

## Research Article

# Periodic Solutions of Second-Order Difference Problem with Potential Indefinite in Sign

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Received 19 January 2013; Accepted 24 February 2013

Academic Editor: Zhenkun Huang

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We investigate the periodic solutions of second-order difference problem with potential indefinite in sign. We consider the compactness condition of variational functional and local linking at 0 by introducing new number  $\lambda_*$ . By using Morse theory, we obtain some new results concerning the existence of nontrivial periodic solution.

## 1. Introduction

We consider the second-order discrete Hamiltonian systems

$$\Delta^2 x_{n-1} + W'(n, x_n) = 0, \quad x_{n+T} = x_n, \quad (1)$$

where  $T \geq 2$  is a given integer,  $n \in \mathbb{Z}$ ,  $x_n \in \mathbb{R}^N$ ,  $\Delta x_n = x_{n+1} - x_n$ ,  $\Delta^2 x_n = \Delta(\Delta x_n)$ ,  $W'$  stands for the gradient of  $W$  with respect to the second variable.  $W \in C^2(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$  is  $T$ -periodic in the first variable and has the form  $W(n, x) = (1/2)a|x|^2 + H(n, x)$ , where  $a = 4\sin^2(m\pi/T)$  for some  $m \in \mathbb{Z}[0, r]$ ,  $r = [T/2]$ ,  $[\cdot]$  stands for the greatest-integer function. For integers  $a \leq b$ , the discrete interval  $\{a, a+1, \dots, b\}$  is denoted by  $\mathbb{Z}[a, b]$ .

In this paper we consider that  $H$  is sign changing, that is,

$$\begin{aligned} H(n, x) &= b(n) \left( \frac{1}{s} |x|^s + \overline{G}_s(n, x) \right) \\ &\triangleq \frac{1}{s} b(n) |x|^s + G_s(n, x), \end{aligned} \quad (2)$$

$\Omega_+ = \{n \in \mathbb{Z}[1, T] | b(n) > 0\}$ ,  $\Omega_- = \{n \in \mathbb{Z}[1, T] | b(n) < 0\}$  are two nonempty subsets of  $\mathbb{Z}[1, T]$ , where  $s > 1$ ,  $b(\cdot)$  is a  $T$ -periodic real function,  $G_s \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ , and  $G_s(n, 0) = 0$ .

Consider the second-order Hamiltonian system

$$\begin{aligned} \ddot{x}(t) + W'(t, x) &= 0, \quad x(0) = x(T), \\ \dot{x}(0) &= \dot{x}(T), \end{aligned} \quad (3)$$

where  $W \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$  is  $T$ -periodic in  $t$ ,  $W(t, x) = (1/2)(A(t)x, x) + H(t, x)$ . Here  $A(\cdot)$  is a continuous,  $T$ -periodic matrix-value function.

Systems (1) and (3) have been investigated by many authors using various methods, see [1–5]. The dynamical behavior of differential and difference equations was studied by using various methods, and many interesting results have obtained, see [6–10] and references therein. The critical point theory [11–14] is a useful tool to investigate differential equations. Morse theory [15–19] has also been used to solve the asymptotically linear problem. By minimax methods in critical point theory, Tang and Wu [4], Antonacci [20, 21] considered the problem (3) with potential indefinite in sign, where  $H$  is superquadratic at zero and infinity. By using Morse theory, Zou and Li [10] study the existence of  $T$ -periodic solution of (3), where  $H$  is asymptotically superquadratic and sign changing. Moroz [19] studies system (3) where  $H$  is asymptotically subquadratic and sign changing. Motivated by [5, 10, 19], we investigate periodic solutions for asymptotically superquadratic or subquadratic discrete system (1).

By expression of  $H(n, x)$ , system (1) possesses a trivial solution  $x = 0$ . Here we are interested in finding the nonzero  $T$ -periodic solution of (1), and we divide the problem into two cases:  $s > 2$  and  $1 < s < 2$ . For  $s = 2$ , one can refer to [22].

*Case 1* (asymptotically superquadratic case:  $s > 2$ ). In this case, we replace  $p$  with  $s$  in (2). Letting  $g_p(n, x) = G'_p(n, x)$ , we rewrite (1) as

$$\begin{aligned} \Delta^2 x_{n-1} + ax_n + b(n) |x_n|^{p-2} x_n + g_p(n, x_n) &= 0, \\ x_{n+T} &= x_n. \end{aligned} \quad (4)$$

Furthermore, for all  $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$ , we assume that  $g_p$  satisfies

- (A1)  $g_p(n, x) = o(|x|)$  as  $|x| \rightarrow \infty$  uniformly in  $n$ ,  
 (A2)  $g_p(n, x) = o(|x|^{p-1})$  as  $|x| \rightarrow 0$  uniformly in  $n$ .

*Case 2* (asymptotically subquadratic case:  $1 < s < 2$ ). Here we replace  $q$  with  $s$  in (2). Letting  $g_q(n, x) = G'_q(n, x)$ , we rewrite (1) as

$$\begin{aligned} \Delta^2 x_{n-1} + ax_n + b(n) |x_n|^{q-2} x_n + g_q(n, x_n) &= 0, \\ x_{n+T} &= x_n. \end{aligned} \quad (5)$$

For all  $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$ , we assume that  $g_q$  satisfies

- (B1)  $g_q(n, x) = o(|x|^{q-1})$  as  $|x| \rightarrow \infty$  uniformly in  $n$ ,  
 (B2)  $g_q(n, x) = o(|x|)$  as  $|x| \rightarrow 0$  uniformly in  $n$ .

Before stating the main results, we introduce space  $E_T = \{x = \{x_n\} \in S | x_{n+T} = x_n, n \in \mathbb{Z}\}$ , where  $S = \{x = \{x_n\} | x_n \in \mathbb{R}^N, n \in \mathbb{Z}\}$ . For any  $x, y \in S, a, b \in \mathbb{R}$ , we define  $ax + by = \{ax_n + by_n\}_{n \in \mathbb{Z}}$ . Then  $S$  is a linear space. Let  $\langle x, y \rangle_{E_T} = \sum_{n=1}^T \langle x_n, y_n \rangle$ ,  $\|x\|_{E_T} = (\sum_{n=1}^T |x_n|^2)^{1/2}$ , for all  $x, y \in E_T$ , where  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  are the usual inner product and norm in  $\mathbb{R}^N$ , respectively. Obviously,  $E_T$  is a Hilbert space with dimension  $NT$  and homeomorphism to  $\mathbb{R}^{NT}$ . For  $r > 1$ , let  $\|x\|_r = (\sum_{n=1}^T |x_n|^r)^{1/r}$ ,  $x \in E_T$ . Moreover, for simplicity, we write  $\langle x, y \rangle$  and  $\|x\|$  instead of  $\langle x, y \rangle_{E_T}$  and  $\|x\|_{E_T}$ , respectively.

**Lemma 1.** *There exist positive numbers  $a_1, a_2$ , such that  $a_1 \|x\|_r \leq \|x\| \leq a_2 \|x\|_r$ .*

*Inspired by [10, 19], one introduces two numbers as follows:*

$$\begin{aligned} \lambda_*(p) &= \inf_{\|x\|=1} \left\{ \|\Delta x\|^2 \mid \sum_{n=1}^T b(n) |x_n|^p = 0 \right\}, \\ \lambda_*(q) &= \inf_{\|x\|=1} \left\{ \|\Delta x\|^2 \mid \sum_{n=1}^T b(n) |x_n|^q = 0 \right\}. \end{aligned} \quad (6)$$

**Theorem 2.** *If  $a < \lambda_*(p)$ , then (4) has a nonzero  $T$ -periodic solution.*

**Theorem 3.** *If  $a < \lambda_*(q)$ , then (5) has a nonzero  $T$ -periodic solution.*

This paper is divided into four sections. Section 2 contains some preliminaries, and the proofs of Theorems 2 and 3 are given in Sections 3 and 4, respectively.

## 2. Preliminaries

**2.1. Variational Functional and (PS) Condition.** For seeking  $T$ -periodic solution of (1), we consider variational functional  $J_p$  associated with (4) as  $J_p(x) = (1/2) \sum_{n=1}^T |\Delta x_n|^2 - (1/2)a \sum_{n=1}^T |x_n|^2 - 1/p \sum_{n=1}^T b(n) |x_n|^p - \sum_{n=1}^T G_p(n, x_n)$ , that is

$$\begin{aligned} J_p(x) &= \frac{1}{2} \|\Delta x\|^2 - \frac{1}{2} a \|x\|^2 - \frac{1}{p} \sum_{n=1}^T b(n) |x_n|^p \\ &\quad - \sum_{n=1}^T G_p(n, x_n), \quad x \in E_T. \end{aligned} \quad (7)$$

Moreover,  $T$ -periodic solution of (5) is associated with the critical point of functional

$$\begin{aligned} J_q(x) &= \frac{1}{2} \|\Delta x\|^2 - \frac{1}{2} a \|x\|^2 - \frac{1}{q} \sum_{n=1}^T b(n) |x_n|^q \\ &\quad - \sum_{n=1}^T G_q(n, x_n), \quad x \in E_T. \end{aligned} \quad (8)$$

We say that a  $C^1$ -functional  $\varphi$  on Hilbert space  $X$  satisfies the Palais-Smale (PS) condition if every sequence  $\{x^{(j)}\}$  in  $X$ , such that  $\{\varphi(x^{(j)})\}$  is bounded and  $\varphi'(x^{(j)}) \rightarrow 0$  as  $j \rightarrow \infty$  contains a convergent subsequence.

**Lemma 4.** *Functional  $J_p$  satisfies (PS) condition if  $a < \lambda_*(p)$ .*

*Proof.* Let  $\{x^{(j)}\} \subset E_T$  be the (PS) sequence for functional  $J_p$ , such that  $J_p(x^{(j)})$  is bounded, and  $J'_p(x^{(j)}) \rightarrow 0$  as  $j \rightarrow \infty$ . Hence, for any  $\varepsilon > 0$ , there exist  $N_\varepsilon > 0$  and constant  $c_1 > 0$ , such that

$$\begin{aligned} |\langle J'_p(x^{(j)}), x^{(j)} \rangle| &\leq \varepsilon \|x^{(j)}\| \quad \text{for } j \geq N_\varepsilon, \\ |J_p(x^{(j)})| &\leq c_1. \end{aligned} \quad (9)$$

To prove that  $J_p$  satisfies (PS) condition, it suffices to show that  $\|x^{(j)}\|$  is bounded in  $E_T$ . Suppose not that there exists a subsequence  $\{x^{(j_k)}\}$ ,  $\|x^{(j_k)}\| \rightarrow \infty$  as  $k \rightarrow \infty$ . For simplicity, we write as  $\{x^{(j)}\}$  instead of  $\{x^{(j_k)}\}$ . Without loss of generality, we assume that there exists  $k \in \mathbb{Z}[1, T]$ , such that

$$\begin{aligned} |x_n^{(j)}| &\rightarrow \infty \quad \text{as } j \rightarrow \infty \quad \text{for } n \in \mathbb{Z}[1, k], \\ x_n^{(j)} &\text{ are bounded for } n \in \mathbb{Z}[k+1, T]. \end{aligned} \quad (10)$$

Therefore for all  $n \in [1, T]$ , by assumption (A1), there exists  $c_2 > 0$  such that

$$\begin{aligned} |G_p(n, x_n^{(j)})| &\leq \varepsilon |x_n^{(j)}|^2 + c_2, \\ |g_p(n, x_n^{(j)})| &\leq \varepsilon |x_n^{(j)}| + c_2 \end{aligned} \quad (11)$$

for large  $j$ . By the previous argument, it follows that

$$\left| \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}) \right| \leq \sum_{n=1}^T |g_p(n, x_n^{(j)})| |x_n^{(j)}| \leq \varepsilon \|x^{(j)}\|^2 + c_2 T \|x^{(j)}\|. \quad (12)$$

By (7), we have

$$\begin{aligned} & pJ_p(x^{(j)}) - \langle J'_p(x^{(j)}), x^{(j)} \rangle \\ &= \left( \frac{p}{2} - 1 \right) (\|\Delta x^{(j)}\|^2 - a \|x^{(j)}\|^2) - p \sum_{n=1}^T G_p(n, x_n^{(j)}) \\ & \quad + \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}). \end{aligned} \quad (13)$$

In terms of (9) and (11), for large  $j$ , it follows that

$$\begin{aligned} & \left( \frac{p}{2} - 1 \right) (\|\Delta x^{(j)}\|^2 - a \|x^{(j)}\|^2) \\ & \leq pc_1 + \varepsilon \|x^{(j)}\| + (p+1)\varepsilon \|x^{(j)}\|^2 + pc_2 T + c_2 T \|x^{(j)}\|. \end{aligned} \quad (14)$$

Set  $y_n^{(j)} = x_n^{(j)} / \|x^{(j)}\|$ . Dividing by  $\|x^{(j)}\|^2$  in the previous formula, it follows that

$$\|\Delta y^{(j)}\|^2 \leq a + \frac{2}{p-2} \left( (p+1)\varepsilon + \frac{c_2 T + \varepsilon}{\|x^{(j)}\|} + \frac{pc_2 T + pc_1}{\|x^{(j)}\|^2} \right) \quad (15)$$

for large  $j$ . Therefore, by  $\varepsilon$  being chosen arbitrarily, there is a subsequence that converges to  $y^0 \in E_T$  such that

$$\|\Delta y^0\|^2 \leq a, \quad \|y^0\| = 1. \quad (16)$$

On the other hand, we have

$$\begin{aligned} & J_p(x^{(j)}) - \frac{1}{2} \langle J'_p(x^{(j)}), x^{(j)} \rangle \\ &= \left( \frac{1}{2} - \frac{1}{p} \right) \sum_{n=1}^T b(n) |x_n^{(j)}|^p - \sum_{n=1}^T G_p(n, x_n^{(j)}) \\ & \quad + \frac{1}{2} \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}). \end{aligned} \quad (17)$$

Then, by (9) and (11), for large  $j$ , we get

$$\begin{aligned} & \left| \left( \frac{1}{2} - \frac{1}{p} \right) \sum_{n=1}^T b(n) |x_n^{(j)}|^p \right| \\ &= \left| J_p(x^{(j)}) - \frac{1}{2} \langle J'_p(x^{(j)}), x^{(j)} \rangle + \sum_{n=1}^T G_p(n, x_n^{(j)}) \right. \\ & \quad \left. - \frac{1}{2} \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}) \right| \\ & \leq c_1 + \frac{\varepsilon}{2} \|x^{(j)}\| + \varepsilon \|x^{(j)}\|^2 + c_2 T + \frac{1}{2} (\varepsilon \|x^{(j)}\|^2 + c_2 T \|x^{(j)}\|). \end{aligned} \quad (18)$$

By dividing by  $\|x^{(j)}\|^p$  in the previous formula, then by  $p > 2$ , we have  $\sum_{n=1}^T b(n) |y_n^{(j)}|^p \rightarrow 0$  as  $j \rightarrow \infty$ , that is,  $\sum_{n=1}^T b(n) |y_n^{(j)}|^p = \lim_{j \rightarrow \infty} \sum_{n=1}^T b(n) |y_n^{(j)}|^p = 0$ . By the definition of  $\lambda_*(p)$ , see (6), we have  $\|\Delta y^0\|^2 \geq \lambda_*(p)$ . This contradicts with (16) and assumption  $a < \lambda_*(p)$ . The proof is completed.  $\square$

**Lemma 5.** Functional  $J_q$  satisfies (PS) condition if  $a < \lambda_*(q)$ .

The proof is similar to that of Lemma 4 and is omitted.

**2.2. Eigenvalue Problem.** Consider eigenvalue problem:

$$-\Delta^2 x_{n-1} = \lambda x_n, \quad x_{n+T} = x_n, \quad x_n \in \mathbb{R}^N, \quad (19)$$

that is,  $x_{n+1} + (\lambda - 2)x_n + x_{n-1} = 0$ ,  $x_{n+T} = x_n$ . By the periodicity, the difference system has complexity solution  $x_n = e^{in\theta} c$  for  $c \in \mathbb{C}^N$ , where  $\theta = 2k\pi/T$ ,  $k \in \mathbb{Z}$ . Moreover,  $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1 - \cos \theta) = 4 \sin^2(k\pi/T)$ . Let  $\eta_k$  denote the real eigenvector corresponding to the eigenvalues  $\lambda_k = 4 \sin^2(k\pi/T)$ , where  $k \in Z[0, r]$  and  $r = [T/2]$ . Since  $a = 4 \sin^2(m\pi/T)$  for some  $m \in Z[0, r]$ , we can split space  $E_T$  as follows:

$$E_T = W^- \oplus W^0 \oplus W^+, \quad (20)$$

where

$$\begin{aligned} W^- &= \text{span} \{ \eta_k \mid k \in Z[0, m-1] \}, \quad W^0 = \text{span} \{ \eta_m \}, \\ W^+ &= \text{span} \{ \eta_k \mid k \in Z[m+1, r] \}. \end{aligned} \quad (21)$$

By means of eigenvalue problem, we have  $|\Delta x_n|^2 - a |x_n|^2 = (\Delta x_n, \Delta x_n) - a(x_n, x_n) = (-\Delta^2 x_{n-1}, x_n) - a(x_n, x_n) = (\lambda - a)(x_n, x_n) = (\lambda - a) |x_n|^2$ . Let

$$\delta = \begin{cases} \min \left\{ 4 \sin^2 \frac{(m+1)\pi}{T} - 4 \sin^2 \frac{m\pi}{T}, \right. \\ \quad \left. 4 \sin^2 \frac{m\pi}{T} - 4 \sin^2 \frac{(m-1)\pi}{T} \right\}, & m \in Z[1, r], \\ 4 \sin^2 \frac{\pi}{T}, & m = 0. \end{cases} \quad (22)$$

Then  $\pm(\|\Delta x\|^2 - a \|x\|^2) \geq \delta \|x\|^2$  for  $x \in W^\pm$ .

On the other hand, associating to numbers  $\lambda_*(p)$  and  $\lambda_*(q)$  (see (6)), we set

$$\begin{aligned}\Lambda_*(p) &= \sum_{n=1}^T b(n) |e_n|^p, \\ \Lambda_*(q) &= \sum_{n=1}^T b(n) |e_n|^q,\end{aligned}\quad (23)$$

where  $e_n = u \in \mathbb{R}^N$  ( $n \in [1, T]$ ) is the real eigenvector corresponding to eigenvalue  $\lambda_0 = 0$ .  $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in E_T$ , where  $\bullet^T$  denotes the transpose of a vector or a matrix. Moreover, letting  $|u| = T^{-1/2}$ , we have  $\|e\| = 1$ ,  $\|\Delta e\| = 0$ . Therefore, by definition of  $\lambda_*(p)$ , if  $\Lambda_*(p) = 0$  then  $\lambda_*(p) = 0$ .

However, by assumption  $\lambda_*(p) > a = 4\sin^2(m\pi/T)$  for some  $m \in Z[0, r]$ , thus  $\lambda_*(p) > 0$ . That is to say the equality  $\Lambda_*(p) = 0$  cannot hold. Therefore our discussion will be distinguished in two cases:  $\Lambda_*(p) > 0$  and  $\Lambda_*(p) < 0$ .

**2.3. Preliminaries.** Let  $X$  be a Hilbert space, and let  $\varphi \in C^1(X, \mathbb{R})$  be a functional satisfying the (PS) condition. Write  $\text{crit}(\varphi) = \{x \in X \mid \varphi'(x) = 0\}$  for the set of critical points of functional  $\varphi$  and  $\varphi^c = \{x \in X \mid \varphi(x) \leq c\}$  for the level set. Denote by  $H_k(A, B)$  the  $k$ th singular relative homology group with integer coefficients. Let  $x_0 \in \text{crit}(\varphi)$  be an isolated critical point with value  $c = \varphi(x_0)$ ,  $c \in \mathbb{R}$ , the group  $C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\})$ , and  $k \in \mathbb{Z}$  is called the  $k$ th critical group of  $\varphi$  at  $x_0$ , where  $U$  is a closed neighbourhood of  $u$ . Due to the excision of homology [13],  $C_k(\varphi, x_0)$  is dependent on  $U$ .

Suppose that  $\varphi(\text{crit}(\varphi))$  is strictly bounded from below by  $a \in \mathbb{R}$ , then the critical groups of  $\varphi$  at infinity are formally defined [11] as  $C_k(\varphi, \infty) = H_k(X, \varphi^a)$ ,  $k \in \mathbb{Z}$ .

**Proposition 6** (Proposition 2.3, [11]). *Assume that  $C^2$ -functional  $\varphi$  satisfying (PS) condition has a local linking at 0 with respect to  $X = X_0^+ \oplus X_0^-$ ; that is, there exists  $\rho > 0$  such that*

$$\begin{aligned}\varphi(x) &\leq \varphi(0) \quad \text{for } x \in X_0^- \text{ and } \|x\| \leq \rho, \\ \varphi(x) &> \varphi(0) \quad \text{for } x \in X_0^+ \text{ and } 0 < \|x\| \leq \rho.\end{aligned}\quad (24)$$

Then  $C_k(\varphi, 0) \neq 0$ ,  $k = \dim X_0^-$ .

By Proposition 6, one proves the following lemmas with respect to  $E_T = X^+ \oplus X^-$ .

**Lemma 7.** *If  $a < \lambda_*(p)$ , then  $C_k(J_p, 0) \neq 0$ ,  $k = \dim X^-$ , where  $X^- = W^- \oplus W^0$  as  $\Lambda_*(p) > 0$ ,  $X^- = W^-$  as  $\Lambda_*(p) < 0$ .  $\Lambda_*(p)$  is defined by (23).*

*Proof.* We first consider the following.

**Case 1** ( $\Lambda_*(p) > 0$  and  $X^+ = W^+$ ,  $X^- = W^- \oplus W^0$ ). By  $p > 2$ ,  $|x|^p = o(|x|^2)$  as  $|x| \rightarrow 0$ , then there exists  $\theta \in (0, 1)$  suitably small, such that  $|x|^p \leq \delta/3(b/p + \varepsilon)|x|^2$  as  $|x| < \theta$ ,

where  $\delta > 0$  see (22) and  $b = \max\{|b(1)|, \dots, |b(T)|\} > 0$ . By assumption (A2) and  $G_p(n, 0) = 0$ , for any given  $\varepsilon > 0$ , there exists  $\rho_n \in (0, \theta)$ , such that  $|G_p(n, x_n)| \leq \varepsilon|x_n|^p$  as  $|x_n| \leq \rho_n$ ,  $n \in Z[1, T]$ . Thus

$$\begin{aligned}\frac{1}{p} \sum_{n=1}^T b(n) |x_n|^p + \sum_{n=1}^T G_p(n, x_n) \\ \leq \left(\frac{b}{p} + \varepsilon\right) \sum_{n=1}^T |x_n|^p \leq \frac{1}{3} \delta \|x\|^2.\end{aligned}\quad (25)$$

Let  $\rho = \min\{\rho_1, \dots, \rho_T\}$ . For  $0 < \|x\| \leq \rho < 1$ , it follows that

$$J_p(x) \geq \frac{1}{2} \delta \|x\|^2 - \frac{1}{3} \delta \|x\|^2 > 0, \quad x \in W^+ = X^+. \quad (26)$$

We need to prove that  $J_p(x) \leq 0$  for  $x \in X^- = W^- \oplus W^0$ ,  $\|x\| \leq \rho$ . We first claim that

$$\sum_{n=1}^T b(n) |x_n|^p > 0, \quad \forall x \in W^- \oplus W^0, \quad x \neq 0. \quad (27)$$

Indeed, by contradiction, assume that  $\sum_{n=1}^T b(n) |x_n|^p \leq 0$ , for some  $x \in W^- \oplus W^0$ ,  $x \neq 0$ . Since  $\Lambda_*(p) = \sum_{n=1}^T b(n) |e_n|^p > 0$ , where  $e = (e_1^T, e_2^T, \dots, e_N^T)^T = (u^T, u^T, \dots, u^T)^T \in W^- \oplus W^0$ , and  $(W^- \oplus W^0) \setminus \{0\}$  is arcwise connected, then there exists a  $x^0 \in (W^- \oplus W^0) \setminus \{0\}$ , such that  $\sum_{n=1}^T b(n) |x_n^0|^p = 0$ . Thus  $\|\Delta x^0\|^2 \geq \lambda_*(p) \|x^0\|^2$  by the definition of  $\lambda_*(p)$ . On the other hand, by the definition of  $W^- \oplus W^0$ , we have  $\|\Delta x^0\|^2 \leq a \|x^0\|^2$ . This is a contradiction with assumption  $a < \lambda_*(p)$ . So the claim (27) holds.

There exists  $c_4 > 0$  by (27), such that  $\sum_{n=1}^T b(n) |x_n|^p \geq c_4 \|x\|_p^p$  for all  $x \in W^- \oplus W^0 \setminus \{0\}$ , where  $\|x\|_p = (\sum_{n=1}^T |x_n|^p)^{1/p}$ . For  $x \in W^- \oplus W^0$ ,  $\|x\| \leq \rho$ ,  $\varepsilon$  sufficiently small, we have

$$\begin{aligned}J_p(x) &\leq -\frac{1}{p} \sum_{n=1}^T b(n) |x_n|^p - \sum_{n=1}^T G_p(n, x_n) \\ &\leq -\frac{c_4}{p} \|x\|_p^p + \varepsilon \|x\|_p^p \leq 0.\end{aligned}\quad (28)$$

Since  $J_p(0) = 0$  and  $J_p$  satisfies (PS) condition by Lemma 4, so by Proposition 6, we obtain that  $C_k(J_p, 0) \neq 0$  for  $k = \dim(W^- \oplus W^0)$ .

**Case 2** ( $\Lambda_*(p) < 0$ ,  $X^+ = W^+ \oplus W^0$ ,  $X^- = W^-$ ). It is easy to see that  $J_p(x) \leq 0$  by  $\|\Delta x\|^2 - a \|x\|^2 \leq -\delta \|x\|^2$  and  $p > 2$ , where  $x \in W^-$  and  $\|x\| \leq \rho$ . We need to claim that  $J_p(x) > 0$ , for  $x \in W^+ \oplus W^0$ ,  $0 < \|x\| \leq \rho$ .

Suppose not that there exists a sequence  $\{x^{(j)}\} \subset E_T$  such that

$$\begin{aligned}\{x^{(j)}\} &\subset W^+ \oplus W^0 \setminus \{0\}, \quad 0 < \|x^{(j)}\| \leq \rho, \\ J_p(x^{(j)}) &\leq 0,\end{aligned}\quad (29)$$

for large  $j$ . For  $\|x^{(j)}\| \leq \rho$ , by Lemma 1, we get

$$\begin{aligned} & \left| \sum_{n=1}^T \left[ \frac{1}{p} b(n) |x_n^{(j)}|^p + G_p(n, x_n^{(j)}) \right] \right| \\ & \leq \sum_{n=1}^T \left[ \frac{1}{p} |x_n^{(j)}|^p + \varepsilon |x_n^{(j)}|^p \right] \leq \left( \frac{b}{p} + \varepsilon \right) \left( \frac{1}{a_1} \right)^p \|x^{(j)}\|^p. \end{aligned} \quad (30)$$

Set  $y_n^{(j)} = x_n^{(j)} / \|x^{(j)}\|$ . Then by (29) and the previous formula, we have

$$\begin{aligned} 0 & \geq \frac{J_p(x^{(j)})}{\|x^{(j)}\|^2} \geq \frac{1}{2} \left( \|\Delta y^{(j)}\|^2 - a \right) \\ & \quad - \left( \frac{b}{p} + \varepsilon \right) \left( \frac{1}{a_1} \right)^p \|x^{(j)}\|^{p-2}. \end{aligned} \quad (31)$$

On the other hand,  $\|\Delta y^{(j)}\|^2 \geq a$  by the definition of  $W^+ \oplus W^0$ . Hence by  $p > 2$ , there exists a subsequence converges to  $y^0 \in E_T$ , such that  $\|\Delta y^0\|^2 = a$ , that is  $y^0 \in W^0$  and  $\|y^0\| = 1$ . Since  $\|\Delta x^{(j)}\|^2 \geq a \|x^{(j)}\|^2$  for  $\{x^{(j)}\} \subset W^+ \oplus W^0$ , it follows from  $J_p(x^{(j)}) \leq 0$  that

$$\begin{aligned} 0 & \leq \frac{1}{p} \sum_{n=1}^T b(n) |x_n^{(j)}|^p + \sum_{n=1}^T G_p(n, x_n^{(j)}) \\ & \leq \frac{1}{p} \sum_{n=1}^T b(n) |x_n^{(j)}|^p + \varepsilon \left( \frac{1}{a_1} \right)^p \|x^{(j)}\|^p. \end{aligned} \quad (32)$$

Dividing by  $\|x^{(j)}\|^p$  in the previous inequality, then  $\sum_{n=1}^T b(n) |y_n^{(j)}|^p = \lim_{j \rightarrow \infty} \sum_{n=1}^T b(n) |y_n^{(j)}|^p \geq 0$ .

Since  $e, y^0 \in W^- \oplus W^0$ ,  $\Lambda_*(p) = \sum_{n=1}^T b(n) |e_n|^p < 0$  and  $(W^- \oplus W^0) \setminus \{0\}$  is arcwise connected, then there exists a  $\bar{y} \in (W^- \oplus W^0) \setminus \{0\}$  such that  $\sum_{n=1}^T b(n) |\bar{y}_n|^p = 0$ . Thus  $\|\Delta \bar{x}\|^2 \geq \lambda_*(p) \|\bar{x}\|^2$  by the definition of  $\lambda_*(p)$ . On the other hand,  $\|\Delta \bar{x}\|^2 \leq a \|\bar{x}\|^2$  by the definition of  $W^- \oplus W^0$ . This is a contradiction with assumption  $a < \lambda_*(p)$ . That is to say, the claim is valid.

By Proposition 6, we obtain  $C_k(J_p, 0) \neq 0$ ,  $k = \dim W^-$ . The proof is completed.  $\square$

**Lemma 8.** If  $a < \lambda_*(q)$ , then  $C_k(J_q, \infty) \neq 0$  for  $k = \dim X^-$ , where  $X^- = W^- \oplus W^0$  as  $\Lambda_*(q) > 0$ ,  $X^- = W^-$  as  $\Lambda_*(q) < 0$ .

The proof is similar to that of Lemma 7 and is omitted.

### 3. Proof of Theorem 2

**Lemma 9.** Let  $a < \lambda_*(p)$ . If there exists  $K_1 > 0$  such that for any  $K > K_1$ ,  $J_p(x) \leq -K$ , then one has  $\sum_{n=1}^T b(n) |x_n|^p > 0$ , and  $(d/dt)J_p(tx)|_{t=1} < 0$ .

*Proof.* We first claim that  $\|x\|$  is sufficiently large, if  $x$  satisfies condition of Lemma 9. Suppose not there exists  $M > 0$  such that  $\|x\| \leq M$ . So there exists  $\{x^{(j)}\} \subset E_T$ ,  $x^0 \in E_T$ ,

such that  $x^{(j)} \rightarrow x^0$  as  $j \rightarrow \infty$ . Since for any  $j > K_1$ , we have  $J_p(x^{(j)}) \leq -j$ , thus  $J_p(x^0) = \lim_{j \rightarrow \infty} J_p(x^{(j)}) = -\infty$ . It is a contradiction with  $J_p(x^0) = c$ .

If  $\|x\|$  is large enough, then we can assume that  $|x_n|$  is large enough for  $n \in Z[1, k]$  and  $|x_n|$  are bounded for  $n \in Z[k+1, T]$ . Therefore, by assumption (A1), for any given  $\varepsilon > 0$ , there exists  $M_1 > 0$  such that

$$\begin{aligned} |g_p(n, x_n)| & \leq \varepsilon |x_n| + \frac{M_1}{T}, \quad |G_p(n, x_n)| \leq \varepsilon |x_n|^2 + \frac{M_1}{T}, \\ \forall (n, x_n) & \in Z[1, T] \times \mathbb{R}^N. \end{aligned} \quad (33)$$

We claim that  $\sum_{n=1}^T b(n) |x_n|^p > 0$ . Suppose not that, for  $j > K_1$ , there exists  $\{x^{(j)}\} \subset E_T$  such that

$$\sum_{n=1}^T b(n) |x_n^{(j)}|^p \leq 0. \quad (34)$$

By  $J_p(x^{(j)}) \leq -j \leq 0$ , (33) and (34), we have

$$\begin{aligned} \frac{1}{2} \|\Delta x^{(j)}\|^2 & \leq \frac{a}{2} \|x^{(j)}\|^2 + \sum_{n=1}^T G_p(n, x_n^{(j)}) \\ & \leq \frac{a}{2} \|x^{(j)}\|^2 + \varepsilon \|x^{(j)}\|^2 + M_1. \end{aligned} \quad (35)$$

Set  $y_n^{(j)} = x_n^{(j)} / \|x^{(j)}\|$  and divided by  $\|x^{(j)}\|^2$  in the previous inequality. Since  $\varepsilon$  can be small enough, then there exists a subsequence that converges to  $y^0 \in E_T$ , such that  $\|\Delta y^0\|^2 \leq a$ ,  $\|y^0\| = 1$ . Moreover, by (33) and (34), we get

$$\begin{aligned} 0 & \geq \frac{1}{p} \sum_{n=1}^T b(n) |x_n^{(j)}|^p \geq j + \frac{1}{2} \|\Delta x^{(j)}\|^2 - \frac{a}{2} \|x^{(j)}\|^2 \\ & \quad - \sum_{n=1}^T G_p(n, x_n^{(j)}) \geq \left( \frac{a}{2} + \varepsilon \right) \|x^{(j)}\|^2 - M_1. \end{aligned} \quad (36)$$

Since  $p > 2$  and  $\lim_{j \rightarrow \infty} \|x^{(j)}\| = \infty$ , divided by  $\|x^{(j)}\|^p$  in the previous inequality, we have  $\sum_{n=1}^T b(n) |y_n^{(j)}|^p = \lim_{j \rightarrow \infty} \sum_{n=1}^T b(n) |y_n^{(j)}|^p = 0$ , that is,  $\|\Delta y^0\| \geq \lambda_*(q)$ , which deduce a contradiction. So the claim  $\sum_{n=1}^T b(n) |x_n|^p > 0$  holds.

Next we prove that  $(d/dt)J_p(tx)|_{t=1} < 0$  holds. By contradiction, there exists a sequence  $\{x^{(j)}\} \subset E_T$  such that, for  $j > K_1$ ,

$$\left. \frac{d}{dt} J_p(tx^{(j)}) \right|_{t=1} \geq 0. \quad (37)$$

Then, by (7), we get

$$\begin{aligned} \left. \frac{d}{dt} J_p(tx^{(j)}) \right|_{t=1} & = \|\Delta x^{(j)}\|^2 - a \|x^{(j)}\|^2 \\ & \quad - \sum_{n=1}^T b(n) |x_n^{(j)}|^p - \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}), \end{aligned} \quad (38)$$

and by (37) and  $J_p(x^{(j)}) \leq -j < 0$ , it follows that

$$\begin{aligned} & \left(1 - \frac{p}{2}\right) \left(\|\Delta x^{(j)}\|^2 - a\|x^{(j)}\|^2\right) \\ & - \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}) + p \sum_{n=1}^T G_p(n, x_n^{(j)}) \quad (39) \\ & = \frac{d}{dt} J_p(tx^{(j)}) \Big|_{t=1} - pJ_p(x^{(j)}) \geq 0. \end{aligned}$$

Set  $y_n^{(j)} = x_n^{(j)} / \|x^{(j)}\|$  and divided by  $\|x^{(j)}\|^2$  in the previous formula; since  $p > 2$  and  $\varepsilon$  can be small enough, then there exists a subsequence converges to  $y^0 \in E_T$  such that  $\|\Delta y^0\|^2 \leq a$ ,  $\|y^0\| = 1$ . Moreover, by (37) and the first claim, we get

$$\begin{aligned} 0 & < \sum_{n=1}^T b(n) |x_n^{(j)}|^p \leq \|\Delta x^{(j)}\|^2 - a\|x^{(j)}\|^2 \\ & - \sum_{n=1}^T (g_p(n, x_n^{(j)}), x_n^{(j)}). \quad (40) \end{aligned}$$

Divided by  $\|x^{(j)}\|^p$  in the previous formula, and by  $p > 2$ , it follows that  $\sum_{n=1}^T b(n) |y_n^0|^p = 0$ . This is a contradiction with the definition of  $\lambda_*(p)$  and condition  $a < \lambda_*(p)$ . So the second claim holds. The proof is completed.  $\square$

Based on Lemma 9, we introduce the following notations:

$$\begin{aligned} J_p^{-K} &= \{x \in E_T : J_p(x) \leq -K\}, \\ E_p^+ &= \left\{x \in E_T : \sum_{n=1}^T b(n) |x_n|^p > 0\right\}, \\ E(\Omega_+) &= \{x \in E_T : x_n = 0 \text{ for } n \in Z[1, T] \setminus \Omega_+\} \setminus \{0\}. \quad (41) \end{aligned}$$

Clearly,  $E(\Omega_+) \subset E_p^+$ . And by Lemma 9, we have  $J_p^{-K} \subset E_p^+$ . In order to describe the  $H_q(E_T, J_p^{-K})$ , we need to show the following lemma.

**Lemma 10.** *If  $a < \lambda_*(p)$ , then there exists  $K_1 > 0$ , such that for any  $K > K_1$ ,  $J_p^{-K}$  is a strong deformation retraction of  $E_p^+$ . Moreover,  $E(\Omega_+)$  and  $E_p^+$  are homotopy equivalent.*

*Proof.* Now we prove that  $J_p^{-K}$  is a strong deformation retraction of  $E_p^+$ .

By Lemma 9, we have  $J_p^{-K} \subset E_p^+$ . Let  $x \in E_p^+$ . By Lemma 9, there exists a unique  $t_p = t_p(x) > 0$  such that  $J_p(t_p x) = -K$ . By applying Implicit Function Theorem,  $t_p(x)$  is a continuous function in  $E_p^+$ . Let  $T_p(x) = \max\{t_p(x), 1\}$  and define  $f_p(s, x) = (1-s)x + sT_p(x)x$ , then  $f_p : [0, 1] \times E_p^+ \rightarrow J_p^{-K}$  is a strong deformation retraction. Thus  $J_p^{-K}$  is a strong deformation retraction of  $E_p^+$ .

We next claim that  $E(\Omega_+)$  is a strong deformation retraction of  $E_p^+$ . Clearly, in terms of the notations, we have  $E(\Omega_+) \subset E_p^+$ . Let  $\xi_p : Z[1, T] \rightarrow \mathbb{R}$  be a function such that

$$\begin{aligned} \xi_p(n) &= 1 \quad \text{if } n \in \Omega_+, \quad \xi_p(n) = 0 \quad \text{if } n \in \Omega_-, \\ \xi_p(n) &\in [0, 1] \quad \text{if } n \in Z[1, T] \setminus (\Omega_+ \cup \Omega_-). \end{aligned} \quad (42)$$

Define

$$\zeta_p(s, x_n) = \begin{cases} (1-2s)x_n + 2s\xi_p(n)x_n & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 2(1-s)\xi_p(n)x_n + 2\left(s - \frac{1}{2}\right)P(\xi_p(n)x_n) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases} \quad (43)$$

where  $P : E_T \rightarrow E(\Omega_+)$  is a projection operator. Then  $\zeta_p : [0, 1] \times E_p^+ \rightarrow E(\Omega_+)$  is a deformation retraction. Indeed,

$$\begin{aligned} \zeta_p(0, x) &= x, \quad \zeta_p(1, x) \in E(\Omega_+), \quad \text{for } x \in E_p^+, \\ \zeta_p(s, x) &= x, \quad \text{for } x \in E(\Omega_+) \text{ and } s \in [0, 1]. \end{aligned} \quad (44)$$

For  $x \in E_p^+$ , if  $s \in [0, 1/2]$ , then

$$\begin{aligned} & \sum_{n=1}^T b(n) |\zeta_p(s, x_n)|^p \\ &= \sum_{n \in \Omega_+} b(n) |x_n|^p + \sum_{n \in \Omega_-} b(n) (1-2s)^p |x_n|^p \quad (45) \\ &\geq \sum_{n=1}^T b(n) |x_n|^p > 0, \end{aligned}$$

where  $0 \leq (1-2s)^p \leq 1$ , that is,  $\zeta_p(s, x) \in E_p^+$ . If  $s \in (1/2, 1]$ , it follows that

$$\begin{aligned} & \sum_{n=1}^T b(n) |\zeta_p(s, x_n)|^p \\ &= \sum_{n \in \Omega_+} b(n) \left| 2(1-s)\xi_p(n)x_n + 2\left(s - \frac{1}{2}\right)P(\xi_p(n)x_n) \right|^p \\ &\geq 0. \end{aligned} \quad (46)$$

We claim that the equality of the previous formula cannot hold. Otherwise,  $Px_n = -((1-s)/(s-(1/2)))x_n$ , for  $n \in \Omega_+$ , which implies that  $Px_n = 0$ . Hence  $x_n = 0$  in  $\Omega_+$ , which contradicts with the fact  $x \in E_p^+$ . So  $\sum_{n=1}^T b(n) |\zeta_p(s, x_n)|^p > 0$ , that is,  $\zeta_p(s, x) \in E_p^+$  as  $s \in (1/2, 1]$ . Therefore,  $\zeta_p$  is a deformation retraction from  $E_p^+$  onto  $E(\Omega_+)$ , and this completes the proof.  $\square$

*Proof of Theorem 2.* Since  $E(\Omega_+)$  is well known to be contractile in itself, and by Lemma 10, it follows that  $J_p^{-K}$  is

homotopically equivalent to  $E(\Omega_+)$  for  $K$  large enough, then the Betti numbers (cf. [11, 13]) are

$$\begin{aligned}\beta_k &= \dim C_k(J_p, \infty) = \dim H_k(E_T, J_p^{-K}) \\ &= \dim H_k(E_T, E(\Omega_+)) = 0, \quad k \in Z[0, NT].\end{aligned}\quad (47)$$

Now we suppose that system (4) has only trivial solution; that is,  $J_p$  has only critical point  $x = 0$ , then we have the Morse-type numbers  $M_k = \dim C_k(J_p, 0)$  for  $k \in Z[0, NT]$  (cf. [13]). Moreover, by Lemma 7,  $C_k(J_p, 0) \neq 0$  for  $k = \dim W^-$  or  $k = \dim(W^- \oplus W^0)$ . Since  $J_p$  satisfies (PS) condition by Lemma 4, then using Morse Relation, we have the following.

$$0 = \sum_{k=0}^{NT} (-1)^k \beta_k = \sum_{k=0}^{NT} (-1)^k M_k \neq 0, \quad (48)$$

which is a contradiction. Therefore,  $J_p$  has at least one critical point  $x^* \neq 0$  and system (4) has at least a nonzero  $T$ -periodic solution.  $\square$

#### 4. Proof of Theorem 3

For convenience, we introduce the following notations:

$$\begin{aligned}J_q^c &= \{x \in E_T : J_q(x) \leq c\}, \quad c \in \mathbb{R}, \\ E_q^+ &= \left\{x \in E_T : \sum_{n=1}^T b(n) |x_n|^q > 0\right\}.\end{aligned}\quad (49)$$

Clearly,  $E_q^+ \cup \{0\}$  is star-shaped with respect to the origin and  $E(\Omega_+) \subset E_q^+$ , where  $E(\Omega_+)$  is given in Section 3. Similarly with the proof of Lemmas 9 and 10, we have the following.

**Lemma 11.** *Let  $a < \lambda_*(q)$ . Then there exists  $\rho > 0$  such that  $(d/dt)J_q(tx)|_{t=1} > 0$  for any  $x \in M_\rho = \{x \in B_\rho \cap E_q^+ : J_q(x) \geq 0\}$ , where  $B_\rho$  stands for the closed ball in  $E_T$  of radius  $\rho > 0$  with the center at zero.*

**Lemma 12.** *Let  $a < \lambda_*(q)$ . Then there exists  $\rho > 0$  such that  $(J_q^0 \cap B_\rho) \setminus \{0\}$  is a retract of  $E_q^+ \cap B_\rho$ , and  $E(\Omega^+)$  is a strong deformation retraction of  $E_q^+$ .*

*Proof of Theorem 3.* We first prove that  $J_q^0 \cap B_\rho$  is contractible in itself. In fact, it is sufficient to show that  $J_q^0 \cap B_\rho$  is starshaped with respect to the origin; that is,  $x \in J_q^0 \cap B_\rho$  implies that  $tx \in J_q^0 \cap B_\rho$  for all  $t \in [0, 1]$ .

Assume, by a contradiction, that there exists  $x_0 \in J_q^0 \cap B_\rho$  and  $t_0 \in (0, 1)$ , such that  $J_q(t_0 x_0) > 0$ . It follows from Lemma 11 that  $(d/dt)J_q(t_0 x_0) > 0$ . By the monotonicity arguments, this implies that

$$J_q(tx_0) > 0 \quad \forall t \in [t_0, 1]. \quad (50)$$

This contradicts the assumption  $x_0 \in J_q^0$ , which implies  $J_q(x_0) \leq 0$ .

On the other hand, since  $E(\Omega_+)$  is contractible in itself, and  $E_q^+ \cup \{0\}$  is starshaped with respect to the origin, then  $E_q^+ \cap B_\rho$  is contractible in itself. The retract of the set which is contractible in itself is also contractible (cf. [19]); it follows that the set  $(J_q^0 \cap B_\rho) \setminus \{0\}$  is contractible by Lemma 12.

Combining the previous argument,  $J_q^0 \cap B_\rho$  and  $(J_q^0 \cap B_\rho) \setminus \{0\}$  are contractible in themselves.

$$\begin{aligned}\dim C_k(J_q, 0) &= \dim H_k(J_q^0 \cap B_\rho, (J_q^0 \cap B_\rho) \setminus \{0\}) = 0, \\ k &\in Z[0, NT].\end{aligned}\quad (51)$$

By Lemma 8,  $C_k(J_q, \infty) \neq 0$  for  $k = \dim(W^- \oplus W^0)$  or  $k = \dim W^-$ . Therefore, by Morse Relation and the same methods in proof of Theorem 2, it follows that  $J_q$  has at least one critical point  $x^* \neq 0$  and system (5) has at least a nonzero  $T$ -periodic solution.  $\square$

#### Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187) and NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), and the Foundation of Education of Fujian Province (JA09152).

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