\operatorname{Re}\left\{e^{i\omega_0\tau_0}\right\} \end{bmatrix}, \tag{81}$$

or

$$E_2 = 2\begin{bmatrix} a_{11}N^* & 0 & a_{12}N^* \\ -\alpha & \alpha & 0 \\ 0 & -a_{21}P^* & a_{22}P^* \end{bmatrix}^{-1} \times \begin{bmatrix} -a_{11}\operatorname{Re}\left\{e^{i\omega_0\tau_0}\right\} - a_{12}\operatorname{Re}\left\{q_2 e^{-i\omega_0\tau_0}\right\} \\ 0 \\ a_{21}\operatorname{Re}\left\{q_1\bar{q}_2\right\} - a_{22}q_2^2\operatorname{Re}\left\{e^{i\omega_0\tau_0}\right\} \end{bmatrix}. \tag{82}$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (68) and (69). Furthermore, g_{21} can be expressed by the parameters and delay. Thus, we can compute the following values:

$$\begin{gathered} c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\left\{c_1(0)\right\}}{\operatorname{Re}\left\{\lambda'(\tau_0)\right\}}, \\ \beta_2 = 2\operatorname{Re}\left\{c_1(0)\right\}, \\ T_2 = -\frac{\operatorname{Im}\left\{c_1(0)\right\} + \mu_2\operatorname{Im}\left\{\lambda'(\tau_0)\right\}}{\omega_0\tau_0}, \end{gathered} \tag{83}$$

which determine the qualities of bifurcating periodic solution in the center manifold at critical value τ_0; that is, μ_2 determine the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$),

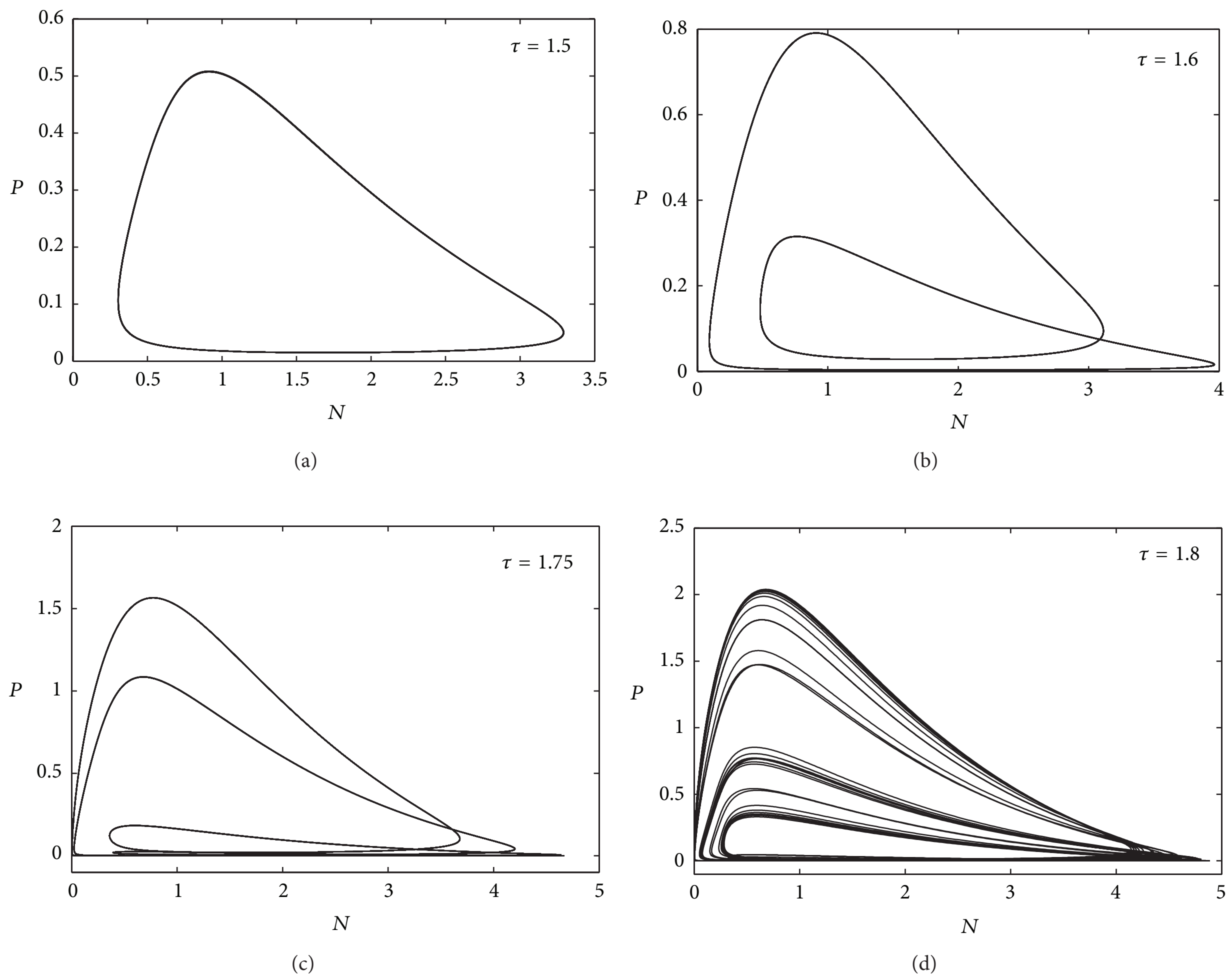

Figure 2: Bifurcation of model (84) with bifurcation parameter τ.

then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0$ ($\tau < \tau_0$); β_2 determines the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solution: the period increases (decreases) if $T_2 > 0$ (< 0).

4. Numerical Investigations and Discussion

In this paper, we propose a two-dimensional predatory-prey model with discrete and distributed delay. Then, by introducing a new variable, the original system is transformed into an equivalent three-dimensional system. In Section 2, we analyze the existence and local stability of the equilibria of the three-dimensional system. The condition for the existence of a Hopf bifurcation is also obtained. In Section 3, by the use of normal form theory and central manifold argument, we establish the formulae for the direction and the stability of the Hopf bifurcation.

In order to confirm our main results obtained in this work, we consider the following special system:

$$\begin{aligned} N'(t) &= N(t)\left[1 - 0.5N(t-\tau) - 2P(t-\tau)\right], \\ P'(t) &= P(t)\left[-1 + 0.8u(t) - 0.5P(t-\tau)\right], \\ u'(t) &= 2N(t) - 2u(t). \end{aligned} \tag{84}$$

By simple calculation, it is easy to see that model (84) exists a unique positive equilibrium E_2 and $E_2 = (50/37, 6/37, 50/37)$. Note that the parameter set provided in model (84) satisfies the conditions of Theorem 2. When $\tau = 1.1$, the positive equilibrium E_2 is asymptotically stable, as shown in Figures 1(a) and 1(c). It follows from the discussion in Section 2 that $\omega_0 \approx 0.7246$, $\tau_0 \approx 1.1746$, and $\lambda'(\tau_0) = 0.09345 - 0.07497i$. Thus, E_2 is stable when $0 \leq \tau < \tau_0$, as indicated in Figures 1(a) and 1(c).

When τ passes through the critical value τ_0, E_2 loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from E_2, as shown in Figures 1(b) and 1(d). Since $\mu_2 > 0$ and $\beta_2 < 0$, the Hopf bifurcation is supercritical and the direction of the bifurcation is $\tau > \tau_0$ and these bifurcating periodic solutions from E_2 are stable; please see Figures 1(b) and 1(d) and Figure 2(a) for $\tau = 1.5$. Note that the model (84) may have very complex dynamics if we choose the time delay τ as a bifurcation parameter. It follows from Figure 2 that the period of periodic solution is doubled as τ increase, and around $\tau = 1.6$ (Figure 2(b)). If the time delay τ is increasing further, a periodic solution with 4-time period appears around $\tau = 1.75$ (Figure 2(c)). Finally, the chaotic solution exists once the time delay reaches around $\tau = 1.8$ (Figure 2(d)).

Both our theoretical and numerical results show that the positive equilibrium is asymptotically stable if $\tau < \tau_0$, which indicates that the dynamical behavior is simple for the considered system. However, if $\tau > \tau_0$, bifurcation and chaos may occur, which means that the considered system can take on very complex dynamics, and this may explain some complex phenomenon in the natural world.

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