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Research Article

The Technique of Measures of Noncompactness in Banach Algebras and Its Applications to Integral Equations

Józef Banaś[1] and Szymon Dudek[2]

[1] *Department of Mathematics, Rzeszów University of Technology, Powstańców Warszawy 8, 35-959 Rzeszów, Poland*
[2] *Department of Mathematics and Natural Sciences, State Higher School of Technology and Economics in Jarosław, Czarnieckiego 16, 37-500 Jarosław, Poland*

Correspondence should be addressed to Józef Banaś; jbanas@prz.edu.pl

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We study the solvability of some nonlinear functional integral equations in the Banach algebra of real functions defined, continuous, and bounded on the real half axis. We apply the technique of measures of noncompactness in order to obtain existence results for equations in question. Additionally, that technique allows us to obtain some characterization of considered integral equations. An example illustrating the obtained results is also included.

1. Introduction

The purpose of the paper is to study the solvability of some functional integral equations in the Banach algebra consisting of real, continuous, and bounded functions defined on an unbounded interval. Equations of such a kind are recently often investigated in the mathematical literature (cf. [1–6]). It is worthwhile mentioning that some functional integral equations of that type find very interesting applications to describe real world problems which appeared in engineering, mathematical physics, radiative transfer, kinetic theory of gases, and so on (cf. [7–13], e.g.).

Functional integral equations considered in Banach algebras have rather complicated form, and the study of such equations requires the use of sophisticated tools. In the approach applied in this paper we will use the technique associated with measures of noncompactness and some fixed point theorems [14]. Such a direction of investigations has been initiated in the paper [2], where the authors introduced the so-called condition (m) related to the operation of multiplication in an algebra and playing a crucial role in the use of the technique of measures of noncompactness in Banach algebras setting. The usefulness of such an approach has been presented in [2], where the solvability of some class of functional integral equations was proved with help of some measures of noncompactness satisfying the mentioned condition (m).

This paper is an extension and continuation of the paper [2]. Here we are going to unify the approach with the use of the technique of measures of noncompactness to some general type of functional integral equations in the Banach algebra described above. Measures of noncompactness used here allow us not only to obtain the existence of solutions of functional integral equations but also to characterize those solutions in terms of stability, asymptotic stability, and ultimate monotonicity, for example.

Let us notice that such an approach to the theory of functional integral equations in Banach algebras is rather new and it was not exploited up to now except from the paper [2] initiating this direction of investigation.

2. Notation, Definitions, and Auxiliary Results

This section is devoted to presenting auxiliary facts which will be used throughout the paper. At the beginning we introduce some notation.

Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. If E is a given real Banach space with the norm $\| \cdot \|$ and the zero element θ, then by $B(x, r)$ we denote the closed ball centered at x and with radius r. We will write B_r to denote

the ball $B(\theta, r)$. If X is a subset of E, then the symbols $\overline{X}$ and Conv$X$ stand for the closure and convex closure of X, respectively. Apart from this the symbol diam X will denote the diameter of a bounded set X while $\|X\|$ denotes the norm of X; that is, $\|X\| = \sup\{\|x\| : x \in X\}$.

Next, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting all relatively compact sets.

In what follows we will accept the following definition of the concept of a measure of noncompactness [14].

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ will be called a *measure of noncompactness* in E if it satisfies the following conditions.

(1°) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(2°) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3°) $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.

(4°) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

(5°) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in (1°) is said to be *the kernel of the measure of noncompactness* μ.

Observe that the set X_∞ from the axiom (5°) is a member of the family $\ker \mu$. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ being satisfied for all $n = 1, 2, \ldots$ we derive that $\mu(X_\infty) = 0$ which means that $X_\infty \in \ker \mu$. This fact will play a key role in our further considerations.

In the sequel we will usually assume that the space E has the structure of Banach algebra. In such a case we write xy in order to denote the product of elements $x, y \in E$. Similarly, we will write XY to denote the product of subsets X, Y of E; that is, $XY = \{xy : x \in X, y \in Y\}$.

Now, we recall a useful concept introduced in [2].

Definition 2. One says that the measure of noncompactness μ defined on the Banach algebra E satisfies condition (m) if for arbitrary sets $X, Y \in \mathfrak{M}_E$ the following inequality is satisfied:

$$\mu(XY) \leq \|X\|\,\mu(Y) + \|Y\|\,\mu(X). \tag{1}$$

It turns out that the above defined condition (m) is very convenient in considerations connected with the use of the technique of measures of noncompactness in Banach algebras. Apart from this the majority of measures of noncompactness satisfy this condition [2]. We recall some details in the next section.

Now, we are going to formulate a fixed point theorem for operators acting in a Banach algebra and satisfying some conditions expressed with help of a measure of noncompactness. To this end we first recall a concept parallel to the concept of Lipschitz continuity (cf. [14]).

Definition 3. Let Ω be a nonempty subset of a Banach space E, and let $F : \Omega \to E$ be a continuous operator which transforms bounded subsets of Ω onto bounded ones. One says that F satisfies the Darbo condition with a constant k with respect to a measure of noncompactness μ if $\mu(FX) \leq k\mu(X)$ for each $X \in \mathfrak{M}_E$ such that $X \subset \Omega$. If $k < 1$, then F is called a contraction with respect to μ.

Now, assume that E is a Banach algebra and μ is a measure of noncompactness on E satisfying condition (m). Then we have the following theorem announced above [2].

Theorem 4. *Assume that Ω is nonempty, bounded, closed, and convex subset of the Banach algebra E, and the operators P and T transform continuously the set Ω into E in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, one assumes that the operator $F = P \cdot T$ transforms Ω into itself. If the operators P and T satisfy on the set Ω the Darbo condition with respect to the measure of noncompactness μ with the constants k_1 and k_2, respectively, then the operator F satisfies on Ω the Darbo condition with the constant*

$$\|P(\Omega)\|\,k_2 + \|T(\Omega)\|\,k_1. \tag{2}$$

Particularly, if $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1 < 1$, then F is a contraction with respect to the measure of noncompactness μ and has at least one fixed point in the set Ω.

Remark 5. It can be shown [14] that the set Fix F of all fixed points of the operator F on the set Ω is a member of the kernel $\ker \mu$.

3. Some Measures of Noncompactness in the Banach Algebra $BC(\mathbb{R}_+)$

In this section we present some measures of noncompactness in the Banach algebra $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous, and bounded on the half axis R_+. The algebra $BC(\mathbb{R}_+)$ is endowed with the usual supremum norm

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\} \tag{3}$$

for $x \in BC(\mathbb{R}_+)$. Obviously, the multiplication in $BC(\mathbb{R}_+)$ is understood as the usual product of real functions. Let us mention that measures of noncompactness, which we intend to present here, were considered in details in [2].

In what follows let us assume that X is an arbitrarily fixed nonempty and bounded subset of the Banach algebra $BC(\mathbb{R}_+)$; that is, $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Choose arbitrarily $\varepsilon > 0$ and $T > 0$. For $x \in X$ denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$; that is,

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}. \tag{4}$$

Next, let us put

$$\begin{aligned}\omega^T(X, \varepsilon) &= \sup\left\{\omega^T(x, \varepsilon) : x \in X\right\}, \\ \omega_0^T(X) &= \lim_{\varepsilon\to 0}\omega^T(X, \varepsilon), \\ \omega_0^\infty(X) &= \lim_{T\to\infty}\omega_0^T(X).\end{aligned} \tag{5}$$

Further, define the set quantity $a(X)$ by putting

$$a(X) = \lim_{T\to\infty} \left\{ \sup_{x\in X} \{\sup \{|x(t) - x(s)| : t, s \geq T\}\} \right\}. \quad (6)$$

Finally, let us put

$$\mu_a(X) = \omega_0^\infty(X) + a(X). \quad (7)$$

It can be shown [2] that the function μ_a is the measure of noncompactness in the algebra $BC(\mathbb{R}_+)$. The kernel $\ker \mu_a$ of this measure contains all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and have finite limits at infinity. Moreover, all functions from the set X tend to their limits with the same rate. It can be also shown that the measure of noncompactness μ_a satisfies condition (m) [2].

In our considerations we will also use another measure of noncompactness which is defined below.

In order to define this measure, similarly as above, fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ and a number $t \in \mathbb{R}_+$. Denote by $X(t)$ the cross-section of the set X at the point t; that is, $X(t) = \{x(t) : x \in X\}$. Denote by $\operatorname{diam} X(t)$ the diameter of $X(t)$. Further, for a fixed $T > 0$ and $x \in X$ denote by $d_T(x)$ the so-called *modulus of decrease* of the function x on the interval $[T, \infty)$, which is defined by the formula

$$d_T(x) = \sup \{|x(t) - x(s)| - [x(t) - x(s)] : T \leq s < t\}. \quad (8)$$

Next, let us put

$$d_T(X) = \sup \{d_T(x) : x \in X\},$$
$$d_\infty(X) = \lim_{T\to\infty} d_T(X). \quad (9)$$

In a similar way we may define *the modulus of increase* of function x and the set X (cf. [2]).

Finally, let us define the set quantity μ_d in the following way:

$$\mu_d(X) = \omega_0^\infty(X) + d_\infty(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \quad (10)$$

Linking the facts established in [2, 15] it can be shown that μ_d is the measure of noncompactness in the algebra $BC(\mathbb{R}_+)$. The kernel $\ker \mu_d$ of this measure consists of all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle $X(t)$ formed by functions from X tends to zero at infinity. Moreover, all functions from X are *ultimately nondecreasing* on $\mathbb{R}_+$ (cf. [16] for details).

Now, we show that the measure μ_d has also an additional property.

Theorem 6. *The measure of noncompactness μ_d defined by (10) satisfies condition (m) on the family of all nonempty and bounded subsets X of Banach algebra $BC(\mathbb{R}_+)$ such that functions belonging to X are nonnegative on $\mathbb{R}_+$.*

Proof. Observe that it is enough to show the second and third terms of the quantity μ_d defined by (10) satisfy condition (m). This assertion is a consequence of the fact that the term ω_0^∞ of the quantity defined by (10) satisfies condition (m) (cf. [2]).

Thus, take sets $X, Y \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ and numbers $T, s, t \in \mathbb{R}_+$ such that $T \leq s < t$. Moreover, assume that functions belonging to X and Y are nonnegative on the interval $\mathbb{R}_+$. Then, fixing arbitrarily $x \in X$ and $y \in Y$, we obtain

$$\begin{aligned}
&|x(t)y(t) - x(s)y(s)| - [x(t)y(t) - x(s)y(s)] \\
&\quad \leq |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\
&\qquad - \{x(t)|y(t) - y(s)| + y(s)|x(t) - x(s)|\} \\
&\quad = |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\
&\qquad - \{|x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)|\} \\
&\quad \leq \|x\| \cdot \{|y(t) - y(s)| - [y(t) - y(s)]\} \\
&\qquad + \|y\| \cdot \{|x(t) - x(s)| - [x(t) - x(s)]\}.
\end{aligned} \quad (11)$$

Consequently, we get

$$d_\infty(XY) \leq \|X\| d_\infty(Y) + \|Y\| d_\infty(X). \quad (12)$$

Next, for arbitrary $x_1, x_2 \in X$, $y_1, y_2 \in Y$, and $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
&|x_1(t)y_1(t) - x_2(t)y_2(t)| \\
&\quad \leq |x_1(t)||y_1(t) - y_2(t)| + |y_2(t)||x_1(t) - x_2(t)| \\
&\quad \leq \|x_1\| \cdot |y_1(t) - y_2(t)| + \|y_2\| \cdot |x_1(t) - x_2(t)|.
\end{aligned} \quad (13)$$

This estimate yields

$$\begin{aligned}
&\limsup_{t\to\infty} \operatorname{diam} (XY)(t) \\
&\quad \leq \|X\| \limsup_{t\to\infty} \operatorname{diam} Y(t) \\
&\quad + \|Y\| \limsup_{t\to\infty} \operatorname{diam} X(t).
\end{aligned} \quad (14)$$

Hence, in view of the fact that the set quantity ω_0^∞ satisfies condition (m), we complete the proof. □

It is worthwhile mentioning that the measure of noncompactness μ_d defined by formula (10) allows us to characterize solutions of considered operator equations in terms of the concept of asymptotic stability.

To formulate precisely that concept (cf. [17]) assume that Ω is a nonempty subset of the Banach algebra $BC(\mathbb{R}_+)$ and $F : \Omega \to BC(\mathbb{R}_+)$ is an operator. Consider the operator equation

$$x(t) = (Fx)(t), \quad t \in \mathbb{R}_+, \quad (15)$$

where $x \in \Omega$.

Definition 7. One says that solutions of (15) are *asymptotically stable* if there exists a ball $B(x_0, r)$ in $BC(\mathbb{R}_+)$ such that $B(x_0, r) \cap \Omega \neq \phi$, and for each $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - y(t)| \leq \varepsilon$ for all solutions $x, y \in B(x_0, r) \cap \Omega$ of (15) and for $t \geq T$.

Let us pay attention to the fact that if solutions of an operator equation considered in the algebra $BC(\mathbb{R}_+)$ belong to a bounded subset being a member of the family $\ker \mu_d$, then from the above given description of the kernel $\ker \mu_d$ we infer that those solutions are asymptotically stable in the sense of Definition 7 (cf. also Remark 5).

4. Existence of Asymptotically Stable and Ultimately Nondecreasing Solutions of a Functional Integral Equation in the Banach Algebra $BC(\mathbb{R}_+)$

This section is devoted to the study of solvability of a functional integral equation in the Banach algebra $BC(\mathbb{R}_+)$. Apart from the existence of solutions of the equation in question we obtain also some characterization of those solutions expressed in terms of asymptotic stability and ultimate monotonicity. Characterizations of such type are possible due to the technique of measures of noncompactness. Obviously, we will apply measures of noncompactness described in the preceding section.

In our considerations we will often use the so-called *superposition operator*. In order to define that operator assume that J is an interval and $f : \mathbb{R}_+ \times J \to \mathbb{R}$ is a given function. Then, to every function $x : \mathbb{R}_+ \to J$ we may assign the function Fx defined by the formula

$$(Fx)(t) = f(t, x(t)), \qquad (16)$$

for $t \in \mathbb{R}_+$. The operator F defined in such a way is called the superposition operator generated by the function $f(t,x)$ (cf. [18]).

Lemma 8 [16] presents a useful property of the superposition operator which is considered in the Banach space $B(\mathbb{R}_+)$ consisting of all real functions defined and bounded on $\mathbb{R}_+$. Obviously, the space $B(\mathbb{R}_+)$ is equipped with the classical supremum norm. Since $B(\mathbb{R}_+)$ has the structure of a Banach algebra, we can consider the Banach algebra $BC(\mathbb{R}_+)$ as a subalgebra of $B(\mathbb{R}_+)$.

Lemma 8. *Assume that the following hypotheses are satisfied.*

(α) *The function f is continuous on the set $\mathbb{R}_+ \times J$.*

(β) *The function $t \mapsto f(t,u)$ is ultimately nondecreasing uniformly with respect to u belonging to bounded subintervals of J, that is,*

$$\lim_{T\to\infty} \{\sup\{|f(t,u) - f(s,u)| - [f(t,u) - f(s,u)] : t > s \geq T,\ u \in J_1\}\} = 0 \qquad (17)$$

for any bounded subinterval $J_1 \subseteq J$.

(γ) *For any fixed $t \in \mathbb{R}_+$ the function $u \mapsto f(t,u)$ is nondecreasing on J.*

(δ) *The function $u \mapsto f(t,u)$ satisfies a Lipschitz condition; that is, there exists a constant $k > 0$ such that*

$$|f(t,u) - f(t,v)| \leq k|u - v| \qquad (18)$$

for all $t \leq 0$ and all $u, v \in J$.

Then the inequality

$$d_\infty(Fx) \leq k d_\infty(x) \qquad (19)$$

holds for any function $x \in B(\mathbb{R}_+)$, where k is the Lipschitz constant from assumption (δ).

Observe that in view of the remark mentioned previously Lemma 8 is also valid in the Banach algebra $BC(\mathbb{R}_+)$.

As we announced before, in this section we will study the solvability of the following integral equation:

$$x(t) = (V_1x)(t)(V_2x)(t), \quad t \in \mathbb{R}_+, \qquad (20)$$

where V_i are the so-called quadratic Volterra-Hammerstein integral operators defined as follows

$$(V_ix)(t) = p_i(t) + f_i(t, x(t)) \times \int_0^t g_i(t,s) h_i(s, x(s))\, ds, \quad (i = 1,2). \qquad (21)$$

Now, we formulate the assumptions under which we will study (20).

(i) $p_i \in BC(\mathbb{R}_+)$ and p_i is ultimately nondecreasing; that is, $d_\infty(p_i) = 0$ $(i = 1,2)$. Moreover, $p_i(t) \geq 0$ for $t \in \mathbb{R}_+$ $(i = 1,2)$.

(ii) $f_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and f_i satisfies assumptions of Lemma 8 for $J = \mathbb{R}_+$ $(i = 1,2)$.

(iii) The function f_i satisfies the Lipschitz condition with respect to the second variable; that is, there exists a constant $k_i > 0$ such that

$$|f_i(t,x) - f_i(t,y)| \leq k_i|x - y| \qquad (22)$$

for $x, y \in \mathbb{R}_+$ and for $t \in \mathbb{R}_+$ $(i = 1,2)$.

(iv) The function $g_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies the following condition:

$$\lim_{T\to\infty}\left\{\sup\left\{\int_0^t \{|g_i(t,\tau) - g_i(s,\tau)| - [g_i(t,\tau) - g_i(s,\tau)]\}\, d\tau : T \leq s < t\right\}\right\} = 0, \quad (i = 1,2). \qquad (23)$$

(v) The function $t \to \int_0^t g_i(t,s)ds$ is bounded on $\mathbb{R}_+$ and

$$\lim_{t\to\infty} \int_0^t g_i(t,s)\, ds = 0 \quad (i = 1,2). \qquad (24)$$

(vi) The function $h_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous for $i = 1,2$.

(vii) The functions $t \to f_i(t,0)$ and $t \to h_i(t,0)$ are bounded on $\mathbb{R}_+$ $(i = 1, 2)$.

(viii) There exists a continuous and nondecreasing function $m_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_i(0) = 0$ and

$$\left|h_i(t,x) - h_i(t,y)\right| \le m_i\left(|x-y|\right) \tag{25}$$

for x, y, and $t \in \mathbb{R}_+$ and $i = 1, 2$.

In view of the above assumptions, we may define the following finite constants:

$$\begin{aligned} \overline{F}_i &= \sup\left\{f_i(t,0) : t \in \mathbb{R}_+\right\}, \\ \overline{H}_i &= \sup\left\{h_i(t,0) : t \in \mathbb{R}_+\right\}, \\ \overline{G}_i &= \sup\left\{\int_0^t g_i(t,s)\,ds : t \in \mathbb{R}_+\right\}, \end{aligned} \tag{26}$$

for $i = 1, 2$.

Using these quantities we formulate the last assumption.

(ix) There existS a solution $r_0 > 0$ of the inequality

$$\begin{aligned} &\left[p + kr\overline{G}m_1(r) + kr\overline{G}\,\overline{H} + \overline{F}\,\overline{G}m_1(r) + \overline{F}\,\overline{G}\,\overline{H}\right] \\ &\quad \times \left[p + kr\overline{G}m_2(r) + kr\overline{G}\,\overline{H} + \overline{F}\,\overline{G}m_2(r) + \overline{F}\,\overline{G}\,\overline{H}\right] \le r \end{aligned} \tag{27}$$

such that

$$\begin{aligned} &pk\overline{G}\left[m_1(r_0) + m_2(r_0) + \overline{H}\right] \\ &\quad + 2k\overline{G}^2\left[kr_0 + \overline{F}\right] \\ &\quad \times \left[m_1(r_0) + \overline{H}\right]\left[m_2(r_0) + \overline{H}\right] < 1, \end{aligned} \tag{28}$$

where $p = \max\{\|p_1\|, \|p_2\|\}$, $\overline{F} = \max\{\overline{F}_1, \overline{F}_2\}$, $\overline{G} = \max\{\overline{G}_1, \overline{G}_2\}$,

$$\begin{aligned} \overline{H} &= \max\left\{\overline{H}_1, \overline{H}_2\right\}, \\ k &= \max\left\{k_1, k_2\right\}. \end{aligned} \tag{29}$$

Now we formulate the main existence result for the functional integral equation (20).

Theorem 9. *Under assumptions* (i)–(ix) (20) *has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$. Moreover all solutions of* (20) *are nonnegative, asymptotically stable, and ultimately nondecreasing.*

Proof. Consider the subset Ω of the algebra $BC(\mathbb{R}_+)$ consisting of all functions being nonnegative on $\mathbb{R}_+$. We will consider operators V_i $(i = 1, 2)$ on the set Ω.

Now, fix an arbitrary function $x \in \Omega$. Then, from assumptions (i), (ii), and (vi) we deduce that the function $V_i x$ is continuous on $\mathbb{R}_+$. Moreover, in view of assumptions (i), (ii), (iv), and (vi) we have that the function $V_i x$ is nonnegative on the interval $\mathbb{R}_+$ for $i = 1, 2$.

Further, for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain

$$\begin{aligned} (V_i x)(t) &\le p_i(t) + f_i(t, x(t)) \int_0^t g_i(t,\tau)\, h_i(\tau, x(\tau))\,d\tau \\ &\le p_i(t) + \left[k_i x(t) + f_i(t,0)\right] \\ &\quad \times \int_0^t g_i(t,\tau)\left[m_i(x(\tau)) + h_i(\tau,0)\right]d\tau \\ &\le \|p_i\| + \left[k_i\|x\| + \overline{F}_i\right]\left[m_i(\|x\|) + \overline{H}_i\right]\overline{G}_i. \end{aligned} \tag{30}$$

Hence we get

$$\begin{aligned} \|V_i x\| &\le \|p_i\| + k_i\|x\|\,\overline{G}_i m_i(\|x\|) \\ &\quad + k_i\|x\|\,\overline{G}_i\overline{H}_i + \overline{F}_i\overline{G}_i m_i(\|x\|) + \overline{F}_i\overline{G}_i\overline{H}_i, \end{aligned} \tag{31}$$

for $i = 1, 2$.

This shows that the function $V_i x$ $(i = 1, 2)$ is bounded on $\mathbb{R}_+$, and, consequently, $V_i x$ is a member of the set Ω. Consequently we infer that the operator W being the product of the operators V_1 and V_2 transforms also the set Ω into itself.

Further, taking into account estimate (31) and assumption (ix) we conclude that the operator W is a self-mapping of the set Ω_{r_0} defined in the following way:

$$\Omega_{r_0} = \left\{x \in BC(\mathbb{R}_+) : 0 \le x(t) \le r_0 \text{ for } t \in \mathbb{R}_+\right\}, \tag{32}$$

where r_0 is the number from assumption (ix).

In the sequel we will work with the measure of noncompactness μ_d defined by formula (10).

At the beginning, let us fix nonempty set $X \subset \Omega_{r_0}$ and numbers $T > 0$, $\varepsilon > 0$. Additionally, let $x \in X$ and $t, s \in [0, T]$ be such that $|t - s| \le \varepsilon$. Without loss of generality we may assume that $t < s$. Then we obtain

$$\begin{aligned} &\left|(V_i x)(t) - (V_i x)(s)\right| \\ &\quad \le \left|p_i(t) - p_i(s)\right| + \left|f_i(t, x(t)) - f_i(s, x(s))\right| \\ &\qquad \times \int_0^t g_i(t,\tau)\, h_i(\tau, x(\tau))\,d\tau \\ &\qquad + \left|f_i(s, x(s))\right|\left|\int_0^t g_i(t,\tau)\, h_i(\tau, x(\tau))\,d\tau \right. \\ &\qquad\qquad \left. - \int_0^s g_i(s,\tau), h_i(\tau, x(\tau))\,d\tau\right| \\ &\quad \le \left|p_i(t) - p_i(s)\right| + \left[\left|f_i(t, x(t)) - f_i(t, x(s))\right|\right. \\ &\qquad\qquad \left. + \left|f_i(t, x(s)) - f_i(s, x(s))\right|\right] \end{aligned}$$

$$\times \int_0^t g_i(t,\tau)\left[h_i(\tau,x(\tau)) - h_i(\tau,0) + h_i(\tau,0)\right] d\tau$$
$$+ \left[\left|f_i(s,x(s)) - f_i(s,0)\right| + \left|f_i(s,0)\right|\right]$$
$$\times \int_0^t \left|g_i(t,\tau) - g_i(s,\tau)\right| h_i(\tau,x(\tau))\, d\tau$$
$$\le \omega^T(p_i,\varepsilon) + \left[k_i\left|x(t)-x(s)\right| + \omega_{r_0}^T(f_i,\varepsilon)\right]$$
$$\times \int_0^t g_i(t,\tau)\left[m_i(|x(\tau)|) + \overline{H}_i\right] d\tau$$
$$+ \left[k_i|x(s)| + \overline{F}_i\right]\int_0^T \omega_1^T(g_i,\varepsilon)\left[m_i(|x(\tau)|) + \overline{H}_i\right] d\tau$$
$$\le \omega^T(p_i,\varepsilon) + \left[k_i\omega^T(x,\varepsilon) + \omega_{r_0}^T(f_i,\varepsilon)\right]\left[m_i(r_0) + \overline{H}_i\right]\overline{G}_i$$
$$+ \left[k_i r_0 + \overline{F}_i\right] T\omega_1^T(g_i,\varepsilon)\left[m_i(r_0) + \overline{H}_i\right], \tag{33}$$

where we denoted

$$\omega_d^T(f_i,\varepsilon) = \sup\left\{\left|f_i(t,y) - f_i(s,y)\right| : t,s\in[0,T],\ y\in[-d,d],\ |t-s|\le\varepsilon\right\},$$
$$\omega_1^T(g_i,\varepsilon) = \sup\left\{\left|g_i(t,\tau) - g_i(s,\tau)\right| : t,s,\tau\in[0,T],\ |t-s|\le\varepsilon\right\}. \tag{34}$$

In view of the uniform continuity of the function f_i on the set $[0,T]\times[-r_0,r_0]$ we infer that $\omega_d^T(f_i,\varepsilon) \to 0$ as $\varepsilon \to 0$. Analogously $\omega_1^T(g_i,\varepsilon) \to 0$ if $\varepsilon \to 0$ since the function g_i is uniformly continuous on the set $[0,T]\times[0,T]$. Hence, in view of estimate (33) we obtain the following evaluation:

$$\omega_0^\infty(V_iX) \le k_i\overline{G}_i\left[m_i(r_0) + \overline{H}_i\right]\omega_0^\infty(X). \tag{35}$$

Similarly as above, for $t\in\mathbb{R}_+$ and $x,y\in X$ we get

$$\left|(V_ix)(t) - (V_iy)(t)\right|$$
$$\le \left|f_i(t,x(t)) - f_i(t,y(t))\right|$$
$$\times \int_0^t g_i(t,\tau)\, h_i(\tau,x(\tau))\, d\tau$$
$$+ f_i(t,y(t))$$
$$\times \int_0^t \left|g_i(t,\tau)\, h_i(\tau,x(\tau)) - g_i(t,\tau)\, h_i(\tau,y(\tau))\right| d\tau$$
$$\le k_i\left|x(t)-y(t)\right|\left[m_i(r_0) + \overline{H}_i\right]\int_0^t g_i(t,\tau)\, d\tau$$
$$+ \left[k_i\left|y(t)\right| + \overline{F}_i\right]\int_0^t g_i(t,\tau)\, m_i\left(\left|x(\tau)-y(\tau)\right|\right) d\tau$$
$$\le k_i\left|x(t)-y(t)\right|\left[m_i(r_0) + \overline{H}_i\right]$$
$$\times \int_0^t g_i(t,\tau)\, d\tau + \left[k_ir_0 + \overline{F}_i\right] m_i(r_0)\int_0^t g_i(t,\tau)\, d\tau. \tag{36}$$

Assumption (v) implies that the right side of the above estimate vanishes at infinity. Hence we obtain

$$\limsup_{t\to\infty} \operatorname{diam}(VX)(t) = 0. \tag{37}$$

This fact helps us prove that V_i is continuous on the set Ω_{r_0}. Indeed, let us fix $\varepsilon > 0$ and take $x,y\in\Omega_{r_0}$ such that $\|x-y\|\le\varepsilon$. In view of (37) we know that there exists a number $T>0$ such that for arbitrary $t\ge T$ we get $|(V_ix)(t) - (V_iy)(t)| \le \varepsilon$. Then, if we take $t\in[0,T]$ we obtain the following estimate:

$$\left|(V_ix)(t) - (V_iy)(t)\right|$$
$$\le \left|f_i(t,x(t)) - f_i(t,y(t))\right|\int_0^t g_i(t,\tau)\, h_i(\tau,x(\tau))\, d\tau$$
$$+ f_i(t,y(t))\int_0^t g_i(t,\tau)\left|h_i(\tau,x(\tau)) - h_i(\tau,y(\tau))\right| d\tau$$
$$\le k_i\left|x(t)-y(t)\right|\left[m_i(\|x\|) + \overline{H}_i\right]\int_0^t g_i(t,\tau)\, d\tau$$
$$+ \left[k_i\left|y(t)\right| + \overline{F}_i\right]\int_0^t g_i(t,\tau)\, m_i\left(\left|x(\tau)-y(\tau)\right|\right) d\tau$$
$$\le k_i\varepsilon\left[m_i(r_0) + \overline{H}_i\right]\overline{G}_i + \left[k_ir_0 + \overline{F}_i\right] m_i(\varepsilon)\,\overline{G}_i. \tag{38}$$

Observe that continuity of the function m_i yields that for each $T>0$ the expression on the right hand side of estimate (38) can be sufficiently small.

In what follows let us fix $T>0$ and $t>s\ge T$. Then, if $x\in\Omega_{r_0}$, we derive the following estimates:

$$\left|(V_ix)(t) - (V_ix)(s)\right| - \left[(V_ix)(t) - (V_ix)(s)\right]$$
$$\le \left|p_i(t) - p_i(s)\right|$$
$$+ \left|f_i(t,x(t))\int_0^t g_i(t,\tau)\, h_i(\tau,x(\tau))\, d\tau - f_i(s,x(s))\int_0^s g_i(s,\tau)\, h_i(\tau,x(\tau))\, d\tau\right|$$
$$- \left[p_i(t) - p_i(s)\right]$$
$$- \left[f_i(t,x(t))\int_0^t g_i(t,\tau)\, h_i(\tau,x(\tau))\, d\tau - f_i(s,x(s))\int_0^s g_i(s,\tau)\, h_i(\tau,x(\tau))\, d\tau\right]$$

$$\leq d_T(p_i) + \left|f_i(t,x(t)) - f_i(s,x(s))\right| \times \int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau + f_i(s,x(s)) \left|\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right| - \left[f_i(t,x(t)) - f_i(s,x(s))\right] \int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - f_i(s,x(s)) \left[\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right]$$
$$\leq d_T(p_i) + \left\{\left|f_i(t,x(t)) - f_i(s,x(s))\right| - \left[f_i(t,x(t)) - f_i(s,x(s))\right]\right\} \times \left[m_i(\|x\|) + \overline{H}_i\right] \int_0^t g_i(t,\tau)\,d\tau + f_i(s,x(s)) \left\{\left|\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right| - \left[\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right]\right\}. \tag{39}$$

Observe that in virtue of imposed assumptions we have

$$\left|\int_0^t g_i(t,\tau) h_i(\tau,x(\tau)) - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right| - \left[\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^s g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right]$$
$$\leq \left|\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^t g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right| + \int_s^t g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau - \left[\int_0^t g_i(t,\tau) h_i(\tau,x(\tau))\,d\tau - \int_0^t g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau\right] - \int_s^t g_i(s,\tau) h_i(\tau,x(\tau))\,d\tau$$
$$\leq \left[m_i(\|x\|) + \overline{H}_i\right] \times \int_0^t \left\{\left|g_i(t,\tau) - g_i(s,\tau)\right| - \left[g_i(t,\tau) - g_i(s,\tau)\right]\right\} d\tau. \tag{40}$$

Linking the above estimate and (39), we obtain

$$\begin{aligned} \sup\{&\left|(V_i x)(t) - (V_i x)(s)\right| - \left[(V_i x)(t) - (V_i x)(s)\right] : t > s \geq T\} \\ &\leq d_T(p_i) + d_T(f_i)\,\overline{G}_i\left[m_i(r_0) + \overline{H}_i\right] \\ &\quad + \left[k_i r_0 + \overline{F}_i\right]\left[m_i(r_0) + \overline{H}_i\right] d_T^c(g_i), \end{aligned} \tag{41}$$

where we denoted

$$\begin{aligned} d_T(f_i) &= \sup\{\left|f_i(t,x(t)) - f_i(s,x(s))\right| - \left[f_i(t,x(t)) - f_i(s,x(s))\right] : t > s \geq T\}, \\ d_T^c(g_i) &= \sup\left\{\int_0^t \{\left|g_i(t,\tau) - g_i(s,\tau)\right| - \left[g_i(t,\tau) - g_i(s,\tau)\right]\} : t > s \geq T\right\}. \end{aligned} \tag{42}$$

In view of assumptions (i), (iv) and Lemma 8, if T tends to infinity, then the above obtained estimate allows us to deduce the following inequality:

$$d_\infty(V_i X) \leq k_i \overline{G}_i\left[m_i(r_0) + \overline{H}_i\right] d_\infty(X). \tag{43}$$

Now, linking (35), (37) and (43) we obtain

$$\mu_d(V_i X) \leq k_i \overline{G}_i\left[m_i(r_0) + \overline{H}_i\right] \mu_d(X) \tag{44}$$

for $i \in \{1,2\}$.

Hence, if we use Theorem 4, we obtain that the operator $W = V_1 V_2$ is a contraction with respect to the measure of noncompactness μ_d and fulfills the Darbo condition with the below indicated constant

$$\begin{aligned} L = {} & pk\overline{G}\left[m_1(r_0) + m_2(r_0) + \overline{H}\right] \\ & + 2k\overline{G}^2\left[kr_0 + \overline{F}\right]\left[m_1(r_0) + \overline{H}\right]\left[m_2(r_0) + \overline{H}\right]. \end{aligned} \tag{45}$$

Taking into account the second part of assumption (ix) we have additionally that $L < 1$.

Finally, invoking Theorem 4 we deduce that the operator W has at least one fixed point $x = x(t)$ in the set Ω_{r_0}. Obviously, the function x is a solution of (20). From Remark 5 we conclude that x is asymptotically stable and ultimately nondecreasing. Obviously, x is nonnegative on $\mathbb{R}_+$.

The proof is complete. □

5. The Existence of Solutions of a Quadratic Fractional Integral Equation in the Banach Algebra $BC(\mathbb{R}_+)$

In this section we will investigate the existence of solutions of the quadratic fractional integral equation having the form

$$x(t) = (U_1 x)(t)(U_2 x)(t), \tag{46}$$

where

$$(U_i x)(t) = m_i(t) + f_i(t, x(t)) \int_0^t \frac{v_i(t, s, x(s))}{(t-s)^{\alpha_i}} ds \tag{47}$$

for $t \in \mathbb{R}_+$ and $i = 1, 2$. Here we assume that $\alpha_i \in (0, 1)$ is a fixed number for $i = 1, 2$.

Our investigations will be conducted, similarly as previously, in the Banach algebra $BC(\mathbb{R}_+)$.

For further purposes we define a few operators on the space $BC(\mathbb{R}_+)$ by putting

$$\begin{gathered}(F_i x)(t) = f_i(t, x(t)), \\ (V_i x)(t) = \int_0^t \frac{v_i(t, s, x(s))}{(t-s)^{\alpha_i}} ds\end{gathered} \tag{48}$$

for $i = 1, 2$. Obviously, we have

$$(U_i x)(t) = m_i(t) + (F_i x)(t)(V_i x)(t) \tag{49}$$

for $i = 1, 2$ and for $t \in \mathbb{R}_+$.

Now, we are going to formulate assumptions imposed on functions involved in (46).

(i) The function m_i is nonnegative, bounded, continuous, and ultimately nondecreasing ($i = 1, 2$).

(ii) The function $f_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions (α)–(γ) of Lemma 8 for $i = 1, 2$.

(iii) The functions f_i ($i = 1, 2$) satisfy the Lipschitz condition with respect to the second variable; that is, there exists a constant $k_i > 0$ such that

$$|f_i(t, x) - f_i(t, y)| \leq k_i |x - y| \tag{50}$$

for, $t \in \mathbb{R}_+ (i = 1, 2)$.

(iv) $v_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ ($i = 1, 2$).

(v) There exist a continuous and nondecreasing function $G_i : \mathbb{R}_+ \to \mathbb{R}_+$ and a bounded and continuous function $g_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $v_i(t, s, x) = g_i(t, s) G_i(|x|)$ for $t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$, and $i = 1, 2$.

(vi) The function $t \mapsto \int_0^t (g_i(t, s)/(t-s)^{\alpha_i}) ds$ is bounded on $\mathbb{R}_+$ and

$$\lim_{t \to \infty} \int_0^t \frac{g_i(t, s)}{(t-s)^{\alpha_i}} ds = 0, \tag{51}$$

for $i = 1, 2$.

(vii) The function g_i ($i = 1, 2$) satisfies the following condition ($i = 1, 2$):

$$\begin{aligned}\lim_{T \to \infty} \Bigg\{ \sup \Bigg\{ \int_0^t \Bigg\{ & \left| \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} - \frac{g_i(s, \tau)}{(s-\tau)^{\alpha_i}} \right| \\ & - \left[\frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} - \frac{g_i(s, \tau)}{(s-\tau)^{\alpha_i}} \right] \Bigg\} d\tau : \\ & T \leq s < t \Bigg\} \Bigg\} = 0.\end{aligned} \tag{52}$$

In view of the above assumptions we may define the following constants ($i = 1, 2$):

$$\begin{gathered}\overline{F}_i = \sup \{ |f_i(t, 0)| : t \in \mathbb{R}_+ \}, \\ \overline{G}_i = \sup \left\{ \int_0^t \frac{g_i(t, s)}{(t-s)^{\alpha_i}} ds : t \in \mathbb{R}_+ \right\}, \\ \overline{g}_i = \sup \{ g_i(t, s) : t, s \in \mathbb{R}_+ \}, \\ \overline{F} = \max \{ \overline{F}_1, \overline{F}_2 \}, \\ k = \max \{ k_1, k_2 \}, \\ m = \max \{ \|m_1\|, \|m_2\| \}.\end{gathered} \tag{53}$$

The last assumption has the form 0.

(viii) There exists a solution $r_0 > 0$ of the inequality

$$\begin{aligned}&\left[m + k\overline{G}_1 r G_1(r) + \overline{F}\, \overline{G}_1 G_1(r) \right] \\ &\quad \times \left[m + k\overline{G}_2 r G_2(r) + \overline{F}\, \overline{G}_2 G_2(r) \right] \leq r\end{aligned} \tag{54}$$

such that

$$\begin{aligned}&mk \left(\overline{G}_1 G_1(r_0) + \overline{G}_2 G_2(r_0) \right) \\ &\quad + 2k\overline{F}\, \overline{G}_1 G_1(r_0) \overline{G}_2 G_2(r_0) \\ &\quad + 2k^2 r_0 \overline{G}_1 G_1(r_0) \overline{G}_2 G_2(r_0) < 1.\end{aligned} \tag{55}$$

Now we are prepared to formulate and prove the existence result concerning the functional integral equation (46).

Theorem 10. *Under assumptions* (i)–(viii) (46) *has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ which is nonnegative, asymptotically stable, and ultimately nondecreasing.*

Proof. Similarly as in the proof of Theorem 9 let us consider the subset Ω of the Banach algebra $BC(\mathbb{R}_+)$ which consists of all functions being nonnegative on $\mathbb{R}_+$. Further, choose arbitrary function $x \in \Omega$. Then, applying assumptions (i), (ii), (iv), and (v) we infer that the function $U_i x$ is nonnegative on $\mathbb{R}_+$ ($i = 1, 2$).

Next, for $t \in \mathbb{R}_+$, in view of (47) and the imposed assumptions, we get

$$\begin{aligned}(U_i x)(t) &\le m_i(t) + [k_i x(t) + f_i(t,0)] \int_0^t \frac{v_i(t,s,x(s))}{(t-s)^{\alpha_i}} ds \\ &\le m_i(t) + [k_i x(t) + f_i(t,0)] G_i(\|x\|) \int_0^t \frac{g_i(t,s)}{(t-s)^{\alpha_i}} ds \\ &\le \|m_i\| + k_i \overline{G}_i \|x\| G_i(\|x\|) + \overline{F}\, \overline{G}_i G_i(\|x\|),\end{aligned} \tag{56}$$

for $i = 1, 2$.

This estimate yields that the function $U_i x$ is bounded on $\mathbb{R}_+$ $(i = 1, 2)$.

Furthermore, let us observe that in view of the properties of the superposition operator [18] and assumption (ii) we derive that the function $F_i x$ is continuous on $\mathbb{R}_+$ $(i = 1, 2)$. Thus, in order to show that $U_i x$ is continuous on the interval $\mathbb{R}_+$, it is sufficient to show that the function $V_i x$ is continuous on $\mathbb{R}_+$.

To this end fix $T > 0$ and $\varepsilon > 0$. Next, choose arbitrarily $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$. Without loss of generality we may assume that $s < t$. Then we have

$$\begin{aligned}&\left|(V_i x)(t) - (V_i x)(s)\right| \\ &\le \left| \int_0^t \frac{v_i(t,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \right| \\ &\quad + \left| \int_0^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^s \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \right| \\ &\quad + \left| \int_0^s \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^s \frac{v_i(s,\tau,x(\tau))}{(s-\tau)^{\alpha_i}} d\tau \right| \\ &\le \int_0^t \frac{\left|v_i(t,\tau,x(\tau)) - v_i(s,\tau,x(\tau))\right|}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad + \int_s^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad + \int_0^s v_i(s,\tau,x(\tau)) \left[\frac{1}{(s-\tau)^{\alpha_i}} - \frac{1}{(t-\tau)^{\alpha_i}} \right] d\tau \\ &\le \omega_{\|x\|}^T(v_i,\varepsilon) \int_0^t \frac{1}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad + G_i(\|x\|)\, \overline{g}_i \int_s^t \frac{1}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad + G_i(\|x\|)\, \overline{g}_i \int_0^s \left[\frac{1}{(s-\tau)^{\alpha_i}} - \frac{1}{(t-\tau)^{\alpha_i}} \right] d\tau \\ &\le \omega_{\|x\|}^T(v_i,\varepsilon) \frac{t^{1-\alpha_i}}{1-\alpha_i} + G_i(\|x\|)\, \overline{g}_i \frac{(t-s)^{1-\alpha_i}}{1-\alpha_i} \\ &\quad + G_i(\|x\|)\, \overline{g}_i \left[\frac{s^{1-\alpha_i}}{1-\alpha_i} - \frac{t^{1-\alpha_i}}{1-\alpha_i} + \frac{(t-s)^{1-\alpha_i}}{1-\alpha_i} \right] \\ &\le \omega_{\|x\|}^T(v_i,\varepsilon) \frac{T^{1-\alpha_i}}{1-\alpha_i} + 2G_i(\|x\|)\, \overline{g}_i \frac{\varepsilon^{1-\alpha_i}}{1-\alpha_i},\end{aligned} \tag{57}$$

where we denoted

$$\omega_d^T(v_i,\varepsilon) = \sup \left\{ \left|v_i(t,\tau,x) - v_i(s,\tau,x)\right| : t, s, \tau \in [0,T],\ |t-s| \le \varepsilon,\ x \in [-d,d] \right\}. \tag{58}$$

From the above estimate and the fact that function v_i is uniformly continuous on the set $[0,T] \times [0,T] \times [-\|x\|, \|x\|]$ we infer that the function $V_i x$ is continuous on the real half axis $\mathbb{R}_+$.

Gathering the above established facts and estimate (57) we conclude that the operator U_i $(i = 1, 2)$ transforms the set Ω into itself.

Apart from this, in view of (56) and assumption (vii) we infer that there exists a number $r_0 > 0$ such that the operator $S = U_1 U_2$ transforms into itself the set Ω_{r_0} defined in the following way:

$$\Omega_{r_0} = \left\{ x \in BC(\mathbb{R}_+) : 0 \le x(t) \le r_0 \text{ for } t \in \mathbb{R}_+ \right\}. \tag{59}$$

Moreover, the following inequality is satisfied:

$$\left\| U_i \Omega_{r_0} \right\| \le m + k_i \overline{G} r_0 G_i(r_0) + \overline{F}\, \overline{G} G_i(r_0). \tag{60}$$

In the sequel we will work with the measure of noncompactness μ_d. Thus, let us fix a nonempty subset X of the set Ω_{r_0} and choose arbitrary numbers $T > 0$ and $\varepsilon > 0$. Then, for $x \in X$ and for $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$ and $t \ge s$ we have

$$\begin{aligned}&\left|(U_i x)(t) - (U_i x)(s)\right| \\ &\le \omega^T(m_i,\varepsilon) \\ &\quad + \left|(F_i x)(t)(V_i x)(t) - (F_i x)(s)(V_i x)(s)\right| \\ &\le \omega^T(m_i,\varepsilon) + \left|(F_i x)(t)(V_i x)(t) - (F_i x)(s)(V_i x)(t)\right| \\ &\quad + \left|(F_i x)(s)(V_i x)(t) - (F_i x)(s)(V_i x)(s)\right| \\ &\le \omega^T(m_i,\varepsilon) + \left|(F_i x)(t) - (F_i x)(s)\right| \left|(V_i x)(t)\right| \\ &\quad + \left|(F_i x)(s)\right| \left|(V_i x)(t) - (V_i x)(s)\right|.\end{aligned} \tag{61}$$

In the similar way we obtain the estimate

$$\begin{aligned}&\left|(F_i x)(t) - (F_i x)(s)\right| \\ &\le \left|f_i(t,x(t)) - f_i(t,x(s))\right| + \left|f_i(t,x(s)) - f_i(s,x(s))\right| \\ &\le k_i |x(t) - x(s)| + \left|f_i(t,x(s)) - f_i(s,x(s))\right| \\ &\le k_i \omega^T(x,\varepsilon) + \omega_{\|x\|}^T(f_i,\varepsilon),\end{aligned} \tag{62}$$

where we denoted

$$\omega_d^T(f_i,\varepsilon) = \sup \left\{ \left|f_i(t,x) - f_i(s,x)\right| : t, s \in [0,T],\ |t-s| \le \varepsilon,\ x \in [-d,d] \right\}. \tag{63}$$

Moreover, we derive the following evaluations:

$$\begin{gathered}\left|\left(V_i x\right)(t)\right| \leq G_i(\|x\|) \int_0^t \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} d\tau \leq G_i(\|x\|) \overline{G}_i, \\ \left|\left(F_i x\right)(s)\right| \leq k_i|x(s)|+\left|f_i(s, 0)\right| \leq k_i r_0+\overline{F}_i,\end{gathered} \tag{64}$$

which hold for an arbitrary $t, s \in \mathbb{R}_+$.

Further, linking (61), (62) with the above obtained evaluation, we arrive at the following estimate:

$$\begin{aligned}&\left|\left(U_i x\right)(t)-\left(U_i x\right)(s)\right| \\ &\quad \leq \omega^T\left(m_i, \varepsilon\right) \\ &\qquad+\left[k_i \omega^T(x, \varepsilon)+\omega_{\|x\|}^T\left(f_i, \varepsilon\right)\right] G_i(\|x\|) \overline{G}_i \\ &\qquad+\left[k_i r_0+\overline{F}_i\right]\left[\omega_{\|x\|}^T\left(u_i, \varepsilon\right) \frac{T^{1-\alpha_i}}{1-\alpha_i}+2 G_i(\|x\|) \overline{g}_i \frac{\varepsilon^{1-\alpha_i}}{1-\alpha_i}\right].\end{aligned} \tag{65}$$

Observe that the terms $\omega_{\|x\|}^T(f_i, \varepsilon)$ and $\omega_{\|x\|}^T(u_i, \varepsilon)$ tend to zero as $\varepsilon \rightarrow 0$ since the functions f_i and u_i are uniformly continuous on the set $[0, T] \times[-\|x\|,\|x\|]$ and $[0, T] \times[0, T] \times [-\|x\|,\|x\|]$, respectively. Hence we obtain

$$\omega_0^T\left(U_i X\right) \leq k_i \overline{G}_i G_i\left(r_0\right) \omega_0^T(X) \tag{66}$$

and consequently

$$\omega_0^{\infty}\left(U_i X\right) \leq k_i \overline{G}_i G_i\left(r_0\right) \omega_0^{\infty}(X). \tag{67}$$

In what follows, let us choose arbitrarily $x, y \in X$ and $t \in \mathbb{R}_+$. Then, based on our assumptions, we obtain

$$\begin{aligned}&\left|\left(U_i x\right)(t)-\left(U_i y\right)(t)\right| \\ &\quad \leq\left|f_i(t, x(t))-f_i(t, y(t))\right|\left|\int_0^t \frac{u_i(t, \tau, x(\tau))}{(t-\tau)^{\alpha_i}} d\tau\right| \\ &\qquad+f_i(t, y(t)) \int_0^t \frac{\left|u_i(t, \tau, x(\tau))-u_i(t, \tau, y(\tau))\right|}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad \leq k_i|x(t)-y(t)| G_i(\|x\|) \int_0^t \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} d\tau \\ &\qquad+\left[k_i y(t)+f_i(t, 0)\right] \\ &\qquad \times \int_0^t \frac{g_i(t, \tau)\left[G_i(x(\tau))-G_i(y(\tau))\right]}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad \leq k_i|x(t)-y(t)| G_i\left(r_0\right) \int_0^t \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} d\tau \\ &\qquad+\left[k_i r_0+\overline{F}_i\right] 2 G_i\left(r_0\right) \int_0^t \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} d\tau.\end{aligned} \tag{68}$$

Hence, keeping in mind assumption (vi), we derive the following equality:

$$\limsup_{t \rightarrow \infty} \operatorname{diam}\left(U_i X\right)(t)=0. \tag{69}$$

Now, we show that U_i is continuous on the set Ω_{r_0}. To this end fix $\varepsilon>0$ and take $x, y \in \Omega_{r_0}$ such that $\|x-y\| \leq \varepsilon$. In view of (69) we know that we may find a number $T>0$ such that for arbitrary $t \geq T$ we get $|(U_i x)(t)-(U_i y)(t)| \leq \varepsilon$. On the other hand, if we take $t \in[0, T]$, we derive the following estimate:

$$\begin{aligned}&\left|\left(U_i x\right)(t)-\left(U_i y\right)(t)\right| \\ &\quad \leq\left|f_i(t, x(t))-f_i(t, y(t))\right|\left|\int_0^t \frac{u_i(t, \tau, x(\tau))}{(t-\tau)^{\alpha_i}} d\tau\right| \\ &\qquad+f_i(t, x(t)) \int_0^t \frac{\left|u_i(t, \tau, x(\tau))-u_i(t, \tau, y(\tau))\right|}{(t-\tau)^{\alpha_i}} d\tau \\ &\quad \leq k_i|x(t)-y(t)| G_i(\|x\|) \int_0^t \frac{g_i(t, \tau)}{(t-\tau)^{\alpha_i}} d\tau \\ &\qquad+\left[k_i|y(t)|+f_i(t, 0)\right] \omega_{r_0}^T\left(u_i, \varepsilon\right) \int_0^t \frac{d\tau}{(t-\tau)^{\alpha_i}} \\ &\quad \leq k \varepsilon \overline{G}_i G_i\left(r_0\right)+\left(k r_0+\overline{F}\right) \frac{T^{1-\alpha_i}}{1-\alpha_i} \omega_{r_0}^T\left(u_i, \varepsilon\right),\end{aligned} \tag{70}$$

where we denoted

$$\begin{aligned}\omega_d^T\left(u_i, \varepsilon\right)= & \sup \left\{\left|u_i(t, s, x)-u_i(t, s, y)\right|:\right. \\ & \left.t, s \in[0, T], x, y \in[-d, d],|x-y| \leq \varepsilon\right\}.\end{aligned} \tag{71}$$

In view of the uniform continuity of the function u_i on the set $[0, T] \times[0, T] \times\left[-r_0, r_0\right]$ we have that $\omega_{r_0}^T(v, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This yields that we can find $T>0$ such that last term in the above estimate is sufficiently small for $t \geq T$ and $i=1,2$.

Next, fix arbitrarily $T>0$, and choose t, s such that $t> s \geq T$. Then, for an arbitrary $x \in X$ we obtain

$$\begin{aligned}&\left|\left(U_i x\right)(t)-\left(U_i x\right)(s)\right|-\left[\left(U_i x\right)(t)-\left(U_i x\right)(s)\right] \\ &\quad \leq\left|m_i(t)-m_i(s)\right|-\left[m_i(t)-m_i(s)\right] \\ &\qquad+\left|\left(F_i x\right)(t)\left(V_i x\right)(t)-\left(F_i x\right)(s)\left(V_i x\right)(t)\right| \\ &\qquad+\left|\left(F_i x\right)(s)\left(V_i x\right)(t)-\left(F_i x\right)(s)\left(V_i x\right)(s)\right| \\ &\qquad-\left[\left(F_i x\right)(t)\left(V_i x\right)(t)-\left(F_i x\right)(s)\left(V_i x\right)(t)\right] \\ &\qquad-\left[\left(F_i x\right)(s)\left(V_i x\right)(t)-\left(F_i x\right)(s)\left(V_i x\right)(s)\right] \\ &\quad \leq d_T\left(m_i\right)+d_T\left(F_i x\right)\left(V_i x\right)(t) \\ &\qquad+\left(F_i x\right)(s)\left\{\left|\left(V_i x\right)(t)-\left(V_i x\right)(s)\right|\right. \\ &\qquad\qquad \left.-\left[\left(V_i x\right)(t)-\left(V_i x\right)(s)\right]\right\}.\end{aligned} \tag{72}$$

On the other hand we get

$$\begin{aligned}&\left|\left(V_i x\right)(t)-\left(V_i x\right)(s)\right|-\left[\left(V_i x\right)(t)-\left(V_i x\right)(s)\right] \\ &\quad \leq\left|\int_0^t \frac{g_i(t, \tau) G_i(x(\tau))}{(t-\tau)^{\alpha_i}} d\tau-\int_0^t \frac{g_i(s, \tau) G_i(x(\tau))}{(s-\tau)^{\alpha_i}} d\tau\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_s^t \frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}d\tau\right| \\
&-\left[\int_0^t \frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\int_0^t \frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}d\tau\right] \\
&-\int_s^t \frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}d\tau \\
\le &\int_0^t \left|\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}\right|d\tau \\
&-\int_0^t \left[\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}\right]d\tau \\
\le &G_i(\|x\|)\int_0^t \left\{\left|\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right|\right. \\
&\qquad\left.-\left[\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right]\right\}d\tau.
\end{aligned}
\tag{73}
$$

Now, taking into account assumptions (i), (v), and (vii) and estimate (72), we obtain

$$d_\infty(U_ix)\le d_\infty(F_ix)G_i(r_0)\overline{G}_i, \tag{74}$$

for $i=1,2$. Hence, in view of Lemma 8, we derive the following inequality:

$$d_\infty(U_ix)\le k_i\overline{G}_iG_i(r_0)d_\infty(x) \tag{75}$$

($i=1,2$). Further, combining the above inequality and (62), (67), and (72), we obtain

$$\mu_d(U_iX)\le k_i\overline{G}_iG_i(r_0)\mu_d(X) \tag{76}$$

for $i=1,2$.

Next, applying Theorem 4 we derive that the operator $S=U_1U_2$ is a contraction with respect to the measure of noncompactness μ_d with the constant L given by the formula

$$
\begin{aligned}
L=&mk\left(\overline{G}_1G_1(r_0)+\overline{G}_2G_2(r_0)\right)\\
&+2k\overline{F}\,\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0)\\
&+2k^2r_0\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0).
\end{aligned}
\tag{77}
$$

Observe that assumption (viii) implies that $L<1$. Thus, in view of Theorem 4 we infer that the operator S has at least one fixed point $x=x(t)$ belonging to the set Ω_{r_0}. Moreover, in view of Remark 5 we conclude that x is nonnegative on $\mathbb{R}_+$, asymptotically stable, and ultimately nondecreasing.

This completes the proof. □

Now we provide an example illustrating Theorem 10.

Example 11. Consider the quadratic fractional integral equation having form of (46) with the operators U_1,U_2 defined by the following formulas

$$
\begin{aligned}
(U_1x)(t)&=\frac{t}{3t+1}+\arctan\left(t^2+x(t)\right)\int_0^t\frac{e^{-(t+s)}\sqrt{|x(t)|}}{(t-s)^{1/3}}ds,\\
(U_2x)(t)&=\frac{1-e^{-t}}{4}+\frac{1}{2}\ln(x(t)+1)\\
&\quad\times\int_0^t\frac{8x^4(t)}{5(t+s+2)^3(t-s)^{1/5}}ds
\end{aligned}
\tag{78}
$$

for $t\in\mathbb{R}_+$.

Observe that in this case the functions involved in (46) have the form

$$
\begin{gathered}
m_1(t)=\frac{t}{3t+1},\qquad m_2(t)=\frac{1-e^{-t}}{4},\\
f_1(t,x)=\arctan\left(t^2+x\right),\qquad f_2(t,x)=\frac{1}{2}\ln(x+1),\\
v_1(t,s,x)=e^{-(t+s)}\sqrt{|x|},\\
v_2(t,s,x)=\frac{8x^4}{5(t+s+2)^3}.
\end{gathered}
\tag{79}
$$

Moreover, $\alpha_1=1/3$, $\alpha_2=1/5$.

It is easy to check that for the above functions there are satisfied assumptions of Theorem 10. Indeed, we have that the function $m_i=m_i(t)$ is nonnegative, bounded, p and con-tin-u-ous on $\mathbb{R}_+$ ($i=1,2$). Since m_1 and m_2 are increasing on $\mathbb{R}_+$ we derive that they are also ultimately nondecreasing on $\mathbb{R}_+$. Moreover, $\|m_1\|=1/3$ and $\|m_2\|=1/4$. Thus, there is satisfied assumption (i). Further notice that the functions f_i ($i=1,2$) transform continuously the set $\mathbb{R}_+\times\mathbb{R}_+$ into $\mathbb{R}_+$. Moreover, f_1 is nondecreasing with respect to both variables and satisfies the Lipschitz condition (with respect to the second variable) with the constant $k_1=1$. Similarly, the function $f_2=f_2(t,x)$ is increasing with respect to x and satisfies the Lipschitz condition with the constant $k_2=1/2$. Apart from this it is easily seen that $\overline{F}_1=\pi/2$, $\overline{F}_2=0$.

Summing up, we see that functions f_1 and f_2 satisfy assumptions (ii) and (iii).

Next, let us note that the function $v_i(t,s,x)$ is continuous on the set $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}$ and transforms the set $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$ into $\mathbb{R}_+$ for $i=1,2$. Apart from this the function v_i can be represented in the form $v_i(t,s,x)=g_i(t,s)G_i(|x|)$ ($i=1,2$), where $g_1(t,s)=e^{-(t+s)}$, $G_1(x)=\sqrt{|x|}$, $g_2(t,s)=8/5(t+s+2)^3$, and $G_2(x)=x^4$. It is easily seen that assumptions (iv) and (v) are satisfied for the functions v_1 and v_2.

Further on, we have

$$
\begin{aligned}
\int_0^t\frac{g_1(t,s)}{(t-s)^{\alpha_1}}ds&=\int_0^t\frac{e^{-t-s}}{(t-s)^{1/3}}ds\\
&\le e^{-t}\int_0^t\frac{ds}{(t-s)^{1/3}}=\frac{3}{2}e^{-t}t^{2/3}.
\end{aligned}
\tag{80}
$$

Hence we see that

$$\lim_{t\to\infty}\int_0^t \frac{g_1(t,s)}{(t-s)^{\alpha_1}}ds = 0. \tag{81}$$

Moreover, we get

$$\begin{aligned}\int_0^t \frac{g_2(t,s)}{(t-s)^{\alpha_2}}ds &= \int_0^t \frac{8}{5(t+s+2)^3(t-s)^{1/5}}ds \\ &\le \frac{8}{5(t+2)^3}\int_0^t \frac{ds}{(t-s)^{1/5}} \\ &= \frac{2t^{4/5}}{(t+2)^3}.\end{aligned} \tag{82}$$

Thus, we have

$$\lim_{t\to\infty}\int_0^t \frac{g_2(t,s)}{(t-s)^{\alpha_2}}ds = 0. \tag{83}$$

This shows that assumption (vi) is satisfied.

In order to show that the function $g_i(t,s)$ satisfies assumption (vii) let us fix arbitrarily $T > 0$. Then, for $T \le s < t$ we obtain

$$\begin{aligned}&\int_0^t \left\{\left|\frac{g_1(t,\tau)}{(t-\tau)^{\alpha_1}} - \frac{g_1(s,\tau)}{(s-\tau)^{\alpha_1}}\right| - \left[\frac{g_1(t,\tau)}{(t-\tau)^{\alpha_1}} - \frac{g_1(s,\tau)}{(s-\tau)^{\alpha_1}}\right]\right\} d\tau \\ &= \int_0^t \left\{\left|\frac{e^{-(t+\tau)}}{(t-\tau)^{1/3}} - \frac{e^{-(s+\tau)}}{(s-\tau)^{1/3}}\right| - \left[\frac{e^{-(t+\tau)}}{(t-\tau)^{1/3}} - \frac{e^{-(s+\tau)}}{(s-\tau)^{1/3}}\right]\right\} d\tau \\ &= 2\int_0^t \left[\frac{e^{-(s+\tau)}}{(s-\tau)^{1/3}} - \frac{e^{-(t+\tau)}}{(t-\tau)^{1/3}}\right] d\tau \\ &\le 2e^{-s}\int_0^t \frac{d\tau}{(s-\tau)^{1/3}} - 2e^{-2t}\int_0^t \frac{d\tau}{(t-\tau)^{1/3}} \\ &= 3e^{-s}\left(\sqrt[3]{s^2} - \sqrt[3]{(s-t)^2}\right) - 3e^{-2t}\sqrt[3]{t^2} \le 3e^{-s}\sqrt[3]{s^2}.\end{aligned} \tag{84}$$

In the similar way, we get

$$\begin{aligned}&\int_0^t \left\{\left|\frac{g_2(t,\tau)}{(t-\tau)^{\alpha_2}} - \frac{g_2(s,\tau)}{(s-\tau)^{\alpha_2}}\right| - \left[\frac{g_2(t,\tau)}{(t-\tau)^{\alpha_2}} - \frac{g_2(s,\tau)}{(s-\tau)^{\alpha_2}}\right]\right\} d\tau \\ &= \int_0^t \left\{\left|\frac{8}{5(t+\tau+2)^3(t-\tau)^{1/5}} - \frac{8}{5(s+\tau+2)^3(s-\tau)^{1/5}}\right| - \left[\frac{8}{5(t+\tau+2)^3(t-\tau)^{1/5}} - \frac{8}{5(s+\tau+2)^3(s-\tau)^{1/5}}\right]\right\} d\tau \\ &= 2\int_0^t \left[\frac{8}{5(s+\tau+2)^3(s-\tau)^{1/5}} - \frac{8}{5(t+\tau+2)^3(t-\tau)^{1/5}}\right] d\tau \\ &\le \frac{16}{5(s+2)^3}\int_0^t \frac{d\tau}{(s-\tau)^{1/5}} - \frac{16}{5(2t+2)^3}\int_0^t \frac{d\tau}{(t-\tau)^{1/5}} \\ &= \frac{4}{(s+2)^3}\left(\sqrt[5]{s^4} - \sqrt[5]{(s-t)^4}\right) - \frac{4}{(2t+2)^3}\sqrt[5]{t^4} \\ &\le \frac{4}{(s+2)^3}\sqrt[5]{s^4}.\end{aligned} \tag{85}$$

In view of the above obtained estimates we conclude that assumption (vii) is also satisfied.

Finally, let us notice that taking into account the above established facts we have that $m = \max\{\|m_1\|, \|m_2\|\} = 1/3$, $k = \max\{k_1, k_2\} = 1$, and $\overline{F} = \max\{\overline{F}_1, \overline{F}_2\} = \pi/2$. Thus, the first inequality from assumption (viii) has the form

$$\left[\frac{1}{3} + \overline{G}_1\left(r\sqrt{r} + \frac{\pi}{2}r\right)\right]\left[\frac{1}{3} + \overline{G}_2\left(r^5 + \frac{\pi}{2}r^4\right)\right] \le r. \tag{86}$$

It can be shown that the number $r_0 = 1/2$ is a solution of the above inequality such that it satisfies also the second inequality from assumption (viii).

Applying Theorem 10 we infer that the quadratic fractional integral equation considered in this example has a solution belonging to the set

$$\Omega_{1/2} = \left\{x \in BC(\mathbb{R}_+) :\ 0 \le x(t) \le \frac{1}{2} \text{ for } t \in \mathbb{R}_+\right\}, \tag{87}$$

which is asymptotically stable and ultimately nondecreasing.

6. The Existence of Solutions Having Limits at Infinity of an Integral Equation of Mixed Type in the Banach Algebra $BC(\mathbb{R}_+)$

In this final section we are going to investigate the following integral equation of mixed type:

$$x(t) = (Vx)(t)(Ux)(t), \quad t \in \mathbb{R}_+, \tag{88}$$

where V denotes the nonlinear Volterra integral operator of the form

$$(Vx)(t) = p_1(t) + f_1(t, x(t))\int_0^t v(t,s,x(s))\,ds, \tag{89}$$

while U is the Urysohn integral operator having the form

$$(Ux)(t) = p_2(t) + f_2(t, x(t))\int_0^\infty u(t,s,x(s))\,ds. \tag{90}$$

Equation (88) will be considered in the Banach algebra $BC(\mathbb{R}_+)$. The existence result concerning (88), which we intend to present here, creates an extension of a result obtained in [2].

In our considerations we will use the measure of noncompactness μ_a defined in Section 3. The use of that measure enables us to obtain a result on the existence of solutions of (88) having (finite) limits at infinity.

In what follows we will study (88) under the below formulated assumptions.

(i) $p_i \in BC(\mathbb{R}_+)$ and $p_i(t) \to 0$ as $t \to \infty$ $(i = 1, 2)$.

(ii) $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and such that $f_i(t, 0) \to 0$ as $t \to \infty$, for $i = 1, 2$.

(iii) The functions f_i $(i = 1, 2)$ satisfy the Lipschitz condition with respect to the second variable; that is, there exists a constant $k_i > 0$ such that

$$\left|f_i(t, x) - f_i(t, y)\right| \le k_i |x - y| \tag{91}$$

for $x, y \in \mathbb{R}$ and for $t \in \mathbb{R}_+$ $(i = 1, 2)$.

(iv) $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous, function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|v(t, s, x)| \le g(t, s) G(|x|) \tag{92}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(v) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous, function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t, s, x)| \le h(t, s) H(|x|) \tag{93}$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(vi) The function $t \to \int_0^t g(t, s)ds$ is bounded on $\mathbb{R}_+$.

(vii) For each $t \in \mathbb{R}_+$ the function $s \to h(t, s)$ is integrable on $\mathbb{R}_+$ and the function $t \to \int_0^\infty h(t, s)ds$ is bounded on $\mathbb{R}_+$.

(viii) The improper integral $\int_0^\infty h(t, s)ds$ is uniformly convergent with respect to $\mathbb{R}_+$; that is,

$$\lim_{T\to\infty} \left\{\sup_{t\in\mathbb{R}_+} \int_T^\infty h(t, s)\, ds\right\} = 0. \tag{94}$$

(ix) There exists a positive solution r_0 of the inequality

$$\begin{aligned}&\left[p + k\overline{G}rG(r) + \overline{F}\ \overline{G}G(r)\right]\\ &\quad\times\left[p + k\overline{H}rH(r) + \overline{F}\ \overline{H}H(r)\right] \le r\end{aligned} \tag{95}$$

such that

$$\begin{aligned}&pk\left(\overline{G}G(r_0) + \overline{H}H(r_0)\right)\\ &\quad + 2k\overline{F}\ \overline{G}\ \overline{H}G(r_0) H(r_0)\\ &\quad + 2k^2 r_0\overline{G}\ \overline{H}G(r_0) H(r_0) < 1,\end{aligned} \tag{96}$$

where the constants involved in the above inequalities are defined as follows:

$$\begin{aligned}\overline{G} &= \sup\left\{\int_0^t g(t, s)\, ds :\ t \in \mathbb{R}_+\right\},\\ \overline{H} &= \sup\left\{\int_0^\infty h(t, s)\, ds :\ t \in \mathbb{R}_+\right\},\\ \overline{F}_i &= \sup\left\{\left|f_i(t, 0)\right| :\ t \in \mathbb{R}_+\right\} \text{ for } i = 1, 2,\\ p &= \max\left\{\|p_1\|, \|p_2\|\right\},\\ \overline{F} &= \max\left\{\overline{F}_1, \overline{F}_2\right\},\\ k &= \max\left\{k_1, k_2\right\}.\end{aligned} \tag{97}$$

It is worthwhile mentioning that a result obtained in the paper [2] asserts that under assumptions (i)–(ix) (88) has at least one solution in the Banach algebra $BC(\mathbb{R}_+)$ such that $x(t) \to 0$ as $t \to \infty$. We generalize that result showing the existence of solutions of (88) which have finite limits at infinity. To this end we will need the following additional hypotheses.

(x) The following conditions hold

$$\begin{aligned}&\lim_{t\to\infty} \int_0^t g_i(t, s)\, ds = 0,\\ &\lim_{t\to\infty} \int_0^\infty h_i(t, s)\, ds = 0\end{aligned} \tag{98}$$

for $i = 1, 2$.

(xi) f_i is bounded function, and for each $x \in \mathbb{R}$ there exists a finite limit $\lim_{t\to\infty} f_i(t, x)$ $(i = 1, 2)$.

Remark 12. Observe that assuming additionally hypothesis (xi) we can dispense with a certain part of assumption (ii).

Now, we can formulate the main result of this section.

Theorem 13. *Under assumptions* (i)–(xi) (88) *has at least one solution* $x = x(t)$ *belonging to the Banach algebra* $BC(\mathbb{R}_+)$ *and such that there exists a finite limit* $\lim_{t\to\infty} x(t)$.

Proof. Observe that based on the paper [2] we infer that operators V and U defined earlier transform the Banach algebra $BC(\mathbb{R}_+)$ into itself. Moreover, recalling [2] again, we have

$$\begin{aligned}\|Vx\| &\le p + k\overline{G}\,\|x\|\, G(\|x\|) + \overline{F}\ \overline{G}G(\|x\|),\\ \|Ux\| &\le p + k\overline{H}\,\|x\|\, H(\|x\|) + \overline{F}\ \overline{H}H(\|x\|).\end{aligned} \tag{99}$$

In virtue of the above estimates and assumption (ix) we deduce that there exists a number $r_0 > 0$ such that the operator $W = VU$ maps the ball B_{r_0} into itself. Moreover, from (99) we derive the following estimates:

$$\begin{aligned}\left\|VB_{r_0}\right\| &\le p + k\overline{G}r_0G(r_0) + \overline{F}\ \overline{G}G(r_0),\\ \left\|UB_{r_0}\right\| &\le p + k\overline{H}r_0H(r_0) + \overline{F}\ \overline{H}H(r_0).\end{aligned} \tag{100}$$

Further, let us take an arbitrary nonempty subset X of the ball B_{r_0}. Then, using some estimates proved in [2] we have

$$\omega_0^{\infty}(VX) \leq k\overline{G}G(r_0)\,\omega_0^{\infty}(X), \tag{101}$$

$$\omega_0^{\infty}(UX) \leq k\overline{H}H(r_0)\,\omega_0^{\infty}(X). \tag{102}$$

Now, let us fix $T > 0$ and take arbitrary numbers $t, s \geq T$. Then we obtain

$$\begin{aligned}
&|(Vx)(t) - (Vx)(s)| \\
&\quad \leq |p_1(t)| + |p_1(s)| + |f_1(t, x(t)) - f_1(s, x(s))| \\
&\qquad \times \int_0^t |v(t, \tau, x(\tau))|\, d\tau + |f_1(s, x(s))| \\
&\qquad \times \left| \int_0^t v(t, \tau, x(\tau))\, d\tau - \int_0^s v(s, \tau, x(\tau))\, d\tau \right| \\
&\quad \leq |p_1(t)| + |p_1(s)| \\
&\qquad + \left[k_1 |x(t) - x(s)| + |f_1(t, x(s)) - f_1(s, x(s))|\right] \\
&\qquad \times \int_0^t |v(t, \tau, x(\tau))|\, d\tau \\
&\qquad + \left[k_1 |x(s)| + f_1(s, 0)\right] \\
&\qquad \times \left[\int_0^t |v(t, \tau, x(\tau))|\, d\tau + \int_0^s |v(s, \tau, x(\tau))|\, d\tau \right] \\
&\quad \leq |p_1(t)| + |p_1(s)| \\
&\qquad + \left[k_1 |x(t) - x(s)| + |f_1(t, x(s)) - f_1(s, x(s))|\right] \\
&\qquad \times G(\|x\|) \int_0^t g(t, \tau)\, d\tau \\
&\qquad + \left[k_1 \|x\| + |f_1(s, 0)|\right] G(\|x\|) \\
&\qquad \times \left[\int_0^t g(t, \tau)\, d\tau + \int_0^s g(s, \tau)\, d\tau \right].
\end{aligned} \tag{103}$$

Hence, we derive the following inequality:

$$\begin{aligned}
&|(Vx)(t) - (Vx)(s)| \\
&\quad \leq \sup\{|p_1(t)| + |p_1(s)| : t, s \geq T\} \\
&\qquad + k_1 \overline{G}G(r_0) \sup\{|x(t) - x(s)| : t, s \geq T\} \\
&\qquad + \overline{G}G(r_0) \sup\{|f_1(t, x(s)) - f_1(s, x(s))| : t, s \geq T\} \\
&\qquad + \left[k_1 r_0 + \overline{F_1}\right] G(r_0) \sup\left\{ \int_0^t g(t, \tau)\, d\tau \right. \\
&\qquad\qquad \left. + \int_0^s g(s, \tau)\, d\tau : t, s \geq T \right\}.
\end{aligned} \tag{104}$$

Combining the above inequality with assumptions (i), (vi), (x), and (xi), we obtain

$$a(VX) \leq k\overline{G}G(r_0)\, a(X). \tag{105}$$

Similarly we can show, based on assumptions (i), (ii), (v), (vii), (viii), (x), and (xi), that the following inequality holds:

$$a(UX) \leq k\overline{H}H(r_0)\, a(X). \tag{106}$$

Further, let us observe that from (101) and (105) we derive

$$\mu_a(VX) \leq k\overline{G}G(r_0)\, \mu_a(X). \tag{107}$$

In the same way, linking (102) and (106), we get

$$\mu_a(UX) \leq k\overline{H}H(r_0)\, \mu_a(X). \tag{108}$$

Finally, taking into account estimates (100), (107), and (108), assumption (ix), and Theorem 4 we deduce that operator $W = VU$ has at least one fixed point $x = x(t)$ in the ball B_{r_0} being a subset of the Banach algebra $BC(\mathbb{R}_+)$. It is clear that the function x is a solution of (88). Moreover, from Remark 5 and the description of the kernel $\ker \mu_a$ we infer that x has a finite limit at infinity.

The proof is complete. □

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