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Research Article

A Generalization of Exponential Class and Its Applications

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A function space, $L^{\theta,\infty)}(\Omega)$, $0 \le \theta < \infty$, is defined. It is proved that $L^{\theta,\infty)}(\Omega)$ is a Banach space which is a generalization of exponential class. An alternative definition of $L^{\theta,\infty)}(\Omega)$ space is given. As an application, we obtain weak monotonicity property for very weak solutions of $\mathscr{A}$-harmonic equation with variable coefficients under some suitable conditions related to $L^{\theta,\infty)}(\Omega)$, which provides a generalization of a known result due to Moscariello. A weighted space $L_w^{\theta,\infty)}(\Omega)$ is also defined, and the boundedness for the Hardy-Littlewood maximal operator M_w and a Calderón-Zygmund operator T with respect to $L_w^{\theta,\infty)}(\Omega)$ is obtained.

1. Introduction

For $1 < p < \infty$ and a bounded open subset $\Omega \subset R^n$, the grand Lebesgue space $L^{p)}(\Omega)$ consists of all functions $f(x) \in \bigcap_{0<\varepsilon\le p-1} L^{p-\varepsilon}(\Omega)$ such that

$$\|f\|_{p),\Omega} = \sup_{0<\varepsilon\le p-1}\left(\varepsilon ⨍_{\Omega}|f|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)} < \infty, \quad (1)$$

where $⨍_{\Omega} = (1/|\Omega|)\int_{\Omega}$ stands for the integral mean over Ω. The grand Sobolev space $W_0^{1,p)}(\Omega)$ consists of all functions $u \in \bigcap_{0<\varepsilon\le p-1} W_0^{1,p-\varepsilon}(\Omega)$ such that

$$\|u\|_{W_0^{1,p)}} = \sup_{0<\varepsilon\le p-1}\left(\varepsilon ⨍_{\Omega}|\nabla f|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)} < \infty. \quad (2)$$

These two spaces, slightly larger than $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively, were introduced in the paper [1] by Iwaniec and Sbordone in 1992 where they studied the integrability of the Jacobian under minimal hypotheses. For $p = n$ in [2], imbedding theorems of Sobolev type were proved for functions $f \in W_0^{1,n)}(\Omega)$. The small Lebesgue space $L^{(p}(\Omega)$ was found by Fiorenza [3] in 2000 as the associate space of the grand Lebesgue space $L^{p)}(\Omega)$. Fiorenza and Karadzhov gave in [4] the following equivalent, explicit expressions for the norms of the small and grand Lebesgue spaces, which depend only on the nondecreasing rearrangement (provided that the underlying measure space has measure 1):

$$\begin{aligned} \|f\|_{L^{(p}} &\approx \int_0^1 (1-\ln t)^{-1/p}\left(\int_0^t [f^*(s)]^p ds\right)^{1/p}\frac{dt}{t}, \\ &\qquad 1 < p < \infty, \\ \|f\|_{L^{p)}} &\approx \sup_{0<t<1}(1-\ln t)^{-1/p}\left(\int_t^1 [f^*(s)]^p ds\right)^{1/p}, \\ &\qquad 1 < p < \infty. \end{aligned} \quad (3)$$

In [5], Greco et al. gave two more general definitions than (1) and (2) in order to derive existence and uniqueness results for p-harmonic operators. For $1 < p < \infty$ and $0 \le \theta < \infty$, the grand L^p space, denoted by $L^{\theta,p)}(\Omega)$, consists of functions $f \in \bigcap_{0<\varepsilon\le p-1} L^{p-\varepsilon}(\Omega)$ such that

$$\|f\|_{\theta,p)} = \sup_{0<\varepsilon\le p-1}\varepsilon^{\theta/p}\|f\|_{p-\varepsilon} < \infty, \quad (4)$$

where

$$\|f\|_{p-\varepsilon} = \left(⨍_{\Omega}|f|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)}. \quad (5)$$

The grand Sobolev space $W^{\theta,p)}(\Omega)$ consists of all functions f belonging to $\bigcap_{0<\varepsilon\le p-1} W^{1,p-\varepsilon}(\Omega)$ and such that $\nabla f \in L^{\theta,p)}(\Omega)$. That is,

$$W^{\theta,p)}(\Omega) = \left\{ f \in \bigcap_{0<\varepsilon\le p-1} W^{1,p-\varepsilon}(\Omega) : \nabla f \in L^{\theta,p)}(\Omega) \right\}. \tag{6}$$

Grand and small Lebesgue spaces are important tools in dealing with regularity properties for very weak solutions of $\mathscr{A}$-harmonic equation as well as weakly quasiregular mappings; see [6, 7].

The aim of the present paper is to provide a generalization $L^{\theta,\infty)}(\Omega)$, $0 \le \theta < \infty$, of exponential class $\mathrm{EXP}(\Omega)$ and prove that it is a Banach space. An alternative definition of $L^{\theta,\infty)}(\Omega)$ is given in terms of weak Lebesgue spaces. As an application, we obtain weak monotonicity property for very weak solutions of $\mathscr{A}$-harmonic equation with variable coefficients under some suitable conditions related to $L^{\theta,\infty)}(\Omega)$. This paper also considers a weighted space $L_w^{\theta,\infty)}(\Omega)$ and some boundedness result for classical operators with respect to this space.

In the sequel, the letter C is used for various constants and may change from one occurrence to another.

2. A Generalization of Exponential Class

Recall that $\mathrm{EXP}(\Omega)$, the exponential class, consists of all measurable functions f such that

$$\int_\Omega e^{\lambda|f|} dx < \infty \tag{7}$$

for some $\lambda > 0$. It is a Banach space under the norm

$$\|f\|_{\mathrm{EXP}} = \inf \left\{ \lambda > 0 : \int_\Omega e^{|f|/\lambda} dx \le 2 \right\}. \tag{8}$$

In this section, we define a space $L^{\theta,\infty)}(\Omega)$, $0 \le \theta < \infty$, which is a generalization of $\mathrm{EXP}(\Omega)$, and prove that it is a Banach space.

Definition 1. For $\theta \ge 0$, the space $L^{\theta,\infty)}(\Omega)$ is defined by

$$L^{\theta,\infty)}(\Omega) = \left\{ f(x) \in \bigcap_{1\le p<\infty} L^p(\Omega) : \sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f(x)|^p dx \right)^{1/p} < \infty \right\}. \tag{9}$$

It is not difficult to see that

$$L^{\theta,\infty)}(\Omega) = \left\{ g(x) \in \bigcap_{1\le p<\infty} L^p(\Omega) : \limsup_{p\to\infty} \frac{1}{p^\theta} \left(⨍_\Omega |g(x)|^p dx \right)^{1/p} < \infty \right\}. \tag{9$'$}$$

There are two special cases of $L^{\theta,\infty)}(\Omega)$ that are worth mentioning since they coincide with two known spaces.

Case 1 ($\theta = 0$). In this case,

$$L^{0,\infty)}(\Omega) = \left\{ f(x) \in \bigcap_{1\le p<\infty} L^p(\Omega) : \sup_{1\le p<\infty} \left(⨍_\Omega |f(x)|^p dx \right)^{1/p} < \infty \right\}. \tag{10}$$

From the fact (see [8, page 12])

$$L^\infty(\Omega) = \left\{ f \in \bigcap_{1\le p<\infty} L^p(\Omega) : \lim_{p\to\infty} \|f\|_p < \infty \right\}, \tag{11}$$

we get $L^{0,\infty)}(\Omega) = L^\infty(\Omega)$.

Case 2 ($\theta = 1$). The following proposition shows that $L^{\theta,\infty}(\Omega)$ can be regarded as a generalization of $\mathrm{EXP}(\Omega)$.

Proposition 2. $L^{1,\infty)}(\Omega) = EXP(\Omega)$.

Proof. In order to realize that a function in the $L^{1,\infty)}(\Omega)$ space is in $\mathrm{EXP}(\Omega)$, it is sufficient to read the last lines of [2]. The vice versa is also true; see for example [9, Chap. VI, exercise no. 17]. □

It is clear that for any $0 \le \theta < \theta' \le \infty$ and any $q < \infty$, we have the inclusions

$$L^\infty(\Omega) \subset L^{\theta,\infty)}(\Omega) \subset L^{\theta',\infty)}(\Omega) \subset L^q(\Omega). \tag{12}$$

The following theorem shows that if $\theta > 0$, then $L^{\theta,\infty)}(\Omega)$ is slightly larger than $L^\infty(\Omega)$.

Theorem 3. *For $\theta > 0$, the space $L^\infty(\Omega)$ is a proper subspace of $L^{\theta,\infty)}(\Omega)$.*

Proof. In the proof of Theorem 3 we always assume $\theta > 0$. Let $f(x) \in L^\infty(\Omega)$, then there exists a constant $M < \infty$, such that $|f(x)| \le M$, a.e. Ω. Thus,

$$\sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f(x)|^p dx \right)^{1/p} \le \sup_{1\le p<\infty} \frac{M}{p^\theta} = M < \infty, \tag{13}$$

which implies that $f(x) \in L^{\theta,\infty)}(\Omega)$.

The following example shows that $L^\infty(\Omega) \subset L^{\theta,\infty)}(\Omega)$ is a proper subset. Since we have the inclusion (12), then it is no loss of generality to assume that $\theta \le 1$. Consider the function $f(x) = (-\ln x)^\theta$ defined in the open interval $(0,1)$. It is obvious that $f(x) \notin L^\infty(0,1)$. We now show that $f(x) \in L^{\theta,\infty)}(0,1)$. In fact, for m a positive integer, integration by parts yields

$$\begin{aligned} \int_0^1 (-\ln x)^m dx &= x(-\ln x)^m \Big|_0^1 - \int_0^1 x d(-\ln x)^m \\ &= -\lim_{x\to 0^+} x(-\ln x)^m + m \int_0^1 (-\ln x)^{m-1} dx. \end{aligned} \tag{14}$$

By L'Hospital's Law, one has

$$\lim_{x\to 0^+} x(-\ln x)^m = \lim_{x\to 0^+} \frac{(-\ln x)^m}{1/x} = \lim_{x\to 0^+} \frac{m(-\ln x)^{m-1}}{1/x} = \cdots = m! \lim_{x\to 0^+} x = 0. \tag{15}$$

This equality together with (14) yields

$$\int_0^1 (-\ln x)^m dx = m\int_0^1 (-\ln x)^{m-1} dx. \tag{16}$$

By induction,

$$\int_0^1 f^m(x)\,dx = m\int_0^1 (-\ln x)^{m-1} dx = \cdots = m! \int_0^1 dx = m!. \tag{17}$$

Recall that the function

$$p \longmapsto \left(⨍_\Omega |f(x)|^p dx \right)^{1/p} \tag{18}$$

is nondecreasing; thus (17) yields

$$\begin{aligned}
&\sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_0^1 |f(x)|^p dx \right)^{1/p} \\
&= \sup_{1\le p<\infty} \left[\frac{1}{p} \left(\int_0^1 (-\ln x)^{p\theta} dx \right)^{1/p\theta} \right]^\theta \\
&\le \sup_{1\le p<\infty} \left[\frac{1}{p} \left(\int_0^1 (-\ln x)^{[p\theta]+1} dx \right)^{1/([p\theta]+1)} \right]^\theta \\
&= \sup_{1\le p<\infty} \left[\frac{([p\theta]+1)!^{1/([p\theta]+1)}}{p} \right]^\theta \le \sup_{1\le p<\infty} \left[\frac{[p\theta]+1}{p} \right]^\theta \le 2,
\end{aligned} \tag{19}$$

where we have used the assumption $\theta \le 1$ and $[p\theta]$ is the integer part of $p\theta$. The proof of Theorem 3 has been completed. □

For functions $f_1(x), f_2(x) \in L^{\theta,\infty)}(\Omega)$, and $\alpha \in R$, the addition $f_1(x) + f_2(x)$ and the multiplication $\alpha f_1(x)$ are defined as usual.

Theorem 4. *$L^{\theta,\infty)}(\Omega)$ is a linear space on R.*

Proof. This theorem is easy to prove, so we omit the details. □

For $f(x) \in L^{\theta,\infty)}(\Omega)$, we define

$$\|f\|_{\theta,\infty),\Omega} = \sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f(x)|^p dx \right)^{1/p}. \tag{20}$$

We drop the subscript Ω from $\|\cdot\|_{\theta,\infty),\Omega}$ when there is no possibility of confusion.

Theorem 5. *$\|\cdot\|_{\theta,\infty)}$ is a norm.*

Proof. (1) It is obvious that $\|f\|_{\theta,\infty)} \ge 0$ and $\|f\|_{\theta,\infty)} = 0$ if and only if $f = 0$ a.e. Ω.

(2) For any $f_1(x), f_2(x) \in L^{\theta,\infty)}(\Omega)$, Minkowski's inequality in $L^p(\Omega)$ yields

$$\begin{aligned}
\|f_1 + f_2\|_{\theta,\infty)} &= \sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f_1 + f_2|^p dx \right)^{1/p} \\
&\le \sup_{1\le p<\infty} \frac{1}{p^\theta} \left[\left(⨍_\Omega |f_1|^p dx \right)^{1/p} + \left(⨍_\Omega |f_2|^p dx \right)^{1/p} \right] \\
&\le \sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f_1|^p dx \right)^{1/p} + \sup_{1\le p<\infty} \frac{1}{p^\theta} \left(⨍_\Omega |f_2|^p dx \right)^{1/p} \\
&= \|f_1\|_{\theta,\infty)} + \|f_2\|_{\theta,\infty)}.
\end{aligned} \tag{21}$$

(3) For all $\lambda \in R$ and all $f(x) \in L^{\theta,\infty)}(\Omega)$, it is obvious that $\|\lambda f\|_{\theta,\infty)} = |\lambda| \, \|f\|_{\theta,\infty)}$. □

Theorem 6. *$(L^{\theta,\infty)}(\Omega), \|\cdot\|_{\theta,\infty)})$ is a Banach space.*

Proof. Suppose that $\{f_n\}_{n=1}^\infty \subset L^{\theta,\infty)}(\Omega)$, and for any positive integer p,

$$\left\| f_{n+p} - f_n \right\|_{\theta,\infty)} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{22}$$

Since Ω is σ-finite, then $\Omega = \bigcup_{m=1}^\infty \Omega_m$ with $|\Omega_m| < \infty$. It is no loss of generality to assume that the Ω_ms are disjoint. Equation (17) implies that for any positive integer p,

$$\int_{\Omega_m} \left| f_{n+p}(x) - f_n(x) \right| dx \longrightarrow 0, \quad n \longrightarrow \infty. \tag{23}$$

Thus, by the completeness of $L^1(\Omega_m)$, there exists $f^{(m)}(x) \in L^1(\Omega_m)$, such that

$$f_n(x) \longrightarrow f^{(m)}(x), \quad n \longrightarrow \infty, \quad \text{in } L^1(\Omega_m). \tag{24}$$

Hence for any positive integer m, there exists a subsequence $\{f_n^{(m)}(x)\}$ of $\{f_n^{m-1}(x)\}$, $\{f_n^{(0)}(x)\} = \{f_n(x)\}$, such that

$$f_n^{(m)}(x) \longrightarrow f^{(m)}(x), \quad n \longrightarrow \infty, \text{ a.e. } x \in \Omega_m. \tag{25}$$

If we let

$$f(x) = f^{(m)}(x), \quad x \in \Omega_m, \; m = 1, 2, \ldots, \tag{26}$$

then

$$f_n^{(n)}(x) \longrightarrow f(x), \quad n \longrightarrow \infty, \text{ a.e. } x \in \Omega. \tag{27}$$

It is no loss of generality to assume that the subsequence $\{f_n^{(n)}(x)\}$ of $\{f_n(x)\}$ is itself; thus

$$f_n(x) \longrightarrow f(x), \quad n \longrightarrow \infty, \text{ a.e. } x \in \Omega. \tag{28}$$

We now prove $f(x) \in L^{\theta,\infty)}(\Omega)$ and $\|f_n - f\|_{\theta,\infty)} \to 0, (n \to \infty)$. In fact, by (22), for any $\varepsilon > 0$, there exists $N = N(\varepsilon)$, such that if $n > N$, then

$$\sup_{1\le q<\infty} \frac{1}{q^\theta}\left(⨍_\Omega \left|f_{n+p}(x) - f_n(x)\right|^q dx\right)^{1/q} < \varepsilon. \tag{29}$$

Let $p \to \infty$; one has

$$\sup_{1\le q<\infty} \frac{1}{q^\theta}\left(⨍_\Omega \left|f_n(x) - f(x)\right|^q dx\right)^{1/q} < \varepsilon, \quad n > N. \tag{30}$$

Hence $f(x) \in L^{\theta,\infty)}(\Omega)$, and $\|f_n(x) - f(x)\|_{\theta,\infty)} \to 0, n \to \infty$. This completes the proof of Theorem 6. □

Definition 7. The grand Sobolev space $W^{\theta,\infty)}(\Omega)$ consists of all functions f belonging to $\bigcap_{1\le p<\infty} W^{1,\infty)}(\Omega)$ and such that $\nabla f \in L^{\theta,\infty)}(\Omega)$. That is,

$$W^{\theta,\infty)}(\Omega) = \left\{f \in \bigcap_{1\le p<\infty} W^{1,\infty)}(\Omega) : \nabla f \in L^{\theta,\infty)}(\Omega)\right\}. \tag{31}$$

This definition will be used in Section 4.

3. An Alternative Definition of $L^{\theta,\infty}(\Omega)$

In this section, we give an alternative definition of $L^{\theta,\infty)}(\Omega)$ in terms of weak Lebesgue spaces. Let us first recall the definition of weak L^p $(0 < p < \infty)$ spaces or the Marcinkiewicz spaces, $L^p_{\text{weak}}(\Omega)$; see [10, Chapter 1, Section 2], [11, Chapter 2, Section 5], or [12, Chapter 2, Section 18].

Definition 8. Let $0 < p < \infty$. We say that $f \in L^p_{\text{weak}}(\Omega)$ if and only if there exists a positive constant $k = k(f)$ such that

$$f_*(t) = \left|\{x \in \Omega : |f(x)| > t\}\right| \le \frac{k}{t^p} \tag{32}$$

for every $t > 0$, where $|E|$ is the n-dimensional Lebesgue measure of $E \subset R^n$ and $f_*(t) = |\{x \in \Omega : |f(x)| > t\}|$ denotes the distribution function of f.

For $p > 1$, we recall that if $f \in L^p_{\text{weak}}(\Omega)$, then $f \in L^q(\Omega)$ for every $1 \le q < p$, and $f \in L^p_{\text{weak}}(\Omega)$ if and only if for every measurable set $E \subset \Omega$, the following inequality holds

$$\int_E |f(x)|\, dx \le c|E|^{(p-1)/p} \tag{33}$$

for some constant $c > 0$.

Equation (32) is equivalent to

$$M_p(f) = \left[\frac{1}{|\Omega|}\sup_{t>0} t^p f_*(t)\right]^{1/p} < \infty. \tag{34}$$

Recall also that

$$\int_\Omega |f(x)|^s dx = s\int_0^\infty t^{s-1} f_*(t)\, dt < \infty. \tag{35}$$

Definition 9. For $\theta \ge 0$, the weak space $L^{\theta,\infty}_{\text{weak}}(\Omega)$ is defined by

$$L^{\theta,\infty}_{\text{weak}}(\Omega) = \left\{f \in \bigcap_{1\le p<\infty} L^p_{\text{weak}}(\Omega) : \sup_{1\le p<\infty} \frac{M_p(f)}{p^\theta} < \infty\right\}. \tag{36}$$

The following theorem shows that $L^{\theta,\infty}_{\text{weak}}(\Omega) = L^{\theta,\infty)}(\Omega)$; thus, $L^{\theta,\infty}_{\text{weak}}(\Omega)$ can be regarded as an alternative definition of the space $L^{\theta,\infty)}(\Omega)$.

Theorem 10.

$$L^{\theta,\infty}_{weak}(\Omega) = L^{\theta,\infty)}(\Omega). \tag{37}$$

Proof. We divided the proof into two steps.

Step 1 ($L^{\theta,\infty}_{\text{weak}}(\Omega) \subset L^{\theta,\infty)}(\Omega)$). If $1 \le s < p$, for each $a > 0$, one can split the integral in the right-hand side of (35) to obtain

$$\begin{aligned}\int_\Omega |f|^s dx &= s\int_0^a t^{s-1} f_*(t)\, dt + s\int_a^\infty t^{s-1} f_*(t)\, dx \\ &\le |\Omega|\, a^s + \frac{sa^{s-p}}{p-s}|\Omega|\, M_p^p(f).\end{aligned} \tag{38}$$

The second integral has been estimated by the inequality $f_*(t) \le |\Omega| t^{-p} M_p^p(f)$, which is a direct consequence of the definition of the constant $M_p(f)$ (see (34)). Setting $a = M_p(f)$, we arrive at

$$⨍_\Omega |f|^s dx \le M_p^s(f) + \frac{s}{p-s} M_p^s(f) = \frac{p}{p-s} M_p(f). \tag{39}$$

This implies that

$$\frac{1}{s^\theta}\left(⨍_\Omega |f|^s dx\right)^{1/s} \le \frac{1}{s^\theta}\left(\frac{p}{p-s}\right)^{1/s} M_p(f). \tag{40}$$

Therefore

$$\begin{aligned}&\sup_{1\le s<\infty} \frac{1}{s^\theta}\left(⨍_\Omega |f|^s dx\right)^{1/s} \\ &\quad = \max\left\{\sup_{1\le s<2} \frac{1}{s^\theta}\left(⨍_\Omega |f|^s dx\right)^{1/s}, \sup_{2\le s<\infty} \frac{1}{s^\theta}\left(⨍_\Omega |f|^s dx\right)^{1/s}\right\} \\ &\quad \le \max\left\{\|f\|_2, \sup_{2\le s=p-1<\infty} \frac{1}{s^\theta}\left(⨍_\Omega |f|^s dx\right)^{1/s}\right\}\end{aligned}$$

$$\leq \max\left\{\|f\|_2, \sup_{2\leq s<\infty}\frac{1}{s^\theta}(s+1)^{1/s}M_{s+1}(f)\right\}$$
$$\leq \max\left\{\|f\|_2, 4\sup_{1\leq s<\infty}\frac{M_s(f)}{s^\theta}\right\}<\infty, \tag{41}$$

here we have used (40) and the definition of $L^\infty_{\text{weak}}(\Omega)$.

Step 2 ($L^{\theta,\infty)}(\Omega)\subset L^\infty_{\text{weak}}(\Omega)$). Since for any $t>0$,

$$\begin{aligned} t^p f_*(t) &= t^p\int_{\{x\in\Omega:|f(x)|>t\}}dx \\ &\leq \int_{\{x\in\Omega:|f(x)|>t\}}|f|^p dx \leq \int_\Omega |f|^p dx, \end{aligned} \tag{42}$$

then

$$\sup_{t>0} t^p f_*(t) \leq \int_\Omega |f|^p dx. \tag{43}$$

This implies that

$$M_p(f) = \left[\frac{1}{|\Omega|}\sup_{t>0} t^p f_*(t)\right]^{1/p} \leq \left(⨍_\Omega |f|^p dx\right)^{1/p}. \tag{44}$$

Thus

$$\sup_{1\leq p<\infty}\frac{M_p(f)}{p^\theta} \leq \sup_{1\leq p<\infty}\frac{1}{p^\theta}\left(⨍_\Omega |f|^p dx\right)^{1/p} < \infty. \tag{45}$$

The proof of Theorem 10 has been completed. □

4. An Application

In this section, we give an application of the space $L^{\theta,\infty)}(\Omega)$ to monotonicity property of very weak solutions of the $\mathscr{A}$-harmonic equation

$$\operatorname{div}\mathscr{A}(x,\nabla u(x)) = 0, \tag{46}$$

where $\mathscr{A}:\Omega\times R^n \to R^n$ is a mapping satisfying the following assumptions:

(1) the mapping $x\mapsto\mathscr{A}(x,\xi)$ is measurable for all $\xi\in R^n$,

(2) the mapping $\xi\mapsto\mathscr{A}(x,\xi)$ is continuous for a.e. $x\in R^n$, for all $\xi\in R^n$, and a.e. $x\in R^n$,

(3)
$$\langle\mathscr{A}(x,\xi),\xi\rangle \geq \gamma(x)|\xi|^p, \tag{47}$$

(4)
$$|\mathscr{A}(x,\xi)| \leq \tau(x)|\xi|^{p-1}, \tag{48}$$

where $1<p<\infty$, $0<\gamma(x)\leq\tau(x)<\infty$, a.e. Ω.

Conditions (1) and (2) insure that the composed mapping $x\mapsto\mathscr{A}(x,g(x))$ is measurable whenever g is measurable. The degenerate ellipticity of the equation is described by condition (3). Finally, condition (4) guarantees that for any $0\leq\theta<\infty$ and any $\varepsilon>0$, $\mathscr{A}(x,\nabla u)$ can be integrated for $u\in W^{\theta,p}(\Omega)$ against functions in $W^{1,(p-\varepsilon)/(1-p\varepsilon)}(\Omega)$ with compact support.

Definition 11. A function $u\in W^{1,r}_{\text{loc}}(\Omega)$, $\max\{1,p-1\}<r\leq p$, is called a very weak solution of (46) if

$$\int_\Omega\langle\mathscr{A}(x,\nabla u(x)),\nabla\varphi(x)\rangle\,dx = 0 \tag{49}$$

for all $\varphi\in W_0^{1,r/(r-p+1)}(\Omega)$.

A fruitful idea in dealing with the continuity properties of Sobolev functions is the notion of monotonicity. In one dimension a function $u:\Omega\to R$ is monotone if it satisfies both a maximum and minimum principle on every subinterval. Equivalently, we have the oscillation bounds $\operatorname{osc}_I u\leq\operatorname{osc}_{\partial I}u$ for every interval $I\subset\Omega$. The definition of monotonicity in higher dimensions closely follows this observation.

A continuous function $u:\Omega\to R^n$ defined in a domain $\Omega\subset R^n$ is monotone if

$$\operatorname{osc}_B u \leq \operatorname{osc}_{\partial B}u \tag{50}$$

for every ball $B\subset R^n$. This definition in fact goes back to Lebesgue [13] in 1973 where he first showed the relevance of the notion of monotonicity in the study of elliptic PDEs in the plane. In order to handle very weak solutions of $\mathscr{A}$-harmonic equation, we need to extend this concept, dropping the assumption of continuity. The following definition can be found in [14]; see also [6, 7].

Definition 12. A real-valued function $u\in W^{1,1}_{\text{loc}}(\Omega)$ is said to be weakly monotone if for every ball $B\subset\Omega$ and all constants $m\leq M$ such that

$$|M-u| - |u-m| + 2u - m - M \in W_0^{1,1}(B), \tag{51}$$

we have

$$m\leq u(x)\leq M \tag{52}$$

for almost every $x\in B$.

For continuous functions (51) holds if and only if $m\leq u(x)\leq M$ on ∂B. Then (52) says we want the same condition in B, that is, the maximum and minimum principles.

Manfredi's paper [14] should be mentioned as the beginning of the systematic study of weakly monotone functions. Koskela et al. obtained in [15] that $\mathscr{A}$-harmonic functions are weakly monotone. In [16], the first author obtained a result which states that very weak solutions $u\in W^{1,p-\varepsilon}_{\text{loc}}(\Omega)$ of the $\mathscr{A}$-harmonic equation are weakly monotone provided that ε is small enough. The objective of this section is to extend the operator $\mathscr{A}$ to spaces slightly larger than $L^p(\Omega)$.

Theorem 13. *Let $\gamma(x)>0$, a.e. Ω, $\tau(x)\in L^{\theta_1,\infty)}(\Omega)$. If $u\in W^{\theta_2,p)}(\Omega)$ is a very weak solution to* (46)*, then it is weakly monotone in Ω provided that $\theta_1+\theta_2<1$.*

Proof. For any ball $B\subset\Omega$ and $0<\varepsilon<1$, let

$$\psi = (u-M)^+ - (m-u)^+ \in W_0^{1,p-\varepsilon}(B). \tag{53}$$

It is obvious that

$$\nabla\psi = \begin{cases} 0, & \text{for } m \le u(x) \le M, \\ \nabla u, & \text{otherwise, say, on a set } E \subset B. \end{cases} \tag{54}$$

Consider the Hodge decomposition (see [6]),

$$\left|\nabla\psi\right|^{-p\varepsilon}\nabla\psi = \nabla\varphi + h. \tag{55}$$

The following estimate holds

$$\|h\|_{(p-\varepsilon)/(1-p\varepsilon)} \le C\varepsilon\left\|\nabla\psi\right\|_{p-\varepsilon}^{1-p\varepsilon}. \tag{56}$$

Definition 11 with φ acting as a test function yields

$$\int_E \left\langle \mathscr{A}(x,\nabla u), |\nabla u|^{-p\varepsilon}\nabla u\right\rangle dx = \int_E \left\langle \mathscr{A}(x,\nabla u), h\right\rangle dx. \tag{57}$$

Hölder's inequality together with the conditions (3) and (4) and equations (56) and (57) yields

$$\begin{aligned}
&\int_E \gamma(x)|\nabla u|^{p(1-\varepsilon)}dx \\
&\quad\le \int_E \left\langle \mathscr{A}(x,\nabla u), |\nabla u|^{-p\varepsilon}\nabla u\right\rangle dx \\
&\quad= \int_E \left\langle \mathscr{A}(x,\nabla u), h\right\rangle dx \\
&\quad\le \int_E \tau(x)|\nabla u|^{p-1}|h|\,dx \\
&\quad\le \|\tau\|_{(p-\varepsilon)/(p-1)\varepsilon}\|\nabla u\|_{p-\varepsilon}^{p-1}\|h\|_{(p-\varepsilon)/(1-p\varepsilon)} \\
&\quad\le C\varepsilon\|\tau\|_{(p-\varepsilon)/(p-1)\varepsilon}\|\nabla u\|_{p-\varepsilon}^{p(1-\varepsilon)} \\
&\quad= C|E|\,\varepsilon\cdot\varepsilon^{-\theta_2(1-\varepsilon)}\left[\frac{p-\varepsilon}{(p-1)\varepsilon}\right]^{\theta_1}\left[\frac{(p-1)\varepsilon}{p-\varepsilon}\right]^{\theta_1} \\
&\qquad\times\left(⨍_E |\tau|^{(p-\varepsilon)/(p-1)\varepsilon}dx\right)^{(p-1)\varepsilon/(p-\varepsilon)} \\
&\qquad\times\varepsilon^{\theta_2(1-\varepsilon)}\left(⨍_E |\nabla u|^{p-\varepsilon}\right)^{p(1-\varepsilon)/(p-\varepsilon)}.
\end{aligned} \tag{58}$$

The condition $\tau \in L^{\theta_1,\infty)}(\Omega)$ implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\left[\frac{(p-1)\varepsilon}{p-\varepsilon}\right]^{\theta_1}\left(⨍_E |\tau|^{(p-\varepsilon)/(p-1)\varepsilon}dx\right)^{(p-1)\varepsilon/(p-\varepsilon)} \\
&\quad\le \|\tau\|_{\theta_1,\infty)} < \infty.
\end{aligned} \tag{59}$$

Since $u \in W^{\theta_2,p)}(\Omega)$, then

$$\lim_{\varepsilon\to 0^+}\varepsilon^{\theta_2(1-\varepsilon)}\left(⨍_E |\nabla u|^{p-\varepsilon}\right)^{p(1-\varepsilon)/(p-\varepsilon)} \le \|\nabla u\|_{\theta_2,p)}^{p} < \infty. \tag{60}$$

By $\theta_1 + \theta_2 < 1$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\varepsilon\cdot\varepsilon^{-\theta_2(1-\varepsilon)}\left[\frac{p-\varepsilon}{(p-1)\varepsilon}\right]^{\theta_1} \\
&\quad= \left(\frac{p}{p-1}\right)^{\theta_1}\lim_{\varepsilon\to 0^+}\varepsilon^{1-\theta_2(1-\varepsilon)-\theta_1} = 0.
\end{aligned} \tag{61}$$

Combining (58)–(61) and taking into account the assumption $\gamma(x) > 0$, a.e. Ω, we arrive at $\nabla u = 0$, a.e. E. This implies that $(u-M)^+ - (m-u)^+$ vanishes a.e. in B, and thus $(u-M)^+ - (m-u)^+$ must be the zero function in B, completing the proof of Theorem 13. □

Remark 14. We remark that the result in Theorem 13 is a generalization of a result due to Moscariello; see [17, Corollary 4.1].

5. A Weighted Version

A weight is a locally integrable function on R^n which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$ and the Lebesgue measure of E by $|E|$. The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by L_w^p with $0 < p < \infty$. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $C > 0$ such that for any cube Q, we have $w(2Q) \le Cw(Q)$, where $2Q$ denotes the cube with the same center as Q whose side length is 2 times that of Q. When w satisfies this condition, we denote $w \in \Delta_2$, for short.

A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if there exists $C > 1$ such that for any cube Q

$$\left(⨍_Q w(x)\,dx\right)\left(⨍_Q w(x)^{1-p'}dx\right)^{p-1} \le C, \tag{62}$$

where $1/p + 1/p' = 1$. We define $A_\infty = \bigcup_{1<p<\infty} A_p$.

Let w be a weight. The Hardy-Littlewood maximal operator with respect to the measure $w(x)dx$ is defined by

$$M_w f(x) = \sup_{Q\ni x}\frac{1}{w(Q)}\int_Q \left|f(x)\right| w(x)\,dx. \tag{63}$$

We say that T is a Calderón-Zygmund operator if there exists a function K which satisfies the following conditions:

$$\begin{aligned}
&Tf(x) = \text{p.v.}\int_{R^n} K(x-y)f(y)\,dy, \\
&|K(x)| \le \frac{C_K}{|x|^n}, \quad |\nabla K(x)| \le \frac{C_K}{|x|^{n+1}}, \quad x \ne 0.
\end{aligned} \tag{64}$$

For w a weight and $0 \le \theta < \infty$, we define the space $L_w^{\theta,\infty)}(\Omega)$ as follows:

$$L_w^{\theta,\infty)}(\Omega) = \left\{f(x) \in \bigcap_{1<p<\infty} L_w^p(\Omega) : \|f\|_{L_w^{\theta,\infty)}(\Omega)} < \infty\right\}, \tag{65}$$

where

$$\|f\|_{L_w^{\theta,\infty)}(\Omega)} = \sup_{1<p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p}. \tag{66}$$

The following lemma comes from [18].

Lemma 15. *If $1 < p < \infty$ and $w \in \Delta_2$, then the operator M_w is bounded on $L_w^p(\Omega)$.*

Theorem 16. *The operator M_w is bounded on $L_w^{\theta,\infty)}(\Omega)$ for $0 \le \theta < \infty$ and $w \in \Delta_2$.*

Proof. By Lemma 15, since for $1 < p < \infty$ and $w \in \Delta_2$, the operator M_w is bounded on $L_w^p(\Omega)$, then

$$\left(\int_\Omega |M_w f(x)|^p w(x)\,dx\right)^{1/p} \le C\left(\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p}. \tag{67}$$

This implies that

$$\begin{aligned}\|M_w f\|_{L_w^{\theta,\infty)}(\Omega)} &= \sup_{1<p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |M_w f(x)|^p w(x)\,dx\right)^{1/p} \\ &\le C \sup_{1<p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p} \\ &= \|f\|_{L_w^{\theta,\infty)}(\Omega)},\end{aligned} \tag{68}$$

completing the proof of Theorem 16. □

The following lemma can be found in [19].

Lemma 17. *If $w \in A_\infty$, then there exists $q \in (1,\infty)$ such that $w \in A_q$.*

The following lemma can be found in [20, 21].

Lemma 18. *If $1 < p < \infty$ and $w \in A_p$, then a Calderón-Zygmund operator T is bounded on $L_w^p(\Omega)$.*

Theorem 19. *A Calderón-Zygmund operator T is bounded on $L_w^{\theta,\infty)}(\Omega)$ for $0 \le \theta < \infty$ and $w \in A_\infty$.*

Proof. By $w \in A_\infty$ and Lemma 17, one has $w \in A_q$ for some $q \in (1,\infty)$. For $1 < p < q < \infty$, Hölder's inequality yields

$$\begin{aligned}\int_\Omega |Tf(x)|^p w(x)\,dx &= \int_\Omega |Tf(x)|^p w(x)^{p/q} w(x)^{(q-p)/p}\,dx \\ &\le \left(\int_\Omega |Tf(x)|^q w(x)\,dx\right)^{p/q}\left(\int_\Omega w(x)\,dx\right)^{(q-p)/q}.\end{aligned} \tag{69}$$

Thus,

$$\begin{aligned}&\frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)dx\right)^{1/p} \\ &\quad\le \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^q w(x)dx\right)^{p/q} \\ &\qquad\times\left(\frac{1}{w(\Omega)}\int_\Omega w(x)dx\right)^{(q-p)/q} \\ &\quad= \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^q w(x)dx\right)^{p/q}.\end{aligned} \tag{70}$$

Lemma 18 yields

$$\begin{aligned}&\|Tf\|_{L_w^{\theta,\infty)}(\Omega)} \\ &\quad= \max\left\{\sup_{1<p<q} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)dx\right)^{1/p},\right. \\ &\qquad\left.\sup_{q\le p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)dx\right)^{1/p}\right\} \\ &\quad= \max\left\{\sup_{1<p<q} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^q w(x)\,dx\right)^{p/q},\right. \\ &\qquad\left.\sup_{q\le p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)dx\right)^{1/p}\right\} \\ &\quad\le \max\left\{\sup_{1<p<q}\left(\frac{q}{p}\right)^\theta, 1\right\} \\ &\qquad\times \sup_{q\le p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)\,dx\right)^{1/p} \\ &\quad\le Cq^\theta \sup_{q\le p<\infty} \frac{1}{p^\theta}\left(\frac{1}{w(\Omega)}\int_\Omega |Tf(x)|^p w(x)\,dx\right)^{1/p} \\ &\quad\le Cq^\theta \|f\|_{L_w^{\theta,\infty)}(\Omega)},\end{aligned} \tag{71}$$

as desired. □

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