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*Research Article*

# Dynamics of a Family of Nonlinear Delay Difference Equations

**Qiuli He,[1] Taixiang Sun,[2] and Hongjian Xi[3]**

[1] *College of Electrical Engineering, Guangxi University, Nanning, Guangxi 530004, China*
[2] *College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China*
[3] *Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China*

Correspondence should be addressed to Taixiang Sun; stx1963@163.com

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We study the global asymptotic stability of the following difference equation: $x_{n+1} = f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t})$, $n = 0, 1, \ldots$, where $0 \le k_1 < k_2 < \cdots < k_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive, and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$. We give sufficient conditions under which the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable.

## 1. Introduction

In this note, we consider a nonlinear difference equation and deal with the question of whether the unique positive equilibrium $\overline{x}$ of that equation is globally asymptotically stable. Recently, there has been much interest in studying the global attractivity, the boundedness character, and the periodic nature of nonlinear difference equations; for example, see [1–22].

Amleh et al. [1] studied the characteristics of the difference equation:

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}. \tag{E1}$$

They confirmed a conjecture in [13] and showed that the unique positive equilibrium $\overline{x} = p + 1$ of (E1) is globally asymptotically stable provided $p > 1$.

Fan et al. [8] investigated the following difference equation:

$$x_{n+1} = f(x_n, x_{n-k}). \tag{E2}$$

They showed that the length of finite semicycle of (E2) is less than or equal to $k$ and gave sufficient conditions under which every positive solution of (E2) converges to the unique positive equilibrium.

Kulenović et al. [11] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{E3}$$

where the initial values $x_{-1}, x_0 \in R_+ \equiv (0, +\infty)$ and $p_n$ is the period-two sequence

$$p_n = \begin{cases} \alpha, & \text{if } n \text{ is even}, \\ \beta, & \text{if } n \text{ is odd}, \end{cases} \quad \text{with } \alpha, \beta \in R_+. \tag{1}$$

Sun and Xi [20] studied the more general equation

$$x_{n+1} = f(x_{n-s}, x_{n-t}), \quad n = 0, 1, 2, \ldots, \tag{E4}$$

where $s, t \in \{0, 1, 2, \ldots\}$ with $s < t$, the initial values $x_{-t}, x_{-t+1}, \ldots, x_0 \in R_+$ and gave sufficient conditions under which every positive solution of (E4) converges to the unique positive equilibrium.

In this paper, we study the global asymptotic stability of the following difference equation:

$$x_{n+1} = f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_s}; x_{n-m_1}, x_{n-m_2}, \ldots, x_{n-m_t}), \quad n = 0, 1, \ldots, \tag{2}$$

where $0 \le k_1 < k_2 < \cdots < k_s$ and $0 \le m_1 < m_2 < \cdots < m_t$ with $\{k_1, k_2, \ldots, k_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, the initial values are positive and $f \in C(E^{s+t}, (0, +\infty))$ with $E \in \{(0, +\infty), [0, +\infty)\}$ and $a = \inf_{(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t) \in E^{s+t}} \times f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \in E$ satisfies the following conditions:

$(H_1)$ $f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t)$ is decreasing in $u_i$ for any $i \in \{1, 2, \ldots, s\}$ and increasing in $v_j$ for any $j \in \{1, 2, \ldots, t\}$.

$(H_2)$ Equation (2) has the unique positive equilibrium, denoted by $\overline{x}$.

$(H_3)$ The function $f(a, a, \ldots, a; x, x, \ldots, x)$ has only fixed point in the interval $(a, +\infty)$, denoted by $A$.

$(H_4)$ For any $y \in E$, $f(y, \ldots, y; x, \ldots, x)/x$ is nonincreasing in $x \in (0, +\infty)$.

$(H_5)$ If $(x, y) \in E \times E$ is a solution of the system

$$\begin{aligned} y &= f(x, \ldots, x; y, \ldots, y), \\ x &= f(y, \ldots, y; x, \ldots, x), \end{aligned} \tag{3}$$

then $x = y$.

## 2. Main Result

**Theorem 1.** *Assume that $(H_1)$–$(H_5)$ hold. Then the unique positive equilibrium $\overline{x}$ of (2) is globally asymptotically stable.*

*Proof.* Let $l = \max\{m_t, k_s\}$. Since

$$\begin{aligned} a &= \inf_{(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_t) \in E^{s+t}} f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) \\ &\in E, \end{aligned} \tag{4}$$

we have

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}) > f(\overline{x} + 1, \overline{x}, \ldots, \overline{x}) \ge a. \tag{5}$$

*Claim 1.* $f(A, \ldots, A; a, \ldots, a) < \overline{x} < A$.

*Proof of Claim 1.* Assume on the contrary that $\overline{x} \ge A$. Then it follows from $(H_1)$, $(H_3)$, and $(H_4)$ that

$$\begin{aligned} A &= f(a, \ldots, a; A, \ldots, A) > f(\overline{x}, \ldots, \overline{x}; A, \ldots, A) \\ &= \frac{f(\overline{x}, \ldots, \overline{x}; A, \ldots, A)}{A} A \ge \frac{f(\overline{x}, \ldots, \overline{x})}{\overline{x}} A \\ &= A. \end{aligned} \tag{6}$$

This is a contradiction. Therefore $\overline{x} < A$. Obviously

$$f(A, \ldots, A; a, \ldots, a) < f(\overline{x}, \ldots, \overline{x}; \overline{x}, \ldots, \overline{x}) = \overline{x}. \tag{7}$$

Claim 1 is proven.

*Claim 2.* For any $M \ge A$, $J = [a, M]$ is an invariable interval of (2).

*Proof of Claim 2.* For any $x_0, x_{-1}, \ldots, x_{-l} \in J$, we have from $(H_4)$ that

$$\begin{aligned} a &\le x_1 \\ &= f(x_{-k_1}, x_{-k_2}, \ldots, x_{-k_s}; x_{-m_1}, x_{-m_2}, \ldots, x_{-m_t}) \\ &\le \frac{f(a, \ldots, a; M, \ldots, M)}{M} M \le \frac{f(a, \ldots, a; A, \ldots, A)}{A} M \\ &= M. \end{aligned} \tag{8}$$

By induction, we may show that $x_n \in J$ for any $n \ge 1$. Claim 2 is proven.

Let $m_0 = a$, $M_0 = M \ge A$ and for any $i \ge 0$,

$$\begin{aligned} m_{i+1} &= f(M_i, \ldots, M_i; m_i, \ldots, m_i), \\ M_{i+1} &= f(m_i, \ldots, m_i; M_i, \ldots, M_i). \end{aligned} \tag{9}$$

*Claim 3.* For any $n \ge 0$, we have

$$\begin{gathered} m_n \le m_{n+1} < \overline{x} < M_{n+1} \le M_n, \\ \lim_{n \to \infty} M_n = \lim_{n \to \infty} m_n = \overline{x}. \end{gathered} \tag{10}$$

*Proof of Claim 3.* From Claim 2, we obtain

$$\begin{aligned} m_0 \le m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1 \le M_0, \\ m_1 &= f(M_0, \ldots, M_0; m_0, \ldots, m_0) \\ &\le f(M_1, \ldots, M_1; m_1, \ldots, m_1) = m_2 \\ &< f(\overline{x}, \ldots, \overline{x}) = \overline{x} \\ &< f(m_1, \ldots, m_1; M_1, \ldots, M_1) = M_2 \\ &\le f(m_0, \ldots, m_0; M_0, \ldots, M_0) \\ &= M_1. \end{aligned} \tag{11}$$

By induction, we have that for $n \ge 0$,

$$m_n \le m_{n+1} < \overline{x} < M_{n+1} \le M_n. \tag{12}$$

Set

$$\beta = \lim_{n \to \infty} m_n \quad \text{and} \quad \alpha = \lim_{n \to \infty} M_n. \tag{13}$$

Then

$$\beta = f(\alpha, \ldots, \alpha; \beta, \ldots, \beta), \quad \alpha = f(\beta, \ldots, \beta; \alpha, \ldots, \alpha). \tag{14}$$

This with $(H_2)$ and $(H_5)$ implies $\alpha = \beta = \overline{x}$. Claim 3 is proven.

*Claim 4.* The equilibrium $\overline{x}$ of (2) is locally stable.

*Proof of Claim 4.* Let $M = A$ and $m_n, M_n$ be the same as Claim 3. For any $\varepsilon > 0$ with $0 < \varepsilon < \min\{A - \overline{x}, \overline{x} - a\}$, there exists $n > 0$ such that

$$\overline{x} - \varepsilon < m_n < \overline{x} < M_n < \overline{x} + \varepsilon. \tag{15}$$

Set $0 < \delta = \min\{\overline{x} - m_n, M_n - \overline{x}\}$. Then for any $x_0, x_{-1}, \ldots, x_{-l} \in (\overline{x} - \delta, \overline{x} + \delta)$, we have

$$\begin{aligned} x_1 &= f\left(x_{-k_1}, \ldots, x_{-k_s}; x_{-m_1}, \ldots, x_{-m_t}\right) \\ &\leq f(m_n, \ldots, m_n; M_n, \ldots, M_n) \\ &= M_{n+1} \leq M_n, \\ x_1 &= f\left(x_{-k_1}, \ldots, x_{-k_s}; x_{-m_1}, \ldots, x_{-m_t}\right) \\ &\geq f(M_n, \ldots, M_n; m_n, \ldots, m_n) \\ &= m_{n+1} \geq m_n. \end{aligned} \tag{16}$$

In similar fashion,we can show that for any $k \geq 1$,

$$x_k \in [m_n, M_n] \subset (\overline{x} - \varepsilon, \overline{x} + \varepsilon). \tag{17}$$

Claim 4 is proven.

*Claim 5.* $\overline{x}$ is the global attractor of (2).

*Proof of Claim 5.* Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (2), and let $M = \max\{x_1, \ldots, x_{l+1}, A\}$ and $m_n, M_n$ be the same as Claim 3. From Claim 2, we have $x_n \in [m_0, M_0] = [a, M]$ for any $n \geq 1$. Moreover, we have

$$\begin{aligned} &x_{l+2} \\ &\quad = f\left(x_{l+1-k_1}, \ldots, x_{l+1-k_s}; x_{l+1-m_1}, \ldots, x_{l+1-m_t}\right) \\ &\quad \leq f(m_0, \ldots, m_0; M_0, \ldots, M_0) = M_1, \\ &x_{l+2} \\ &\quad = f\left(x_{l+1-k_1}, \ldots, x_{l+1-k_s}; x_{l+1-m_1}, \ldots, x_{l+1-m_t}\right) \\ &\quad \geq f(M_0, \ldots, M_0; m_0, \ldots, m_0) = m_1. \end{aligned} \tag{18}$$

In similar fashion, we may show $x_n \in [m_1, M_1]$ for any $n \geq l + 2$. By induction, we obtain

$$x_n \in [m_k, M_k] \quad \text{for } n \geq k(l+1) + 1. \tag{19}$$

It follows from Claim 3 that $\lim_{n\to\infty} x_n = \overline{x}$. Claim 5 is proven.

From Claims 4 and 5, Theorem 1 follows. □

## 3. Applications

In this section, we will give two applications of Theorem 1.

*Example 2.* Consider equation

$$x_{n+1} = p + \frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}} + \sqrt{\frac{\sum_{i=1}^{t} a_i x_{n-m_i}}{\sum_{k=1}^{s} b_k x_{n-n_k}}}, \quad n = 0, 1, \ldots, \tag{20}$$

where $0 \leq n_1 < n_2 < \cdots < n_s$ and $0 \leq m_1 < m_2 < \cdots < m_t$ with $\{n_1, n_2, \ldots, n_s\} \bigcap \{m_1, m_2, \ldots, m_t\} = \emptyset$, $p > 0$, $a_i > 0$ for any $i \in \{1, 2, \ldots, t\}$ and $b_k > 0$ for any $k \in \{1, 2, \ldots, s\}$, and the initial conditions $x_{-l}, \ldots, x_0 \in (0, \infty)$ with $l = \max\{m_t, n_s\}$. Write $A = \sum_{i=1}^{t} a_i$ and $B = \sum_{k=1}^{s} b_k$. If $pB > A$, then the unique positive equilibrium $\overline{x}$ of (20) is globally asymptotically stable.

*Proof.* Let $E = (0, +\infty)$. It is easy to verify that $(H_1)$, $(H_2)$, and $(H_4)$ hold for (20). Note that $a = \inf_{(u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_t)\in E^{s+t}} f(u_1, u_2, \ldots, u_s; v_1, v_2, \ldots, v_t) = p$. Then

$$x = f(a, a, \ldots, a; x, x, \ldots, x) = p + \frac{Ax}{Bp} + \sqrt{\frac{Ax}{Bp}} \tag{21}$$

has only solution

$$x = \sqrt{\left[\sqrt{pAB} + \sqrt{pAB + 4p^2B(Bp - A)}\right] / 2(Bp - a)} \tag{22}$$

in the interval $(p, +\infty)$, which implies that $(H_3)$ holds for (20). In addition, let

$$x = p + \frac{xA}{yB} + \sqrt{\frac{xA}{yB}}, \quad y = p + \frac{yA}{xB} + \sqrt{\frac{yA}{xB}}, \tag{23}$$

then

$$\frac{x}{y} = \frac{p + xA/yB + \sqrt{xA/yB}}{p + yA/xB + \sqrt{yA/xB}}. \tag{24}$$

Therefore $x/y = 1$, which implies that (23) has unique solution

$$x = y = \overline{x} = p + \frac{A}{B} + \sqrt{A/B}. \tag{25}$$

Thus $(H_5)$ holds for (20). It follows from Theorem 1 that the equilibrium $\overline{x} = p + A/B + \sqrt{A/B}$ of (20) is globally asymptotically stable. □

*Example 3.* Consider equation

$$x_{n+1}=\frac{q+\sum_{i=1}^{t}a_i x_{n-m_i}}{p+\sum_{k=1}^{s}b_k x_{n-n_k}},\quad n=0,1,\ldots,\tag{26}$$

where $0\le n_1<n_2<\cdots<n_s$ and $0\le m_1<m_2<\cdots<m_t$ with $\{n_1,n_2,\ldots,n_s\}\bigcap\{m_1,m_2,\ldots,m_t\}=\emptyset$, $p>0$, $q>0$, $a_i>0$ for any $1\le i\le t$ and $b_j>0$ for any $1\le j\le s$, and the initial conditions $x_{-l},\ldots,x_0\in(0,\infty)$ with $l=\max\{m_t,n_s\}$. Write $A=\sum_{i=1}^{t}a_i$ and $B=\sum_{k=1}^{s}b_k$. If $p>A$, then the unique positive equilibrium $\overline{x}$ of (26) is globally asymptotically stable.

*Proof.* Let $E=[0,+\infty)$. It is easy to verify that $(H_1)$–$(H_4)$ hold for (26). In addition, the following equation

$$\begin{aligned} x&=\frac{q+xA}{p+yB},\\ y&=\frac{q+yA}{p+xB}\end{aligned}\tag{27}$$

has unique solution

$$x=y=\overline{x}=\frac{A-p+\sqrt{(p-A)^2+4Bq}}{2B},\tag{28}$$

which implies that $(H_5)$ holds for (26). It follows from Theorem 1 that the equilibrium $\overline{x}=(A-p+\sqrt{(p-A)^2+4Bq})/2B$ of (26) is globally asymptotically stable. □

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