Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 540108, 10 pages
http://dx.doi.org/10.1155/2013/540108

Research Article

Strong Convergence of Iterative Algorithm for a New System of Generalized $H(\cdot,\cdot)-\eta$-Cocoercive Operator Inclusions in Banach Spaces

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Received 10 September 2013; Revised 6 November 2013; Accepted 20 November 2013

Academic Editor: Mohammad Mursaleen

We introduce and study a new system of generalized $H(\cdot,\cdot)-\eta$-cocoercive operator inclusions in Banach spaces. Using the resolvent operator technique associated with $H(\cdot,\cdot)-\eta$-cocoercive operators, we suggest and analyze a new generalized algorithm of nonlinear set-valued variational inclusions and establish strong convergence of iterative sequences produced by the method. We highlight the applicability of our results by examples in function spaces.

1. Introduction

The resolvent operator technique is a powerful tool to study the approximation solvability of nonlinear variational inequalities and variational inclusions, which have been applied widely to optimization and control, mechanics and physics, economics and transportation equilibrium, and engineering sciences, see, for example, [1–4] and the references therein.

In a series of papers [5–8], the authors investigated (A,η)-accretive and $H(\cdot,\cdot)$-accretive operators for solving variational inclusions in Banach spaces. Convergence and stability of iterative algorithms for the systems of (A,η)-accretive operators have been studied in [9, 10]. The notion of $(H,\phi)-\eta$-monotone operators has been introduced and investigated by the authors in [11]. Generalized mixed variational inclusions involving $(H(\cdot,\cdot),\eta)$-monotone operators have been discussed in [12]. Some results on $H((\cdot,\cdot),\eta)$-accretive operators and application for solving set-valued variational inclusions in Banach spaces have been proved in [7]. Some other related articles on the variational inclusion problems can be found in [13–22].

Very recently, Ahmad et al. [23] introduced a new $H(\cdot,\cdot)-\eta$-cocoercive operator and its resolvent operator in the setting of Banach spaces. The authors proposed concrete examples in support of $H(\cdot,\cdot)-\eta$-cocoercive operators and they also proved the Lipschitz continuity of resolvent operator associated with $H(\cdot,\cdot)-\eta$-cocoercive operator. Motivated and inspired by the research works mentioned above, in this paper, we introduce and study a new system of $H(\cdot,\cdot)-\eta$-cocoercive mapping inclusions in Banach spaces. Using the resolvent operator associated with $H(\cdot,\cdot)-\eta$-cocoercive mapping, we suggest and analyze a new general algorithm and establish the existence and uniqueness of solutions for this system of $H(\cdot,\cdot)-\eta$-cocoercive mappings.

2. Preliminaries

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$. Let X be a Banach space with the norm $\|\cdot\|$ and the dual space X^*. For any $x\in X$, we denote the value of $x^*\in X^*$ at x by $\langle x,x^*\rangle$. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x\in X$ by $x_n\to x$ as $n\to\infty$. We denote by 2^X the family of all nonempty subsets of X. Let

$CB(X)$ be the family of all nonempty, closed, and bounded subsets of X. The Hausdörff metric on $CB(X)$ [24] is defined by

$$D(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\}, \quad A,B \in CB(X), \tag{1}$$

where $d(x,B) = \inf_{b\in B}\|x-b\|$ and $d(A,y) = \inf_{a\in A}\|a-y\|$.

Definition 1 (see [25]). A continuous and strictly increasing function $\phi : [0,+\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$ is called a gauge function.

Definition 2 (see [25]). Let X be a Banach space. Given a gauge function ϕ, the mapping $J_\phi : X \to 2^{X^*}$ corresponding to ϕ defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\,\|x^*\|, \quad \|x^*\| = \phi(\|x\|)\}, \quad \forall x \in X, \tag{2}$$

is called the duality mapping with gauge function ϕ.

In particular, if $\phi(t) = t$, the duality map $J = J_\phi$ is called the normalized duality mapping.

Lemma 3 (see [26]). *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{3}$$

for all $j(x+y) \in J(x+y)$.

Definition 4. Let X be a Banach space. Let $A : X \to X$ and $\eta : X\times X \to X$ be two mappings and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, A is called

(i) η-cocoercive, if there exists a constant $\mu_1 > 0$ such that

$$\langle Ax - Ay, j(\eta(x,y))\rangle \ge \mu_1\|Ax-Ay\|^2, \quad \forall x,y \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{4}$$

(ii) η-accretive, if

$$\langle Ax - Ay, j(\eta(x,y))\rangle \ge 0, \quad \forall x,y \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{5}$$

(iii) η-strongly accretive, if there exists a constant $\beta_1 > 0$ such that

$$\langle Ax - Ay, j(\eta(x,y))\rangle \ge \beta_1\|x-y\|^2, \quad \forall x,y \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{6}$$

(iv) η-relaxed cocoercive, if there exists a constant $\gamma_1 > 0$ such that

$$\langle Ax - Ay, j(\eta(x,y))\rangle \ge (-\gamma_1)\|Ax-Ay\|^2, \quad \forall x,y \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{7}$$

(v) Lipschitz continuous, if there exists a constant $\lambda_A > 0$ such that

$$\|Ax - Ay\| \le \lambda_A\|x-y\|, \quad \forall x,y \in X, \tag{8}$$

(vi) α-expansive, if there exists a constant $\alpha > 0$ such that

$$\|Ax - Ay\| \ge \alpha\|x-y\|, \quad \forall x,y \in X, \tag{9}$$

(vii) η is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \le \tau\|x-y\|, \quad \forall x,y \in X. \tag{10}$$

Definition 5. Let X be a Banach space. Let $A, B : X \to X$, $H : X\times X \to X$, $\eta : X\times X \to X$ be four single-valued mappings and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then,

(i) $H(A,\cdot)$ is said to be η-cocoercive with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), j(\eta(x,y))\rangle \ge \mu\|Ax-Ay\|^2, \quad \forall x,y,u \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{11}$$

(ii) $H(\cdot,B)$ is said to be η-relaxed cocoercive with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u,Bx) - H(u,By), j(\eta(x,y))\rangle \ge (-\gamma)\|Bx-By\|^2, \quad \forall x,y,u \in X, \quad j(\eta(x,y)) \in J(\eta(x,y)), \tag{12}$$

(iii) $H(A,\cdot)$ is said to be r_1-Lipschitz continuous with respect to A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \le r_1\|x-y\|, \quad \forall x,y,u \in X, \tag{13}$$

(iv) $H(\cdot,B)$ is said to be r_2-Lipschitz continuous with respect to B, if there exists a constant $r_2 > 0$ such that

$$\|H(u,Bx) - H(u,By)\| \le r_2\|x-y\|, \quad \forall x,y,u \in X. \tag{14}$$

Definition 6. Let X be a Banach space. A set-valued mapping $M : X \to 2^X$ is said to be η-cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle u - v, j(\eta(x,y))\rangle \ge \mu_2\|u-v\|^2, \quad \forall x,y \in X, \quad u \in M(x), \quad v \in M(y), \quad j(\eta(x,y)) \in J(\eta(x,y)). \tag{15}$$

Definition 7. Let X be a Banach space. A mapping $T : X \to CB(X)$ is said to be $\mathscr{D}$-Lipschitz continuous, if there exists a constant $\lambda_T > 0$ such that

$$\mathscr{D}(Tx,Ty) \le \lambda_T\|x-y\|, \quad x,y \in X. \tag{16}$$

Definition 8. Let X be a Banach space. Let $T, Q : X \to CB(X)$ be the mappings. A mapping $N : X \times X \to X$ is said to be

(i) Lipschitz continuous in the first argument with respect to T, if there exists a constant $t_1 > 0$ such that

$$\|N(w_1, \cdot) - N(w_2, \cdot)\| \le t_1 \|w_1 - w_2\|, \quad \forall u_1, u_2 \in X, \quad w_1 \in T(u_1), \quad w_2 \in T(u_2), \tag{17}$$

(ii) Lipschitz continuous in the second argument with respect to Q, if there exists a constant $t_2 > 0$ such that

$$\|N(\cdot, v_1) - N(\cdot, v_2)\| \le t_2 \|v_1 - v_2\|, \quad \forall u_1, u_2 \in X, \quad v_1 \in Q(u_1), \quad v_2 \in Q(u_2), \tag{18}$$

(iii) η-relaxed Lipschitz in the first argument with respect to T, if there exists a constant $\tau_1 > 0$ such that

$$\begin{aligned} &\langle N(w_1, \cdot) - N(w_2, \cdot), j(\eta(u_1, u_2))\rangle \\ &\quad \le (-\tau_1)\|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in X, \\ &\quad w_1 \in T(u_1), \quad w_2 \in T(u_2), \\ &\quad j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2)), \end{aligned} \tag{19}$$

(iv) η-relaxed Lipschitz in the second argument with respect to Q, if there exists a constant $\tau_2 > 0$ such that

$$\begin{aligned} &\langle N(\cdot, v_1) - N(\cdot, v_2), j(\eta(u_1, u_2))\rangle \\ &\quad \le (-\tau_2)\|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in X, \ v_1 \in Q(u_1), \\ &\quad v_2 \in Q(u_2), \quad j(\eta(u_1, u_2)) \in J(\eta(u_1, u_2)). \end{aligned} \tag{20}$$

Definition 9. Let X be a Banach space. Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be four single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. M is said to be $H(\cdot,\cdot) - \eta$-cocoercive operator with respect to A and B, if M is η-cocoercive and $(H(A, B) + \lambda M)(X) = X$, for every $\lambda > 0$.

Example 10. Let $X = \mathbb{R} \times \mathbb{R}$ and $A, B : X \to X$ be defined by

$$\begin{aligned} &A(x_1, x_2) = (2x_1 - x_2, x_1 - x_2), \\ &B(y_1, y_2) = (-2y_2, y_1 - y_2), \quad \forall (x_1, x_2), (y_1, y_2) \in X. \end{aligned} \tag{21}$$

Assume now that $H(A, B), \eta : X \times X \to X$ are defined by

$$H(Ax, By) = Ax + By, \quad \eta(x, y) = x - y, \quad \forall x, y \in X. \tag{22}$$

Let $M = I$, where I is the identity mapping. Then, M is $H(\cdot,\cdot) - \eta$-cocoercive with respect to A and B.

Example 11. Let $X = C[0, 1]$, the space of all real valued continuous functions defined on closed interval $[0, 1]$ with the norm

$$\|f\| = \max_{t \in [0,1]} |f(t)|. \tag{23}$$

Let $A, B : X \to X$ be defined by

$$A(f) = \sin^2(f), \quad B(g) = \cos^2(g), \quad \forall f, g \in X. \tag{24}$$

Let $H(A, B) : X \times X \to X$ be defined by

$$H(A(f), B(g)) = A(f) + B(g), \quad \forall f, g \in X. \tag{25}$$

Suppose that $M(f) = f^2$, where $f^2(t) = f(t)f(t)$ for all $t \in [0, 1]$. Then, for $\lambda = 1$, we conclude that

$$\begin{aligned} \|(H(A, B) + M)(f)\| &= \|A(f) + B(f) + f^2\| \\ &= \max_{t \in [0,1]} \left|\sin^2(f(t)) + \cos^2(f(t)) + f^2(t)\right| \\ &= 1 + f^2(t) > 0. \end{aligned} \tag{26}$$

This proves that $0 \notin (H(A, B) + M)(X)$ and M is not $H(A, B) - \eta$-cocoercive with respect to A and B.

Proposition 12 (see [23]). *Let $H(A, B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A be α-expansive and B be β-Lipschitz continuous $\mu > \gamma$ and $\alpha > \beta$. Let $M : X \to 2^X$ be $H(A, B) - \eta$-cocoercive operator. Suppose that*

$$\langle x - y, j(\eta(u, v))\rangle \ge 0, \quad \forall (v, y) \in Graph(M), \quad j(\eta(u, v)) \in J(\eta(u, v)). \tag{27}$$

Then, $x \in Mu$, where $Graph(M) = \{(u, x) \in X \times X : x \in Mu\}$.

Theorem 13 (see [23]). *Let $H(A, B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A be α-expansive and B be β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(\cdot,\cdot) - \eta$-cocoercive operator with respect to A and B. Then, for each $\lambda > 0$, the operator $(H(A, B) + \lambda M)^{-1}$ is single-valued.*

Definition 14. Let X be a Banach space. Let $H(A, B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A be α-expansive B be β-Lipschitz continuous and η be β-Lipschitz continuous, $\mu > \gamma$, and $\alpha > \beta$. Let M be a $H(\cdot,\cdot) - \eta$-cocoercive operator with respect to A and B. Then, the resolvent $R_{\lambda,M}^{H(\cdot,\cdot)-\eta} : X \to X$ is defined by

$$R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X. \tag{28}$$

Theorem 15 (see [23]). *Let X be a Banach space. Let $H(A, B)$ be η-cocoercive with respect to A with constant $\mu > 0$ and η-relaxed cocoercive with respect to B with constant $\gamma > 0$, A be α-expansive B be β-Lipschitz continuous, and η be ρ-Lipschitz continuous; $\mu > \gamma$ and $\alpha > \beta$. Let M be $H(\cdot,\cdot) - \eta$-cocoercive*

operator with respect to A and B. Then, the resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta}: X \to X$ *is* $\rho/(\mu\alpha^2-\gamma\beta^2)$*-Lipschitz continuous, that is,*

$$\left\|R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u)-R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\right\| \le \frac{\rho}{\mu\alpha^2-\gamma\beta^2}\|u-v\|, \quad \forall u,v\in X. \tag{29}$$

3. Strong Convergence Theorem

In this section, using the resolvent operator technique associated with $H(\cdot,\cdot)-\eta$-cocoercive operators, we propose a new generalized algorithm of nonlinear set-valued variational inclusions and establish strong convergence of iterative sequences produced by the method.

For $i=1,2$, let X_i be real Banach spaces with the norm $\|\cdot\|_i$. Let $A_i, B_i: X_i \to X_i$, $H_i: X_i\times X_i \to X_i$, $\eta_i: X_i\times X_i \to X_i$, $F: X_1\times X_2 \to X_1$, and $G: X_1\times X_2 \to X_2$ be single-valued mappings, and $T: X_1 \to CB(X_1)$, $Q: X_2 \to CB(X_2)$ be set-valued mappings. Let $M: X_1\times X_1 \to 2^{X_1}$, $N: X_2\times X_2 \to 2^{X_2}$ be $H_1(\cdot,\cdot)-\eta_1$-cocoercive and $H_2(\cdot,\cdot)-\eta_2$-cocoercive operators with respect to (A_1,B_1) and (A_2,B_2), respectively. We consider the following problem.

Find $(x,y)\in X_1\times X_2$, $w\in T(x)$, and $v\in Q(y)$ such that

$$\begin{aligned} &0\in M(x,x)+F(w,v),\\ &0\in N(y,y)+G(w,v). \end{aligned} \tag{30}$$

We call problem (30) a system of generalized $H(\cdot,\cdot)-\eta$-cocoercive operator inclusions.

Under the assumptions mentioned above, we have the following key and simple lemma.

Lemma 16. *$(x,y)\in X_1\times X_2$, $w\in T(x)$, $v\in Q(y)$ is a solution of problem* (30) *if and only if*

$$\begin{aligned} x&=R_{\lambda_1,M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta}\left[H_1(A_1x,B_1x)-\lambda_1F(w,v)\right],\\ y&=R_{\lambda_2,N(\cdot,y)}^{H_2(\cdot,\cdot)-\eta}\left[H_2(A_2y,B_2y)-\lambda_2G(w,v)\right], \end{aligned} \tag{31}$$

where $R_{\lambda_1,M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta}=(H_1(A_1x,B_1x)+\lambda_1M(\cdot,x))^{-1}$, $R_{\lambda_1,N(\cdot,y)}^{H_2(\cdot,\cdot)-\eta}=(H_2(A_2y,B_2y)+\lambda_2N(\cdot,y))^{-1}$, *and* $\lambda_1,\lambda_2>0$ *are constants.*

Proof. This is an easy and direct consequence of Definition 14. □

Algorithm 17. For $i=1,2$, let X_i be real Banach spaces with the norm $\|\cdot\|_i$. Let $A_i,B_i: X_i\to X_i$, $H_i: X_i\times X_i\to X_i$, $\eta_i: X_i\times X_i\to X_i$, $F: X_1\times X_2\to X_1$, and $G: X_1\times X_2\to X_2$ be single-valued mappings, and $T: X_1\to CB(X_1)$, $Q: X_2\to CB(X_2)$ be set-valued mappings. Let $M: X_1\times X_1\to 2^{X_1}$, $N: X_2\times X_2\to 2^{X_2}$ be such that, for each fixed $x\in X_1$, $y\in X_2$, $M(\cdot,x)$ and $N(\cdot,y)$ are $H_1(\cdot,\cdot)-\eta_1$-cocoercive and $H_2(\cdot,\cdot)-\eta_2$-cocoercive operators with respect to (A_1,B_1) and (A_2,B_2), respectively. For any given constants $\lambda_i>0$ $(i=1,2)$, define the mappings $S_1: X_1\times X_2\to X_1$ and $S_2: X_1\times X_2\to X_2$ by

$$\begin{aligned} S_1(x,y)&=\bigcup_{w\in T(x)}\bigcup_{v\in Q(y)}R_{\lambda_1,M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta_1}\left[H_1(A_1x,B_1x)-\lambda_1F(w,v)\right],\\ S_2(x,y)&=\bigcup_{w\in T(x)}\bigcup_{v\in Q(y)}R_{\lambda_2,N(\cdot,y)}^{H_2(\cdot,\cdot)-\eta_2}\left[H_2(A_2y,B_2y)-\lambda_2G(w,v)\right]. \end{aligned} \tag{32}$$

For any given $(x_0,y_0)\in X_1\times X_2$, $w_0\in T(x_0)$, $v_0\in Q(y_0)$, let

$$\begin{aligned} z_0&=R_{\lambda_1,M(\cdot,x_0)}^{H_1(\cdot,\cdot)-\eta_1}\\ &\quad\times\left[H_1(A_1x_0,B_1x_0)-\lambda_1F(w_0,v_0)\right]\in S_1(x_0,y_0),\\ u_0&=R_{\lambda_2,N(\cdot,y_0)}^{H_2(\cdot,\cdot)-\eta_2}\\ &\quad\times\left[H_2(A_2y_0,B_2y_0)-\lambda_2G(w_0,v_0)\right]\in S_2(x_0,y_0). \end{aligned} \tag{33}$$

Since $w_0\in T(x_0)\subset CB(X_1)$ and $v_0\in Q(y_0)\subset CB(X_2)$, in view of Nadler's theorem [24], there exist $w_1\in T(x_1)$ and $v_1\in Q(y_1)$ such that

$$\begin{aligned} \|w_1-w_0\|_1&\le(1+1)D(T(x_1),T(x_0)),\\ \|v_1-v_0\|_2&\le(1+1)D(Q(y_1),Q(y_0)). \end{aligned} \tag{34}$$

By induction, we define iterative sequences $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, and $\{v_n\}$ as follows:

$$\begin{aligned} x_{n+1}&=R_{\lambda_1,M(\cdot,x_n)}^{H_1(\cdot,\cdot)-\eta_1}\\ &\quad\times\left[H_1(A_1x_n,B_1x_n)-\lambda_1F(w_n,v_n)\right]\in S_1(x_n,y_n),\\ y_{n+1}&=R_{\lambda_2,N(\cdot,y_n)}^{H_2(\cdot,\cdot)-\eta_2}\\ &\quad\times\left[H_2(A_2y_n,B_2y_n)-\lambda_2G(w_n,v_n)\right]\in S_2(x_n,y_n),\\ w_n&\in T(x_n),\\ \|w_{n+1}-w_n\|_1&\le\left(1+\frac{1}{n+1}\right)D(T(x_{n+1}),T(x_n)),\\ v_n&\in Q(y_n),\\ \|v_{n+1}-v_n\|_2&\le\left(1+\frac{1}{n+1}\right)D(Q(y_{n+1}),Q(y_n)), \end{aligned} \tag{35}$$

where $n=0,1,2,\ldots$, and $\lambda_1,\lambda_2>0$ are constants.

Theorem 18. *For $i=1,2$, let X_i be real Banach spaces with the norm $\|\cdot\|_i$. Let $A_i,B_i: X_i\to X_i$, $H_i: X_i\times X_i\to X_i$, $\eta_i: X_i\times X_i\to X_i$, $F: X_1\times X_2\to X_1$, and $G: X_1\times X_2\to X_2$ be single-valued mappings, and $T: X_1\to CB(X_1)$, $Q: X_2\to$*

$CB(X_2)$ be set-valued mappings. Let $M : X_1 \times X_1 \to 2^{X_1}$ and $N : X_2 \times X_2 \to 2^{X_2}$ be such that, for each fixed $x \in X_1$, $y \in X_2$, $M(\cdot, x)$ and $N(\cdot, y)$ are $H_1(\cdot,\cdot)-\eta_1$-cocoercive and $H_2(\cdot,\cdot)-\eta_2$-cocoercive operators with respect to (A_1, B_1) and (A_2, B_2), respectively. Suppose that the following conditions are satisfied.

(i) *$H_i(A_i, B_i)$ is η_i-cocoercive with respect to A_i with constant μ_i and η_i-relaxed cocoercive with respect to B_i with constant γ_i, $i = 1, 2$.*

(ii) *A_i is α_i-expansive and B_i is β_i-Lipschitz continuous, $i = 1, 2$.*

(iii) *$H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and s_i-Lipschitz continuous with respect to B_i, $i = 1, 2$.*

(iv) *T is D-Lipschitz continuous with constant λ_T and Q is D-Lipschitz continuous with constant λ_Q.*

(v) *F is t_1-Lipschitz continuous with respect to T in the first argument and t_2-Lipschitz continuous with respect to Q in the second argument.*

(vi) *G is l_1-Lipschitz continuous with respect to T in the first argument and l_2-Lipschitz continuous with respect to Q in the second argument.*

(vii) *η_i is ρ_i-Lipschitz continuous, $i = 1, 2$.*

(viii) *F is η_1-relaxed Lipschitz continuous with respect to T in the first argument and η_1-relaxed Lipschitz continuous with respect to Q in the second argument with constants τ_1 and τ_2, respectively.*

(ix) *G is η_2-relaxed Lipschitz continuous with respect to T in the first argument and η_2-relaxed Lipschitz continuous with respect to Q in the second argument with constants ϵ_1 and ϵ_2, respectively. Furthermore, assume that there exist constants $\sigma_1, \sigma_2 > 0$ such that*

$$\left\| R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_1)}(x) - R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_2)}(x) \right\|_1 \le \sigma_1 \|x_1 - x_2\|_1, \quad \forall x, x_1, x_2 \in X_1, \tag{36}$$

$$\left\| R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2, N(\cdot,y_1)}(y) - R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2, N(\cdot,y_2)}(y) \right\|_2 \le \sigma_2 \|y_1 - y_2\|_2, \quad \forall y, y_1, y_2 \in X_2, \tag{37}$$

and $\lambda_1, \lambda_2 > 0$ are constants satisfying the following conditions:

$$\begin{gathered}
p_0 = \sqrt{r_1^2 + 2\lambda_1 t_1 \lambda_T [r_1 + \lambda_1 t_1 \lambda_T + \rho_1] - 2\lambda_1 \tau_1}, \\
q_0 = \sqrt{r_2^2 + 2\lambda_2 l_2 \lambda_Q [r_2 + \lambda_2 l_2 \lambda_Q + \rho_2] - 2\lambda_2 \epsilon_2}, \\
r_1^2 + 2\lambda_1 t_1 \lambda_T [r_1 + \lambda_1 t_1 \lambda_T + \rho_1] > 2\lambda_1 \tau_1, \\
r_2^2 + 2\lambda_2 l_2 \lambda_Q [r_2 + \lambda_2 l_2 \lambda_Q + \rho_2] > 2\lambda_2 \epsilon_2, \\
\mu_i > \gamma_i, \quad \alpha_i > \beta_i, \quad i = 1, 2, \\
\theta_1 = \frac{\rho_1}{\mu_1 \alpha_1^2 - \gamma_1 \beta_1^2}, \qquad \theta_2 = \frac{\rho_2}{\mu_2 \alpha_2^2 - \gamma_2 \beta_2^2}, \\
\sigma_1 + \theta_1 p_0 + \theta_1 s_1 + \lambda_2 \theta_2 l_1 \lambda_T < 1, \\
\sigma_2 + \theta_2 q_0 + \theta_2 s_2 + \lambda_1 \theta_1 t_2 \lambda_Q < 1.
\end{gathered} \tag{38}$$

Then, the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, and $\{v_n\}$ generated by Algorithm 17 converge strongly to x, y, w, and v, respectively, and (x, y, w, v) is a solution of problem (30).

Proof. In view of Theorem 13, the resolvent operator $R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M}$ is θ_1-Lipschitz continuous. This, together with Algorithm 17 and (36), implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1 \\
&= \Big\| R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_n)} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \\
&\quad - R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_{n-1})} \\
&\quad \times [H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 F(w_{n-1}, v_{n-1})] \Big\|_1 \\
&\le \Big\| R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_n)} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \\
&\quad - R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_{n-1})} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \Big\|_1 \\
&\quad + \Big\| R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_{n-1})} [H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)] \\
&\quad - R^{H_1(\cdot,\cdot)-\eta_1}_{\lambda_1, M(\cdot,x_{n-1})} \\
&\quad \times [H_1(A_1 x_{n-1}, B_1 x_{n-1}) - \lambda_1 F(w_{n-1}, v_{n-1})] \Big\|_1 \\
&\le \sigma_1 \|x_n - x_{n-1}\|_1 \\
&\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) \\
&\quad - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_{n-1}))\|_1 \\
&\le \sigma_1 \|x_n - x_{n-1}\|_1 \\
&\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_{n-1}) \\
&\quad - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1 \\
&\quad + \lambda_1 \theta_1 \|F(w_{n-1}, v_n) - F(w_{n-1}, v_{n-1})\|_1 \\
&\le \sigma_1 \|x_n - x_{n-1}\|_1 \\
&\quad + \theta_1 \|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x_{n-1}, B_1 x_n) \\
&\quad - \lambda_1 (F(w_n, v_n) - F(w_{n-1}, v_n))\|_1
\end{aligned}$$

$$+\theta_1\|H_1(A_1x_{n-1},B_1x_n)-H_1(A_1x_{n-1},B_1x_{n-1})\|_1 + \lambda_1\theta_1\|F(w_{n-1},v_n)-F(w_{n-1},v_{n-1})\|_1. \tag{39}$$

Since F is t_1-Lipschitz continuous with respect to T in the first argument and t_2-Lipschitz continuous in the second argument, T is λ_T-Lipschitz continuous, and Q is λ_Q-Lipschitz continuous, by Algorithm 17, we get

$$\begin{aligned}\|F(w_n,v_n)-F(w_{n-1},v_n)\|_1 &\le t_1\|w_n-w_{n-1}\|_1\\ &\le t_1\left(1+\frac{1}{n}\right)D(T(x_n),T(x_{n-1}))\\ &\le t_1\lambda_T\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|_1,\end{aligned} \tag{40}$$

$$\begin{aligned}\|F(w_{n-1},v_n)-F(w_{n-1},v_{n-1})\|_1 &\le t_2\|v_n-v_{n-1}\|_2\\ &\le t_2\left(1+\frac{1}{n}\right)D(Q(y_n),Q(y_{n-1}))\\ &\le t_2\lambda_Q\left(1+\frac{1}{n}\right)\|y_n-y_{n-1}\|_2.\end{aligned} \tag{41}$$

As $H_1(\cdot,\cdot)$ is r_1-Lipschitz continuous with respect to A_1, we obtain

$$\|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\|_1 \le r_1\|x_n-x_{n-1}\|_1. \tag{42}$$

Since η_1 is ρ_1-Lipschitz continuous, we conclude that

$$\|\eta_1(x_n,x_{n-1})\|_1 \le \rho_1\|x_n-x_{n-1}\|_1. \tag{43}$$

Since $H_1(\cdot,\cdot)$ is η_1-relaxed Lipschitz continuous with respect to T and η_2-relaxed Lipschitz continuous with respect to Q in the first and second arguments with constants τ_1 and τ_2, respectively, we have

$$\begin{aligned}&\langle F(w_n,v_n)-F(w_{n-1},v_n),j(\eta_1(x_n,x_{n-1}))\rangle\\ &\quad\le -\tau_1\|x_n-x_{n-1}\|_1^2.\end{aligned} \tag{44}$$

Employing Lemma 3 and taking into account (39)–(44), we obtain

$$\begin{aligned}&\|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\\ &\quad-\lambda_1(F(w_n,v_n)-F(w_{n-1},v_n))\|_1^2\\ &\le \|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\|_1^2\\ &\quad-2\lambda_1\langle F(w_n,v_n)-F(w_{n-1},v_n),\\ &\qquad j[H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\\ &\qquad\quad-\lambda_1(F(w_n,v_n)-F(w_{n-1},v_n))]\rangle\\ &= \|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\|_1^2\\ &\quad-2\lambda_1\langle F(w_n,v_n)-F(w_{n-1},v_n),\\ &\qquad j[H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\\ &\qquad\quad-\lambda_1(F(w_n,v_n)-F(w_{n-1},v_n))]\\ &\qquad\quad+j(\eta_1(x_n,x_{n-1}))\rangle\\ &\quad+2\lambda_1\langle F(w_n,v_n)-F(w_{n-1},v_n),j(\eta_1(x_n,x_{n-1}))\rangle\\ &\le \|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\|_1^2\\ &\quad+2\lambda_1\|F(w_n,v_n)-F(w_{n-1},v_n)\|_1\\ &\quad\times\Big[\|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\|_1\\ &\qquad+\lambda_1\|F(w_n,v_n)-F(w_{n-1},v_n)\|_1\\ &\qquad+\|\eta_1(x_n,x_{n-1})\|_1\Big]\\ &\quad+2\lambda_1\langle F(w_n,v_n)-F(w_{n-1},v_n),j(\eta_1(x_n,x_{n-1}))\rangle\\ &\le r_1^2\|x_n-x_{n-1}\|_1^2+2\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|_1\\ &\quad\times\left[r_1\|x_n-x_{n-1}\|_1+\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)\right.\\ &\qquad\left.\times\|x_n-x_{n-1}\|_1+\rho_1\|x_n-x_{n-1}\|_1\right]\\ &\quad-2\lambda_1\tau_1\|x_n-x_{n-1}\|_1^2\\ &= \left[r_1^2+2\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)\right.\\ &\qquad\left.\times\left[r_1+\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)+\rho_1\right]-2\lambda_1\tau_1\right]\|x_n-x_{n-1}\|_1^2.\end{aligned} \tag{45}$$

This implies that

$$\begin{aligned}&\|H_1(A_1x_n,B_1x_n)-H_1(A_1x_{n-1},B_1x_n)\\ &\quad-\lambda_1(F(w_n,v_n)-F(w_{n-1},v_n))\|_1\\ &\le \sqrt{r_1^2+2\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)\left[r_1+\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)+\rho_1\right]-2\lambda_1\tau_1}\\ &\quad\times\|x_n-x_{n-1}\|_1\\ &= p_n\|x_n-x_{n-1}\|_1,\end{aligned} \tag{46}$$

where

$$p_n = \sqrt{r_1^2+2\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)\left[\left[r_1+\lambda_1t_1\lambda_T\left(1+\frac{1}{n}\right)+\rho_1\right]-2\lambda_1\tau_1\right]}. \tag{47}$$

Using s_1-Lipschitz continuity of $H_1(\cdot,\cdot)$ with respect to B_1, we deduce that

$$\begin{aligned}&\left\|H_1\left(A_1x_{n-1},B_1x_n\right)-H_1\left(A_1x_{n-1},B_1x_{n-1}\right)\right\|_1\\&\quad\le s_1\left\|x_n-x_{n-1}\right\|_1.\end{aligned}\tag{48}$$

In view of (41), (46), (48), (39) becomes

$$\begin{aligned}&\left\|x_{n+1}-x_n\right\|_1\\&\quad\le\sigma_1\left\|x_n-x_{n-1}\right\|_1+\theta_1p_n\left\|x_n-x_{n-1}\right\|_1\\&\qquad+\theta_1s_1\left\|x_n-x_{n-1}\right\|_1\\&\qquad+\lambda_1\theta_1t_2\lambda_Q\left(1+\frac{1}{n}\right)\left\|y_n-y_{n-1}\right\|_2\\&\quad=\left(\sigma_1+\theta_1p_n+\theta_1s_1\right)\left\|x_n-x_{n-1}\right\|_1\\&\qquad+\lambda_1\theta_1t_2\lambda_Q\left(1+\frac{1}{n}\right)\left\|y_n-y_{n-1}\right\|_2.\end{aligned}\tag{49}$$

Similarly, we have

$$\begin{aligned}&\left\|y_{n+1}-y_n\right\|_2\\&\quad=\Big\|R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_n)}\left[H_2\left(A_2y_n,B_2y_n\right)-\lambda_2G\left(w_n,v_n\right)\right]\\&\qquad-R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_{n-1})}\left[H_2\left(A_2y_{n-1},B_2y_{n-1}\right)\right.\\&\qquad\qquad\left.-\lambda_2G\left(w_{n-1},v_{n-1}\right)\right]\Big\|_2\\&\quad\le\Big\|R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_n)}\left[H_2\left(A_2y_n,B_2y_n\right)-\lambda_2G\left(w_n,v_n\right)\right]\\&\qquad-R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_{n-1})}\left[H_2\left(A_2y_n,B_2y_n\right)-\lambda_2G\left(w_n,v_n\right)\right]\Big\|_2\\&\qquad+\Big\|R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_{n-1})}\left[H_2\left(A_2y_n,B_2y_n\right)-\lambda_2G\left(w_n,v_n\right)\right]\\&\qquad\qquad-R^{H_2(\cdot,\cdot)-\eta_2}_{\lambda_2,N(\cdot,y_{n-1})}\left[H_2\left(A_2y_{n-1},B_2y_{n-1}\right)\right.\\&\qquad\qquad\qquad\left.-\lambda_2G\left(w_{n-1},v_{n-1}\right)\right]\Big\|_2\\&\quad\le\sigma_2\left\|y_n-y_{n-1}\right\|_2\\&\qquad+\theta_2\left\|H_2\left(A_2y_n,B_2y_n\right)-H_2\left(A_2y_{n-1},B_2y_{n-1}\right)\right.\\&\qquad\qquad\left.-\lambda_2\left(G\left(w_n,v_n\right)-G\left(w_{n-1},v_{n-1}\right)\right)\right\|_2\\&\quad\le\sigma_2\left\|y_n-y_{n-1}\right\|_2\\&\qquad+\theta_2\left\|H_2\left(A_2y_n,B_2y_n\right)-H_2\left(A_2y_{n-1},B_2y_{n-1}\right)\right.\\&\qquad\qquad\left.-\lambda_2\left(G\left(w_n,v_n\right)-G\left(w_n,v_{n-1}\right)\right)\right\|_2\\&\qquad+\lambda_2\theta_2\left\|G\left(w_n,v_{n-1}\right)-G\left(w_{n-1},v_{n-1}\right)\right\|_2\\&\quad\le\sigma_2\left\|y_n-y_{n-1}\right\|_2\\&\qquad+\theta_2\left\|H_2\left(A_2y_n,B_2y_n\right)-H_2\left(A_2y_{n-1},B_2y_n\right)\right.\\&\qquad\qquad\left.-\lambda_2\left(G\left(w_n,v_n\right)-G\left(w_n,v_{n-1}\right)\right)\right\|_2\\&\qquad+\theta_2\left\|H_2\left(A_2y_{n-1},B_2y_n\right)-H_2\left(A_2y_{n-1},B_2y_{n-1}\right)\right\|_2\\&\qquad+\lambda_2\theta_2\left\|G\left(w_n,v_{n-1}\right)-G\left(w_{n-1},v_{n-1}\right)\right\|_2\\&\quad\le\sigma_2\left\|y_n-y_{n-1}\right\|_2+\theta_2q_n\left\|y_n-y_{n-1}\right\|_2+\theta_2s_2\left\|y_n-y_{n-1}\right\|_2\\&\qquad+\lambda_2\theta_2t_1\lambda_T\left(1+\frac{1}{n}\right)\left\|x_n-x_{n-1}\right\|_1\\&\quad=\left(\sigma_2+\theta_2q_n+\theta_2s_2\right)\left\|y_n-y_{n-1}\right\|_2\\&\qquad+\lambda_2\theta_2l_1\lambda_T\left(1+\frac{1}{n}\right)\left\|x_n-x_{n-1}\right\|_1,\end{aligned}\tag{50}$$

where

$$\begin{aligned}&q_n\\&\quad=\sqrt{r_2^2+2\lambda_2l_2\lambda_Q\left(1+\frac{1}{n}\right)\left[r_2+\lambda_2l_2\lambda_Q\left(1+\frac{1}{n}\right)+\rho_2\right]-2\lambda_2\epsilon_2}.\end{aligned}\tag{51}$$

In view of (49) and (50), we obtain

$$\begin{aligned}&\left\|x_{n+1}-x_n\right\|_1+\left\|y_{n+1}-y_n\right\|_2\\&\quad\le c_n\left\|x_n-x_{n-1}\right\|_1+d_n\left\|y_n-y_{n-1}\right\|_2\\&\quad\le k_n\left(\left\|x_n-x_{n-1}\right\|_1+\left\|y_n-y_{n-1}\right\|_2\right),\end{aligned}\tag{52}$$

where $c_n=\sigma_1+\theta_1p_n+\theta_1s_1+\lambda_2\theta_2l_1\lambda_T(1+1/n)$, $d_n=\sigma_2+\theta_2q_n+\theta_2s_2+\lambda_1\theta_1t_2\lambda_Q(1+1/n)$, and $k_n=\max\{c_n,d_n\}$.

Letting $n\to\infty$, we obtain $k_n\to k$, where

$$\begin{aligned}k=\max\{&\sigma_1+\theta_1p_0+\theta_1s_1+\lambda_2\theta_2l_1\lambda_T,\\&\sigma_2+\theta_2q_0+\theta_2s_2+\lambda_1\theta_1t_2\lambda_Q\}.\end{aligned}\tag{53}$$

Next, we define the norm $\|\cdot\|$ on $X_1\times X_2$ by

$$\|(x,y)\|=\|x\|_1+\|y\|_2,\quad(x,y)\in X_1\times X_2.\tag{54}$$

One can easily check that $(X_1\times X_2,\|\cdot\|)$ is a Banach space.

Define $a_{n+1}=(x_{n+1},y_{n+1})$. Then, we have

$$\|a_{n+1}-a_n\|=\left\|x_{n+1}-x_n\right\|_1+\left\|y_{n+1}-y_n\right\|_2.\tag{55}$$

In view of (38), we conclude that $0<k<1$. This implies that there exist $n_0\in\mathbb{N}$ and $k_0\in(0,1)$ such that $k_n\le k_0$ for all $n\ge n_0$. It follows from (52) and (54) that

$$\|a_{n+1}-a_n\|\le k_0\|a_n-a_{n-1}\|,\quad\forall n\ge n_0.\tag{56}$$

In view of (56), we obtain

$$\|a_{n+1}-a_n\|\le k_0^{n-n_0}\left\|a_{n_0+1}-a_{n_0}\right\|,\quad\forall n\ge n_0.\tag{57}$$

This implies that for any $m \geq n \geq n_0$,

$$\begin{aligned} \|x_m - x_n\|_1 &\leq \|a_m - a_n\| \\ &\leq \sum_{i=n}^{m-1} \|a_{i+1} - a_i\| \leq \sum_{i=n}^{m-1} k_0^{i-n_0} \|a_{n_0+1} - a_{n_0}\|. \end{aligned} \tag{58}$$

Since $0 < k_0 < 1$, it follows from (58) that $\|x_m - x_n\|_1 \to 0$ and $n \to \infty$. This proves that $\{x_n\}$ is a Cauchy sequence in X_1. Similarly, we conclude that $\{y_n\}$ is a Cauchy sequence in X_2. Thus, there exist $x \in X_1$ and $y \in X_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Next, we prove that $w_n \to w \in T(x)$ and $v_n \to v \in Q(y)$. In view of Lipschitz continuity of T and Q and Algorithm 17, we obtain

$$\begin{aligned} \|w_n - w_{n-1}\|_1 &\leq \left(1 + \frac{1}{n}\right) \lambda_T \|x_n - x_{n-1}\|_1, \\ \|v_n - v_{n-1}\|_2 &\leq \left(1 + \frac{1}{n}\right) \lambda_Q \|y_n - y_{n-1}\|_2. \end{aligned} \tag{59}$$

From (59), we deduce that $\{w_n\}, \{v_n\}$ are Cauchy sequences in X_1 and X_2, respectively. Thus, there exist $w \in T(x)$ and $v \in Q(y)$ such that $w_n \to w$ and $v_n \to v$ as $n \to \infty$. Since T is D-Lipschitz continuous with constant λ_T, it is obvious that

$$\begin{aligned} d(w, T(x)) &\leq \|w - w_n\|_1 + d(w_n, T(x)) \\ &\leq \|w - w_{n-1}\|_1 + D(T(x_n), T(x)) \\ &\leq \|w - w_n\|_1 \\ &\quad + \lambda_T \| x_n - x\|_1 \longrightarrow 0 \, (n \longrightarrow \infty). \end{aligned} \tag{60}$$

By the closedness of $T(x)$, we conclude that $w \in T(x)$. Similarly, we have $v \in Q(y)$.

Assume now that

$$\begin{aligned} x_0 &= R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta} \left[H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)\right], \\ y_0 &= R_{\lambda_2, N(\cdot,y)}^{H_2(\cdot,\cdot)-\eta} \left[H_2(A_2 y, B_2 y) - \lambda_2 G(w, v)\right]. \end{aligned} \tag{61}$$

Then, we have

$$\begin{aligned} &\|x_{n+1} - x_0\|_1 \\ &\quad \leq \left\| R_{\lambda_1, M(\cdot,x_n)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)\right] \right. \\ &\qquad \left. - R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)\right] \right\|_1 \\ &\quad \leq \left\| R_{\lambda_1, M(\cdot,x_n)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)\right] \right. \\ &\qquad \left. - R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)\right] \right\|_1 \\ &\qquad + \left\| R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n)\right] \right. \\ &\qquad \left. - R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta_1} \left[H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)\right] \right\|_1 \\ &\quad \leq \sigma_1 \|x_n - x\|_1 \\ &\qquad + \theta_1 \left\| H_1(A_1 x_n, B_1 x_n) - \lambda_1 F(w_n, v_n) \right. \\ &\qquad \left. - \left[H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)\right] \right\|_1 \\ &\quad \leq \sigma_1 \|x_n - x\|_1 \\ &\qquad + \theta_1 \left[\|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \right. \\ &\qquad \left. + \lambda_1 \|F(w_n, v_n) - F(w, v)\|_1 \right] \\ &\quad \leq \sigma_1 \|x_n - x\|_1 \\ &\qquad + \theta_1 \left[\|H_1(A_1 x_n, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \right. \\ &\qquad \left. + \|H_1(A_1 x, B_1 x_n) - H_1(A_1 x, B_1 x)\|_1 \right] \\ &\qquad + \lambda_1 \theta_1 \left[\|F(w_n, v_n) - F(w, v_n)\|_1 \right. \\ &\qquad \left. + \|F(w, v_n) - F(w, v)\|_1 \right] \\ &\quad \leq \sigma_1 \|x_n - x\|_1 + \theta_1 [r_1 + s_1] \|x_n - x\|_1 \\ &\qquad + \lambda_1 \theta_1 \left[t_1 \|w_n - w\|_1 + t_2 \|v_n - v\|_2\right] \\ &\quad = (\sigma_1 + \theta_1 r_1 + \theta_1 s_1) \|x_n - x\|_1 \\ &\qquad + \lambda_1 \theta_1 \left[t_1 \|w_n - w\|_1 + t_2 \|v_n - v\|_2\right]. \end{aligned} \tag{62}$$

Since $x_n \to x$, $w_n \to w$, and $v_n \to v$ as $n \to \infty$, it follows from (62) that

$$\lim_{n \to \infty} \|x_n - x_0\|_1 = 0, \tag{63}$$

and hence $x_0 = x$.

A similar argument shows that $y_0 = y$. Therefore,

$$\begin{aligned} x = x_0 &= R_{\lambda_1, M(\cdot,x)}^{H_1(\cdot,\cdot)-\eta} \left[H_1(A_1 x, B_1 x) - \lambda_1 F(w, v)\right], \\ y = y_0 &= R_{\lambda_2, N(\cdot,y)}^{H_2(\cdot,\cdot)-\eta} \left[H_2(A_2 y, B_2 y) - \lambda_2 G(w, v)\right]. \end{aligned} \tag{64}$$

In view of Lemma 16, we conclude that (x, y, w, v) is a solution of problem (30), which completes the proof. □

At the end of this paper, we include the following simple example in support of Theorem 18.

Example 19. Let $X = \mathbb{R}^2$ with the usual inner product. We define two mappings $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} A(x) &:= \left(\frac{1}{4} x_1 - x_2, x_1 + \frac{1}{4} x_2\right), \\ B(x) &:= \left(-\frac{1}{4} x_1 + \frac{1}{4} x_2, -\frac{1}{4} x_1 - \frac{1}{4} x_2\right), \\ &\qquad \forall x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{65}$$

Let a mapping $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$H(Ax, By) := Ax + By, \quad \forall x, y \in \mathbb{R}^2. \tag{66}$$

By similar arguments, as in Example 4.1 of [27], we can prove the following.

(1) $H(A, B)$ is $4/17$-cocoercive with respect to A and 1-relaxed cocoercive with respect to B.

(2) A is $\sqrt{17}/n$-expansive, for $n = 4, 5$.

(3) B is $1/\sqrt{n}$-expansive, for $n = 1, 2$.

(4) $H(A, B)$ is $\sqrt{17}/n$-Lipschitz continuous with constant $\sqrt{17}/n$ with respect to A and B, for $n = 1, 2, \ldots, 15, 16$.

(5) Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} f(x) &:= \left(8x_1 - \frac{8}{5}x_2, \frac{8}{5}x_1 + 8x_2\right), \\ g(x) &:= \left(-\frac{17}{8}x_1 + \frac{5}{8}x_2, \frac{5}{8}x_1 + \frac{17}{8}x_2\right), \\ &\qquad \forall x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{67}$$

(6) Now, we define a mapping $M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$M(fx, gy) := fx - gy, \quad \forall x, y \in \mathbb{R}^2. \tag{68}$$

Let $R, S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity mappings. It is obvious that these mappings are D-Lipschitz continuous.

(7) Assume that $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$\begin{aligned} F(x) &:= \left(-\frac{1}{4}x_1 - \frac{1}{8}x_2, \frac{1}{8}x_1 - \frac{1}{4}x_2\right), \\ G(x) &:= \left(-\frac{1}{8}x_1 + \frac{1}{5}x_2, -\frac{1}{5}x_1 - \frac{1}{8}x_2\right), \\ &\qquad \forall x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \tag{69}$$

It could easily be seen that all the aspects of the hypotheses of Theorem 18 are satisfied, so we have the desired conclusion.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper was funded by King Abdulaziz University, under Grant no. 23-130-1433-HiCi. The authors, therefore, acknowledge the technical and financial support of KAU. The authors would like to thank the referees for their sincere evaluation and constructive comments which improved the paper considerably.

References

[1] S. Adly, "Perturbed algorithms and sensitivity analysis for a general class of variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.

[2] A. Alotaibi, V. Kumar, and N. Hussain, "Convergence comparison and stability of Jungck-Kirk type algorithms for common fixed point problems," *Fixed Point Theory and Applications*, vol. 2013, article 173, 2013.

[3] L. C. Ceng, N. Husain, A. Latif, and J. C. Yao, "Strong convergence for solving general system of variational inequalities and fixed point problems in Banach spaces," *Journal of Inequalities and Applications*, vol. 2013, article 334, 2013.

[4] H. R. Feng and X. P. Ding, "A new system of generalized nonlinear quasi-variational-like inclusions with A-monotone operators in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 365–373, 2009.

[5] H.-Y. Lan, Y. J. Cho, and R. U. Verma, "Nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings in Banach spaces," *Computers and Mathematics with Applications*, vol. 51, no. 9-10, pp. 1529–1538, 2006.

[6] Y.-P. Fang and N.-J. Huang, "H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.

[7] Z. B. Wang and X. P. Ding, "$H((.,.), \eta)$-accretive operators with an application for solving set-valued variational inclusions in Banach spaces," *Computers and Mathematics with Applications*, vol. 59, no. 4, pp. 1559–1567, 2010.

[8] Y.-Z. Zou and N.-J. Huang, "$H(.,.)$-accretive operator with an application for solving variational inclusions in Banach spaces," *Applied Mathematics and Computation*, vol. 204, no. 2, pp. 809–816, 2008.

[9] M.-M. Jin, "Convergence and stability of iterative algorithm for a new system of (A, η)-accretive mapping inclusions in Banach spaces," *Computers and Mathematics with Applications*, vol. 56, no. 9, pp. 2305–2311, 2008.

[10] H.-Y. Lan, "Stability of iterative processes with errors for a system of nonlinear (A, η)-accretive variational inclusions in Banach spaces," *Computers and Mathematics with Applications*, vol. 56, no. 1, pp. 290–303, 2008.

[11] X.-P. Luo and N.-J. Huang, "$(H, \phi) - \eta$–monotone operators in Banach spaces with an application to variational inclusions," *Applied Mathematics and Computation*, vol. 216, no. 4, pp. 1131–1139, 2010.

[12] Z. Xu and Z. Wang, "A Generalized mixed variational inclusion involving $(H(.,.), \eta)$-monotone operators in Banach spaces," *Journal of Mathematics Research*, vol. 2, no. 3, pp. 47–56, 2010.

[13] R. U. Verma, "Approximation solvability of a class of nonlinear set-valued variational inclusions involving (A, η)-monotone mappings," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 969–975, 2008.

[14] H.-Y. Lan, "Nonlinear parametric multi-valued variational inclusion systems involving (A, η)-accretive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 5-6, pp. 1757–1767, 2008.

[15] H.-Y. Lan, "(A, η)-accretive mappings and set-valued variational inclusions with relaxed cocoercive mappings in Banach spaces," *Applied Mathematics Letters*, vol. 20, no. 5, pp. 571–577, 2007.

[16] C.-L. Su, "Finger shape expansion and recognition by wavelet transform," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1583–1592, 2007.

[17] H.-Y. Lan, Y.-S. Cui, and Y. Fu, "New approximation-solvability of general nonlinear operator inclusion couples involving (A, η, m)-resolvent operators and relaxed cocoercive type operators," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 4, pp. 1844–1851, 2012.

[18] N. Hussain and A. Rafiq, "On modified implicit Mann iteration method involving strictly hemicontractive mappings in smooth Banach spaces," *Journal of Computational Analysis and Applications*, vol. 15, no. 5, pp. 892–902, 2013.

[19] N. Hussain, A. Rafiq, and L. B. Ciric, "Stability of the Ishikawa iteration scheme with errors for two strictly hemicontractive operators in Banach spaces," *Fixed Point Theory and Applications*, vol. 2012, article 160, 2012.

[20] S. H. Khan, A. Rafiq, and N. Hussain, "A three-step iterative scheme for solving nonlinear ϕ-strongly accretive operator equations in Banach spaces," *Fixed Point Theory and Applications*, vol. 2012, article 149, 2012.

[21] L.-C. Ceng, A. R. Khan, Q. H. Ansari, and J.-C. Yao, "Viscosity approximation methods for strongly positive and monotone operators," *Fixed Point Theory*, vol. 10, no. 1, pp. 35–71, 2009.

[22] R. Ahmad and M. Mursaleen, "System of generalized H-resolvent equations and the corresponding system of generalized variational inclusions," *Hacettepe Journal of Mathematics and Statistics*, vol. 41, no. 1, pp. 33–45, 2012.

[23] R. Ahmad, M. Dilshad, M.-M. Wong, and J.-C. Yao, "Generalized set-valued variational-like inclusions involving $H(.,.)$-η-cocoercive operator in Banach spaces," *Journal of Inequalities and Applications*, vol. 2012, article 149, 2012.

[24] S. B. Nadler, "Multivalued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 3, pp. 457–488, 1969.

[25] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[26] W. V. Petryshyn, "A characterization of strict convexity of Banach spaces and other uses of duality mappings," *Journal of Functional Analysis*, vol. 6, no. 2, pp. 282–291, 1970.

[27] R. Ahmad, M. Akram, and J. C. Yao, "Generalized monotone mapping with an application for solving a variational inclusion problem," *Journal of Optimization Theroy and Applications*, vol. 157, no. 2, pp. 324–346, 2013.