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Research Article

Approximate Solution of Inverse Problem for Elliptic Equation with Overdetermination

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A finite difference method for the approximate solution of the inverse problem for the multidimensional elliptic equation with overdetermination is applied. Stability and coercive stability estimates of the first and second orders of accuracy difference schemes for this problem are established. The algorithm for approximate solution is tested in a two-dimensional inverse problem.

1. Introduction

It is well known that inverse problems arise in various branches of science (see [1, 2]). The theory and applications of well-posedness of inverse problems for partial differential equations have been studied extensively by many researchers (see, e.g., [3–17] and the references therein). One of the effective approaches for solving inverse problem is reduction to nonlocal boundary value problem (see, e.g., [6, 8, 11]). Well-posedness of the nonlocal boundary value problems of elliptic type equations was investigated in [18–25] (see also the references therein).

In [4], Orlovsky proved existence and uniqueness theorems for the inverse problem of finding a function u and an element p for the elliptic equation in an arbitrary Hilbert space H with the self-adjoint positive definite operator A:

$$\begin{gathered} -u_{tt}(t) + Au(t) = f(t) + p, \quad 0 < t < T, \\ u(0) = \varphi, \quad u(T) = \psi, \quad u(\lambda) = \xi, \quad 0 < \lambda < T. \end{gathered} \tag{1}$$

In [11], the authors established stability estimates for this problem and studied inverse problem for multidimensional elliptic equation with overdetermination in which the Dirichlet condition is required on the boundary.

In present work, we study inverse problem for multidimensional elliptic equation with Dirichlet-Neumann boundary conditions.

Let $\Omega = (0,\ell) \times \cdots \times (0,\ell)$ be the open cube in the n-dimensional Euclidean space with boundary S and $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, we consider the inverse problem of finding functions $u(t,x)$ and $p(x)$ for the multidimensional elliptic equation

$$\begin{gathered} -u_{tt}(t,x) - \sum_{r=1}^{n} \left(a_r(x) u_{x_r}\right)_{x_r} + \delta u(t,x) = f(t,x) + p(x), \\ x = (x_1, \ldots, x_n) \in \Omega, \quad 0 < t < T, \\ u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad u(\lambda,x) = \xi(x), \\ x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0, \qquad x \in S, \ 0 \le t \le T. \end{gathered} \tag{2}$$

Here, $0 < \lambda < T$ and $\delta > 0$ are known numbers, $a_r(x)$ $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T), x \in \Omega)$ are given smooth functions, and also $a_r(x) \ge a > 0$ $(x \in \Omega)$.

The aim of this paper is to investigate inverse problem (2) for multidimensional elliptic equation with Dirichlet-Neumann boundary conditions. We obtain well-posedness of problem (2). For the approximate solution of problem (2), we construct first and second order of accuracy in

t and difference schemes with second order of accuracy in space variables. Stability and coercive stability estimates for these difference schemes are established by applying operator approach. The modified Gauss elimination method is applied for solving these difference schemes in a two-dimensional case.

The remainder of this paper is organized as follows. In Section 2, we obtain stability and coercive stability estimates for problem (2). In Section 3, we construct the difference schemes for (2) and establish their well-posedness. In Section 4, the numerical results in a two-dimensional case are presented. Section 5 is conclusion.

2. Well-Posedness of Inverse Problem with Overdetermination

It is known that the differential expression [26]

$$A^x u(x) = -\sum_{r=1}^{n}\left(a_r(x)\, u_{x_r}\right)_{x_r} + \delta u(x) \tag{3}$$

defines a self-adjoint positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain $D(A^x) = \{u(x) \in W_2^2(\overline{\Omega}), \partial u/\partial \vec{n} = 0 \text{ on } S\}$.

Let H be the Hilbert space $L_2(\overline{\Omega})$. By using abstract Theorems 2.1 and 2.2 of paper [11], we get the following theorems about well-posedness of problem (2).

Theorem 1. *Assume that A^x is defined by formula (3), $\varphi, \xi, \psi \in D(A^x)$. Then, for the solutions (u, p) of inverse boundary value problem (2), the stability estimates are satisfied:*

$$\begin{aligned}
&\|u\|_{C(L_2(\overline{\Omega}))} \\
&\quad \le \left[M\,\|\varphi\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} + \|\xi\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}\right], \\
&\left\|(A^x)^{-1} p\right\|_{L_2(\overline{\Omega})} \le M\left[\|\varphi\|_{L_2(\overline{\Omega})}\right. \\
&\qquad\qquad \left. + \|\xi\|_{L_2(\overline{\Omega})} + \|f\|_{C(L_2(\overline{\Omega}))}\right], \\
&\|p\|_{L_2(\overline{\Omega})} \le M\left[\|A^x\varphi\|_{L_2(\overline{\Omega})} + \|A^x\psi\|_{L_2(\overline{\Omega})}\right. \\
&\qquad\qquad \left. + \|A^x\xi\|_{L_2(\overline{\Omega})} + \frac{1}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))}\right],
\end{aligned} \tag{4}$$

where M is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t,x)$.

Here, $\mathscr{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))$ is the space obtained by completion of the space of all smooth $L_2(\overline{\Omega})$-valued functions ρ on $[0,T]$ with the norm

$$\begin{aligned}
&\|\rho\|_{\mathscr{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} \\
&\quad = \|\rho\|_{\mathscr{C}(L_2(\overline{\Omega}))} \\
&\qquad + \sup_{0\le t<t+\tau\le T} \frac{(t+\tau)^{\alpha}(T-t)^{\alpha}\|\rho(t+\tau)-\rho(t)\|_{L_2(\overline{\Omega})}}{\tau^{\alpha}}.
\end{aligned} \tag{5}$$

Theorem 2. *Assume that A^x is defined by formula (3), $\varphi, \psi, \xi \in D(A^x)$. Then, for the solution of inverse boundary value problem (2), coercive stability estimate*

$$\begin{aligned}
&\|u''\|_{\mathscr{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|u\|_{\mathscr{C}_{0T}^{\alpha,\alpha}(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})} \\
&\quad \le M\left[\frac{1}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}_{0T}^{\alpha,\alpha}(L_2(\overline{\Omega}))} + \|\varphi\|_{W_2^2(\overline{\Omega})}\right. \\
&\qquad \left. + \|\psi\|_{W_2^2(\overline{\Omega})} + \|\xi\|_{W_2^2(\overline{\Omega})}\right]
\end{aligned} \tag{6}$$

holds, where M is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t,x)$.

3. Difference Schemes and Their Well-Posedness

Suppose that A^x is defined by formula (3). Then (see [26]), $C = (1/2)(\tau A^x + \sqrt{4A^x + \tau^2 (A^x)^2})$ is a self-adjoint positive definite operator and $R = (I + \tau C)^{-1}$ which is defined on the whole space $H = L_2(\overline{\Omega})$ is a bounded operator. Here, I is the identity operator.

Now we present the following lemmas, which will be used later.

Lemma 3. *The following estimates are satisfied (see [27]):*

$$\begin{aligned}
&\left\|R^k\right\|_{H\to H} \le M\left(1+\delta^{1/2}\tau\right)^{-k}, \quad \delta > 0, \\
&\left\|CR^k\right\|_{H\to H} \le \frac{M}{k\tau}, \quad k \ge 1, \\
&\left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \le M.
\end{aligned} \tag{7}$$

Lemma 4. *For $1 \le l \le N-1$ and for the operator $S = R^{2N} + R^l - R^{2N-l} + R^{N-l} - R^{N+l}$, the operator $I - S$ has an inverse $G = (I-S)^{-1}$ and the estimate*

$$\|G\|_{H\to H} \le M \tag{8}$$

is satisfied, where M does not depend on τ.

Proof of Lemma 4 is based on Lemma 3 and representation

$$\begin{aligned}
Q &= I - R^{2N} - R^l + R^{2N-l} - R^{N-l} + R^{N+l} \\
&= \left(I - R^{N-l}\right)\left(I - R^N\right)\left(I - R^l\right).
\end{aligned} \tag{9}$$

Lemma 5. *For $1 \le l \le N-1$ and for the operator*

$$\begin{aligned}
S_1 &= R^{2N} - \left(\frac{\lambda}{\tau} - l - 1\right)\left(R^l - R^{2N-l} + R^{N-l} - R^{N+l}\right) \\
&\quad + \left(\frac{\lambda}{\tau} - l\right)\left(R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}\right),
\end{aligned} \tag{10}$$

the operator $I - S_1$ has an inverse

$$G_1 = \left(I - R^{2N} + \left(\frac{\lambda}{\tau} - l - 1\right) \times \left(R^l - R^{2N-l} + R^{N-l} - R^{N+l}\right) - \left(\frac{\lambda}{\tau} - l\right)\left(R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}\right)\right)^{-1}, \tag{11}$$

and the estimate

$$\|G_1\|_{H\to H} \le M \tag{12}$$

is valid, where M is independent of τ.

Proof. We have that

$$G_1 - G = G_1 G K, \tag{13}$$

where

$$K = -\left(\frac{\lambda}{\tau} - l\right)\left(R^l - R^{2N-l} + R^{N-l} - R^{N+l}\right) + \left(\frac{\lambda}{\tau} - l\right)\left(R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}\right). \tag{14}$$

By using estimates of Lemma 3, we have that

$$\|K\|_{H\to H} = \left\|-\left(\frac{\lambda}{\tau} - l\right)\left(R^l - R^{2N-l} + R^{N-l} - R^{N+l}\right) + \left(\frac{\lambda}{\tau} - l\right)\left(R^{l+1} - R^{2N-l-1} + R^{N-l-1} - R^{N+l+1}\right)\right\|_{H\to H} \le M_1\tau, \tag{15}$$

where M_1 is independent of τ. Using the triangle inequality, formula (13), and estimates (8) and (15), we obtain

$$\|G_1\|_{H\to H} = \|G\|_{H\to H} + \|G_1\|_{H\to H}\|G\|_{H\to H} \le M + \|G_1\|_{H\to H} M M_1 \tau \tag{16}$$

for sufficiently small positive τ. From that it follows estimate (11). Lemma 5 is proved. □

Further, we discretize problem (2) in two steps. In the first step, we define the grid spaces

$$\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \ldots, h_n m_n);\ m = (m_1, \ldots, m_n),\ m_r = 0, \ldots, M_r,\ h_r M_r = \ell,\ r = 1, \ldots, n\}, \quad \Omega_h = \widetilde{\Omega}_h \cap \Omega, \quad S_h = \widetilde{\Omega}_h \cap S. \tag{17}$$

Introduce the Hilbert space $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ of grid functions $\rho^h(x) = \{\rho(h_1 m_1, \ldots, h_n m_n)\}$, defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|\rho^h\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |\rho^h(x)|^2 h_1, \ldots, h_n\right)^{1/2},$$
$$\|\rho^h\|_{W_{2h}^2} = \|\rho^h\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n} |(\rho^h)_{x_r}|^2 h_1, \ldots, h_n\right)^{1/2} + \left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^{n} |(\rho^h(x))_{x_r\overline{x_r}, m_r}|^2 h_1, \ldots, h_n\right)^{1/2}, \tag{18}$$

respectively.

To the differential operator A^x generated by problem (2) we assign the difference operator A_h^x defined by formula (3), acting in the space of grid functions $u^h(x)$, satisfying the condition $D^h u^h(x) = 0$ for all $x \in S_h$. Here, $D^h u^h(x)$ is an approximation of $\partial u/\partial \vec{n}$.

By using A_h^x, for obtaining $u^h(t,x)$ functions, we arrive at problem

$$-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + p^h(x), \quad 0 < t < T,\ x \in \Omega_h,$$
$$u^h(0,x) = \varphi^h(x), \quad u^h(\lambda,x) = \xi^h(x), \quad u^h(T,x) = \psi^h(x), \quad x \in \widetilde{\Omega}_h. \tag{19}$$

For finding a solution $u^h(t,x)$ of problem (19) we apply the substitution

$$u^h(t,x) = v^h(t,x) + \left(A_h^x\right)^{-1} p^h(x), \tag{20}$$

where $v^h(t,x)$ is the solution of nonlocal boundary value problem; a system of ordinary differential equations

$$-\frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < T, \quad x \in \Omega_h,$$
$$v^h(0,x) - v^h(\lambda,x) = \varphi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h,$$
$$v^h(T,x) - v^h(\lambda,x) = \psi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h \tag{21}$$

and unknown function $p^h(x)$ is defined by formula

$$p^h(x) = A_h^x \varphi^h(x) - A_h^x v^h(0,x), \quad x \in \widetilde{\Omega}_h. \tag{22}$$

Thus, we consider the algorithm for solving problem (19) which includes three stages. In the first stage, we get

the nonlocal boundary value problem (21) and obtain $v^h(t,x)$. In the second stage, we put $t = 0$ and find $v^h(0,x)$. Then, using (22), we obtain $p^h(x)$. Finally, in the third stage, we use formula (20) for obtaining the solution $u^h(t,x)$ of problem (19).

In the second step, we approximate (19) in variable t. Let $[0,T]_\tau = \{t_k = k\tau, k = 1,\dots,N, N\tau = T\}$ be the uniform grid space with step size $\tau > 0$, where N is a fixed positive integer. Applying the approximate formulas

$$\begin{aligned} u^h(\lambda,x) &= u^h\left(\left[\frac{\lambda}{\tau}\right]\tau, x\right) + o(\tau), \quad x \in \Omega_h, \\ u^h(\lambda,x) &= u^h\left(\left[\frac{\lambda}{\tau}\right]\tau, x\right) + \left(\frac{\lambda}{\tau} - \left[\frac{\lambda}{\tau}\right]\right) \\ &\quad \times \left(u^h\left(\left[\frac{\lambda}{\tau}\right]\tau + \tau, x\right) - u^h\left(\left[\frac{\lambda}{\tau}\right]\tau, x\right)\right) \\ &\quad + o\left(\tau^2\right), \quad x \in \Omega_h \end{aligned} \tag{23}$$

for $u^h(\lambda,x) = \xi^h(x)$, problem (19) is replaced by first order of accuracy difference scheme

$$\begin{aligned} &-\frac{u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)}{\tau^2} + A^x_h u^h_k(x) \\ &\quad = f^h_k(x) + p^h(x), \\ &\quad f^h_k(x) = f^h(t_k,x), \quad t_k = k\tau, \\ &\quad 1 \le k \le N-1, \quad x \in \Omega_h, \\ &\quad u^h_0(x) = \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\ &\quad u^h_l(x) = \xi^h(x), \quad x \in \widetilde{\Omega}_h, \\ &\quad u^h_N(x) = \psi^h(x), \quad x \in \widetilde{\Omega}_h, \ l = \left[\frac{\lambda}{\tau}\right] \end{aligned} \tag{24}$$

and second order of accuracy difference scheme

$$\begin{aligned} &-\frac{u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)}{\tau^2} + A^x_h u^h_k(x) \\ &\quad = f^h_k(x) + p^h(x), \\ &\quad f^h_k(x) = f^h(t_k,x), \quad t_k = k\tau, \\ &\quad 1 \le k \le N-1, \ x \in \Omega_h, \\ &\quad u^h_0(x) = \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\ &\quad u^h_l(x) + \left(\frac{\lambda}{\tau} - l\right)\left(u^h_{l+1}(x) - u^h_l(x)\right) = \xi^h(x), \\ &\quad x \in \widetilde{\Omega}_h, \\ &\quad u^h_N(x) = \psi^h(x), \quad x \in \widetilde{\Omega}_h, \ l = \left[\frac{\lambda}{\tau}\right]. \end{aligned} \tag{25}$$

For approximate solution of nonlocal problem (21), we have first order of accuracy difference scheme

$$\begin{aligned} &-\frac{v^h_{k+1}(x) - 2v^h_k(x) + v^h_{k-1}(x)}{\tau^2} + A^x_h v^h_k(x) = f^h_k(x), \\ &\quad f^h_k(x) = f^h(t_k,x), \quad t_k = k\tau, \\ &\quad 1 \le k \le N-1, \quad x \in \Omega_h, \\ &\quad v^h_0(x) - v^h_l(x) = \varphi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h, \\ &\quad v^h_N(x) - v^h_l(x) = \psi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h \end{aligned} \tag{26}$$

and second order of accuracy difference scheme

$$\begin{aligned} &-\frac{v^h_{k+1}(x) - 2v^h_k(x) + v^h_{k-1}(x)}{\tau^2} + A^x_h v^h_k(x) = f^h_k(x), \\ &\quad f^h_k(x) = f^h(t_k,x), \\ &\quad t_k = k\tau, \quad 1 \le k \le N, \quad x \in \Omega_h, \\ &\quad v^h_0(x) - \left(\frac{\lambda}{\tau} - l\right) v^h_{l+1}(x) + \left(\frac{\lambda}{\tau} - l - 1\right) v^h_l(x) \\ &\quad = \varphi^h(x) - \xi^h(x), \\ &\quad v^h_N(x) - \left(\frac{\lambda}{\tau} - l\right) v^h_{l+1}(x) + \left(\frac{\lambda}{\tau} - l - 1\right) v^h_l(x) \\ &\quad = \psi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h, \end{aligned} \tag{27}$$

respectively.

Theorem 6. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, for the solutions $(\{u^h_k\}_{K-1}^{N-1}, p^h)$ of difference schemes* (24) *and* (25) *the stability estimates*

$$\begin{aligned} \left\|\left\{u^h_k\right\}_1^{N-1}\right\|_{\mathscr{C}_\tau(L_{2h})} &\le M\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\xi^h\right\|_{L_{2h}}\right. \\ &\quad \left. + \left\|\left\{f^h_k\right\}_1^{N-1}\right\|_{\mathscr{C}_\tau(L_{2h})}\right], \\ \left\|p^h\right\|_{L_{2h}} &\le M\left[\left\|\varphi^h\right\|_{W^2_{2h}} + \left\|\psi^h\right\|_{W^2_{2h}} + \left\|\xi^h\right\|_{W^2_{2h}}\right. \\ &\quad \left. + \frac{1}{\alpha(1-\alpha)}\left\|\left\{f^h_k\right\}_1^{N-1}\right\|_{\mathscr{C}_\tau(L_{2h})}\right] \end{aligned} \tag{28}$$

hold, where M is independent of $\tau, \alpha, h, \varphi^h(x), \psi^h(x)$, $\xi^h(x)$, and $\{f^h_k(x)\}_1^{N-1}$.

Theorem 7. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, for the solutions of difference schemes* (24) *and* (25) *the following almost coercive stability estimate*

$$\begin{aligned}
&\left\| \left\{ \frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k}{\tau^2} \right\}_1^{N-1} \right\|_{\mathscr{C}_\tau(L_{2h})} \\
&\quad + \left\| \left\{ u_k^h \right\}_1^{N-1} \right\|_{\mathscr{C}_\tau(W_{2h}^2)} + \left\| p^h \right\|_{L_{2h}} \\
&\le M \left[\left\| \varphi^h \right\|_{W_{2h}^2} + \left\| \psi^h \right\|_{W_{2h}^2} + \left\| \xi^h \right\|_{W_{2h}^2} \right. \\
&\quad \left. + \ln\left(\frac{1}{\tau + h} \right) \left\| \left\{ f_k^h \right\}_1^{N-1} \right\|_{\mathscr{C}_\tau(L_{2h})} \right]
\end{aligned} \tag{29}$$

holds, where M is independent of $\tau, \alpha, h, \varphi^h(x), \psi^h(x)$, $\xi^h(x)$, and $\{f_k^h(x)\}_1^{N-1}$.

Proofs of Theorems 6 and 7 are based on the symmetry property of operator A^x, on Lemmas 3–5, the formulas

$$\begin{aligned}
u_k^h(x) &= \left(I - R^{2N}\right)^{-1} \\
&\quad \times \left[\left(\left(R^k - R^{2N-k}\right) v_0^h(x) \right. \right. \\
&\qquad \left. + \left(R^{N-k} - R^{N+k}\right) v_N^h(x) \right) \\
&\qquad - \left(R^{N-k} - R^{N+k}\right) (I + \tau C)(2I + \tau C)^{-1} \\
&\qquad \left. \times C^{-1} \sum_{i=1}^{N-1} \left(R^{N-i} - R^{N+i}\right) f_i^h(x)\tau \right] \\
&\quad + (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{|k-i|} - R^{k+i}\right) \\
&\qquad \times f_i^h(x)\tau + \varphi^h(x) - v_0^h(x), \\
p^h(x) &= A_h^x \varphi^h(x) - A_h^x v_0^h(x), \\
v_N^h(x) &= v_0^h(x) + \psi^h(x) - \varphi^h(x),
\end{aligned}$$

$$\begin{aligned}
v_0^h(x) &= -G\left(R^{N-l} - R^{N+l}\right) \\
&\quad \times (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{N-i} - R^{N+i}\right) f_i^h(x)\tau \\
&\quad + G\left(I - R^{2N}\right)(I + \tau C) \\
&\quad \times (2I + \tau C)^{-1} C^{-1} \sum_{i=1}^{N-1} \left(R^{|l-i|} - R^{l+i}\right) f_i^h(x)\tau \\
&\quad + G\left(I - R^{2N}\right)\left(\varphi^h(x) - \xi^h(x)\right) \\
&\quad + G\left(R^{N-l} - R^{N+l}\right)\left(\psi^h(x) - \varphi^h(x)\right),
\end{aligned} \tag{30}$$

for difference scheme (24),

$$\begin{aligned}
v_0^h(x) &= \left(\frac{\lambda}{\tau} - l - 1\right) G_1 \left(R^{N-l} - R^{N+l}\right) \\
&\quad \times (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{N-i} - R^{N+i}\right) f_i^h(x)\tau \\
&\quad - \left(\frac{\lambda}{\tau} - l - 1\right) G_1 \left(I - R^{2N}\right) \\
&\quad \times (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{|l-i|} - R^{l+i}\right) f_i^h(x)\tau \\
&\quad - \left(\frac{\lambda}{\tau} - l\right) G_1 \left(R^{N-l-1} - R^{N+l+1}\right) \\
&\quad \times (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{N-i} - R^{N+i}\right) f_i^h(x)\tau \\
&\quad + \left(\frac{\lambda}{\tau} - l\right) G_1 \left(I - R^{2N}\right) \\
&\quad \times (I + \tau C)(2I + \tau C)^{-1} C^{-1} \\
&\quad \times \sum_{i=1}^{N-1} \left(R^{|l+1-i|} - R^{l+1+i}\right) f_i^h(x)\tau \\
&\quad + G_1 \left(I - R^{2N}\right)\left(\varphi^h(x) - \xi^h(x)\right)
\end{aligned}$$

$$+\left(\left(\frac{\lambda}{\tau}-l-1\right)G_1\left(R^{N-l}-R^{N+l}\right)\right.$$
$$\left.+\left(\frac{\lambda}{\tau}-l\right)G_1\left(R^{N-l-1}-R^{N+l+1}\right)\right)$$
$$\times\left(\psi^h(x)-\varphi^h(x)\right), \tag{31}$$

for difference scheme (25), and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 8 (see [28]). *For the solution of the elliptic difference problem*

$$A_h^x u^h(x)=\omega^h(x), \quad x\in\widetilde{\Omega}_h,$$
$$D^h u^h(x)=0, \quad x\in S_h, \tag{32}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n}\left\|(u_k^h)_{\bar{x}_r\bar{x}_r,j_r}\right\|_{L_{2h}}\le M\left\|\omega^h\right\|_{L_{2h}}, \tag{33}$$

where M does not depend on h and ω^h.

4. Numerical Results

We have not been able to obtain a sharp estimate for the constants figuring in the stability estimates. Therefore, we will give the following results of numerical experiments of the inverse problem for the two-dimensional elliptic equation with Dirichlet-Neumann boundary conditions

$$-\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial}{\partial x}\left((2+\cos x)\frac{\partial u(t,x)}{\partial x}\right)+u(t,x)$$
$$=f(t,x)+p(x), \quad 0<x<\pi,\ 0<t<T,$$
$$f(t,x)=-\exp(-t)\cos(x)$$
$$+\left(\exp(-t)+t\right)(3\cos(x)+\cos(2x)),$$
$$u(0,x)=2\cos(x), \quad 0\le x\le\pi,$$
$$u(T,x)=\left(\exp(-T)+T+1\right)\cos(x), \quad 0\le x\le\pi,$$
$$u(\lambda,x)=\left(\exp(-\lambda)+\lambda+1\right)\cos(x), \quad 0\le x\le\pi,$$
$$u_x(t,0)=u_x(t,\pi)=0, \quad 0\le t\le T,\ \lambda=\frac{3T}{5}. \tag{34}$$

It is clear that $u(t,x)=(\exp(-t)+t+1)\cos(x)$ and $p(x)=\sin(x)+(x+2)\cos(x)$ are the exact solutions of (34).

We can obtain $u(t,x)$ by formula $u(t,x)=v(t,x)+w(t,x)$, where $v(t,x)$ is the solution of the nonlocal boundary value problem

$$-\frac{d^2 v(t,x)}{dt^2}-\frac{\partial}{\partial x}\left((2+\cos x)\frac{\partial v(t,x)}{\partial x}\right)+v(t,x)$$
$$=f(t,x), \quad 0<x<\pi,\ 0<t<T,$$
$$v(0,x)-v(\lambda,x)=\left(1-\exp(-\lambda)-\lambda\right)\cos(x),$$
$$0\le x\le\pi$$
$$v(T,x)-v(\lambda,x)$$
$$=\left(\exp(-T)-\exp(-\lambda)+T-\lambda\right)\cos(x),$$
$$0\le x\le\pi,$$
$$v_x(t,0)=v_x(t,\pi)=0, \quad 0\le t\le T, \tag{35}$$

and $w(t,x)$ is the solution of the boundary value problem

$$-\frac{d^2 w(t,x)}{dt^2}-\frac{\partial}{\partial x}\left((2+\cos x)\frac{\partial w(t,x)}{\partial x}\right)+w(t,x)$$
$$=p(x), \quad 0<x<\pi,\ 0<t<T,$$
$$w(0,x)=\left(\exp(-\lambda)+\lambda+1\right)\cos(x)-v(\lambda,x),$$
$$0\le x\le\pi,$$
$$w(T,x)=\left(\exp(-\lambda)+\lambda+1\right)\cos(x)-v(\lambda,x),$$
$$0\le x\le\pi,$$
$$w_x(t,0)=w_x(t,\pi)=0, \quad 0\le t\le T. \tag{36}$$

Introduce small parameters τ and h such that $N\tau=T$, $Mh=\pi$. For approximate solution of nonlocal boundary value problem (35) we consider the set $[0,T]_\tau\times[0,\pi]_h$ of a family of grid points

$$[0,T]_\tau\times[0,\pi]_h$$
$$=\left\{(t_k,x_n): t_k=k\tau,\right.$$
$$\left.k=0,\dots,N,\ x_n=nh,\ n=0,\dots,M\right\}. \tag{37}$$

Applying (21), we obtain difference schemes of the first order of accuracy in t and the second order of accuracy in x

$$\begin{gathered}\frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+\left(2+\cos\left(x_n\right)\right)\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}\\ -\sin\left(x_n\right)\frac{v_{n+1}^k-v_{n-1}^k}{2h}-v_n^k=\theta_n^k,\\ \theta_n^k=-f\left(t_k,x_n\right),\quad k=1,\ldots,N-1,\ n=1,\ldots,M-1,\\ v_0^k-v_1^k=v_M^k-v_{M-1}^k=0,\quad k=0,\ldots,N,\\ v_n^0-v_n^l=\left(1-\exp\left(-\lambda\right)-\lambda\right)\cos\left(x_n\right),\quad n=0,\ldots,M,\\ v_n^N-v_n^l=\left(\exp\left(-t_N\right)-\exp\left(-\lambda\right)+t_N-\lambda\right)\cos\left(x_n\right),\\ n=0,\ldots,M,\quad l=\left[\frac{\lambda}{\tau}\right],\end{gathered}\tag{38}$$

for the approximate solutions of the nonlocal boundary value problem (35), and

$$\begin{gathered}\frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}+\left(2+\cos\left(x_n\right)\right)\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}\\ -\sin\left(x_n\right)\frac{w_{n+1}^k-w_{n-1}^k}{2h}-w_n^k\\ =-p_n,\quad k=1,\ldots,N-1,\\ p_n=p\left(x_n\right),\ n=1,\ldots,M-1,\\ w_0^k-w_1^k=w_M^k-w_{M-1}^k=0,\quad k=0,\ldots,N,\\ w_n^0=\left(\exp\left(-\lambda\right)+\lambda+1\right)\cos\left(x_n\right)-v_n^l,\\ n=0,\ldots,M,\ l=\left[\frac{\lambda}{\tau}\right],\\ w_n^N=\left(\exp\left(-\lambda\right)+\lambda+1\right)\cos\left(x_n\right)-v_n^l,\\ n=0,\ldots,M,\end{gathered}\tag{39}$$

for the approximate solutions of the boundary value problem (36).

By using (22) and second order of accuracy in x approximation of A, we get the following values of p in grid points:

$$\begin{gathered}p_n=-\frac{\left(2+\cos\left(x_n\right)\right)}{h^2}\left(\left(\varphi_{n+1}-v_{n+1}^0\right)-2\left(\varphi_n-v_n^0\right)\right.\\ \left.+\left(\varphi_{n-1}-v_{n-1}^0\right)\right)+\frac{\sin\left(x_n\right)}{2h}\\ \times\left(\left(\varphi_{n+1}-v_{n+1}^0\right)-\left(\varphi_{n-1}-v_{n-1}^0\right)\right)\\ +\left(\varphi_n-v_n^0\right),\quad n=1,\ldots,M-1.\end{gathered}\tag{40}$$

We can rewrite difference scheme (38) in the matrix form

$$\begin{gathered}A_nv_{n+1}+B_nv_n+C_nv_{n-1}=I\theta_n^k,\quad n=1,\ldots,M-1,\\ v_0=v_1,\qquad v_M=v_{M-1}.\end{gathered}\tag{41}$$

Here, I is the $(N+1)\times(N+1)$ identity matrix, A_n, B_n, C_n are $(N+1)\times(N+1)$ square matrices, and θ_n is a $(N+1)\times 1$ column matrix which are defined by

$$A_n=\begin{bmatrix}0&0&0&0&\cdots&0&0&0&0\\0&a_n&0&0&\cdots&0&0&0&0\\0&0&a_n&0&\cdots&0&0&0&0\\0&0&0&a_n&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&a_n&0&0&0\\0&0&0&0&\cdots&0&a_n&0&0\\0&0&0&0&\cdots&0&0&a_n&0\\0&0&0&0&\cdots&0&0&0&0\end{bmatrix},$$

$$B_n=\begin{bmatrix}1&0&0&0&\cdots&-1&\cdots&0&0&0&0\\d&b_n&d&0&\cdots&0&\cdots&0&0&0&0\\0&d&b_n&d&\cdots&0&\cdots&0&0&0&0\\0&0&d&b_n&\cdots&0&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&0&\cdots&b_n&d&0&0\\0&0&0&0&\cdots&0&\cdots&d&b_n&d&0\\0&0&0&0&\cdots&0&\cdots&0&d&b_n&d\\0&0&0&0&\cdots&-1&\cdots&0&0&0&1\end{bmatrix},\tag{42}$$

$$C_n=\begin{bmatrix}0&0&0&0&\cdots&0&0&0&0\\0&c_n&0&0&\cdots&0&0&0&0\\0&0&c_n&0&\cdots&0&0&0&0\\0&0&0&c_n&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&c_n&0&0&0\\0&0&0&0&\cdots&0&c_n&0&0\\0&0&0&0&\cdots&0&0&c_n&0\\0&0&0&0&\cdots&0&0&0&0\end{bmatrix},\tag{43}$$

$$\begin{gathered}a_n=\frac{2+\cos\left(x_n\right)}{h^2}-\frac{\sin\left(x_n\right)}{2h},\\ b_n=-\frac{2}{\tau^2}-\frac{2\left(2+\cos\left(x_n\right)\right)}{h^2}-1,\\ c_n=\frac{2+\cos\left(x_n\right)}{h^2}+\frac{\sin\left(x_n\right)}{2h},\qquad d=\frac{1}{\tau^2},\end{gathered}\tag{44}$$

$$\theta_n = \begin{bmatrix} \theta_n^0 \\ \vdots \\ \theta_n^N \end{bmatrix},$$

$$\theta_n^0 = (1 - \exp(-\lambda) - \lambda)\cos(x_n),$$

$$\theta_n^N = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda)\cos(x_n),$$

$$n = 1, \ldots, M-1,$$

$$\theta_n^k = -f(t_k, x_n), \quad k = 1, \ldots, N-1, \ n = 1, \ldots, M-1,$$

$$v_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1. \tag{45}$$

For solving (41) we use the modified Gauss elimination method (see [29]). Namely, we seek solution of (41) by the formula

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 1, \tag{46}$$

where $v_M = \vec{0}$, α_n $(n = 1, \ldots, M-1)$ are $(N+1)\times(N+1)$ square matrices and β_n $(n = 1, \ldots, M-1)$ are $(N+1)\times 1$ column matrices. For $\alpha_{n+1}, \beta_{n+1}$, we get formulas

$$\alpha_{n+1} = -(B_n + C_n\alpha_n)^{-1} A_n,$$

$$\beta_{n+1} = -(B_n + C_n\alpha_n)^{-1}(I\theta_n - C_n\beta_n), \quad n = 1, \ldots, M-1, \tag{47}$$

where α_1 is the $(N+1)\times(N+1)$ identity matrix and β_1 is the $(N+1)\times 1$ zero column vector.

Futher, we rewrite difference scheme (39) in the matrix form

$$A_n w_{n+1} + E_n w_n + C_n w_{n-1} = I\eta_n^k,$$

$$n = 1, \ldots, M-1, \tag{48}$$

$$w_0 = w_1, \qquad w_M = w_{M-1}.$$

Here,

$$E_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & b_n & d & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & b_n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & d & b_n & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & b_n & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{49}$$

A_n and C_n are defined by (42) and (43) and $(N+1)\times 1$ column matrix η_n is defined by

$$\eta_n = \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix},$$

$$\eta_n^0 = (\exp(-\lambda) + \lambda + 1)\cos(x_n) - v_n^l,$$

$$\eta_n^N = (\exp(-\lambda) + \lambda + 1)\cos(x_n) - v_n^l, \quad n = 1, \ldots, M-1,$$

$$\eta_n^k = -p_n, \quad k = 1, \ldots, N-1, \ n = 1, \ldots, M-1,$$

$$w_s = \begin{bmatrix} w_s^0 \\ \vdots \\ w_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1. \tag{50}$$

Now we present second order of accuracy in t and x difference schemes for problems (35) and (36). Applying (27) and formulas for sufficiently smooth function ρ

$$\frac{\rho(x_{n+1}) - \rho(x_{n-1})}{2h} - \rho'(x_n) = O(h^2),$$

$$\frac{\rho(x_{n+1}) - 2\rho(x_n) + \rho(x_{n-1})}{h^2} - \rho''(x_n) = O(h^2),$$

$$\frac{10\rho(0) - 15\rho(h) + 6\rho(2h) - \rho(3h)}{h^3} - \rho'''(0) = O(h^2),$$

$$\frac{-3\rho(0) + 4\rho(h) - \rho(2h)}{2h} - \rho'(0) = O(h^2),$$

$$\frac{10\rho(\pi) - 15\rho(\pi - h) + 6\rho(\pi - 2h) - \rho(\pi - 3h)}{h^3}$$

$$- \rho'''(\pi) = O(h^2),$$

$$\frac{-3\rho(\pi) + 4\rho(\pi - h) - \rho(\pi - 2h)}{2h} - \rho'(\pi) = O(h^2), \tag{51}$$

we get

$$\frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + (2 + \cos(x_n))\frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2}$$

$$- \sin(x_n)\frac{v_{n+1}^k - v_{n-1}^k}{2h} - v_n^k = \theta_n^k,$$

$$\begin{aligned}
&\theta_n^k = -f(t_k, x_n), \quad k = 1, \ldots, N-1, \quad n = 1, \ldots, M-1, \\
&\quad -3v_0^k + 4v_1^k - v_2^k \\
&\quad = -3v_M^k + 4v_{M-1}^k - v_{M-2}^k = 0, \quad k = 0, \ldots, N, \\
&\quad 10v_0^k - 15v_1^k + 6v_2^k - v_3^k \\
&\quad = 10v_M^k - 15v_{M-1}^k + 6v_{M-2}^k - v_{M-3}^k = 0, \\
&v_n^0 + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1} \\
&\quad = (1 - \exp(-\lambda) - \lambda) \cos(x_n), \quad n = 0, \ldots, M, \\
&v_n^N + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1} \\
&\quad = (\exp(-t_N) - \exp(-\lambda) + t_N - \lambda) \cos(x_n), \\
&\qquad n = 0, \ldots, M,
\end{aligned} \tag{52}$$

difference scheme for nonlocal problem (35), and

$$\begin{aligned}
&\frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + (2 + \cos(x_n)) \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} \\
&\quad - \sin(x_n) \frac{w_{n+1}^k - w_{n-1}^k}{2h} = -p_n, \\
&k = 1, \ldots, N-1, \quad p_n = p(x_n), \quad n = 1, \ldots, M-1, \\
&\quad -3w_0^k + 4w_1^k - w_2^k = -3w_M^k + 4w_{M-1}^k - w_{M-2}^k = 0, \\
&\qquad k = 0, \ldots, N, \\
&\quad 10w_0^k - 15w_1^k + 6w_2^k - w_3^k \\
&\quad = 10w_M^k - 15w_{M-1}^k + 6w_{M-2}^k - w_{M-3}^k = 0, \\
&w_n^0 = (\exp(-\lambda) + \lambda + 1) \cos(x_n) \\
&\quad + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1}, \quad n = 0, \ldots, M, \\
&w_n^N = (\exp(-\lambda) + \lambda + 1) \cos(x_n) \\
&\quad + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1}, \\
&\xi_n = \xi(x_n), \quad n = 0, \ldots, M,
\end{aligned} \tag{53}$$

difference scheme for boundary value problem (36).

By difference scheme (52), we write in matrix form

$$\begin{aligned}
&A_n v_{n+1} + B_n v_n + C_n v_{n-1} = I\theta_n^k, \quad n = 1, \ldots, M-1, \\
&-3v_0 + 4v_1 - v_2 = 0, \\
&-3v_M + 4v_{M-1} - v_{M-2} = 0,
\end{aligned} \tag{54}$$

where A_n, C_n are defined by (42), (43), (44), and B_n is defined by

$$B_n = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 0 \\
d & b_n & d & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & d & b_n & d & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & d & b_n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & b_n & d & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d & b_n & d & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & d & b_n & d \\
0 & 0 & 0 & 0 & \cdots & 0 & y & z & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$\begin{aligned}
&b_n = -\frac{2}{\tau^2} - \frac{2(2 + \cos(x_n))}{h^2} - 1, \\
&d = \frac{1}{\tau^2}, \qquad y = \left(\frac{\lambda}{\tau} - l - 1\right), \qquad z = -\left(\frac{\lambda}{\tau} - l\right).
\end{aligned} \tag{55}$$

We seek solution of (54) by the formula

$$v_n = \alpha_n v_{n+1} + \beta_n v_{n+2} + \gamma_n, \quad n = M-2, \ldots, 0, \tag{56}$$

where α_n, β_n $(n = 0, \ldots, M-2)$ are $(N+1) \times (N+1)$ square matrices and γ_n $(n = 0, \ldots, M-2)$ are $(N+1) \times 1$ column matrices. For the solution of difference equation (41) we need to use the following formulas for α_n, β_n:

$$\begin{aligned}
&\alpha_n = -(B_n + C_n \alpha_{n-1})^{-1} (A_n + C_n \beta_{n-1}), \\
&\beta_n = 0, \\
&\gamma_n = -(B_n + C_n \alpha_{n-1})^{-1} (I\theta_n - C_n \gamma_{n-1}), \quad n = 1, \ldots, M-1,
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
&\alpha_0 = \frac{4}{3} I, \qquad \beta_0 = -\frac{1}{3} I, \\
&\alpha_1 = \frac{8}{5} I, \qquad \beta_1 = -\frac{3}{5} I, \\
&\alpha_{M-2} = 4I, \qquad \beta_{M-2} = -3I, \\
&\alpha_{M-3} = \frac{8}{3} I, \qquad \beta_{M-3} = -\frac{5}{3} I,
\end{aligned} \tag{58}$$

and $\gamma_0, \gamma_1, \gamma_{M-2}, \gamma_{M-3}$ are the $(N+1) \times 1$ zero column vector. For v_M and v_{M-1} we have

$$\begin{aligned}
&v_M = \left(Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}\right)^{-1} \left(G_1 - Q_{12} Q_{22}^{-1} G_2\right), \\
&v_{M-1} = Q_{22}^{-1} (G_2 - Q_{21} v_M),
\end{aligned} \tag{59}$$

where

$$Q_{11} = -3A_{M-2} - 8B_{M-2} - 8C_{M-2}\alpha_{M-3} - 3C_{M-2}\beta_{M-3},$$
$$Q_{12} = 4A_{M-2} + 9B_{M-2} + 9C_{M-2}\alpha_{M-3} + 4C_{M-2}\beta_{M-3},$$
$$Q_{21} = -3B_{M-1} - 8C_{M-1},$$
$$Q_{22} = A_{M-1} + 4B_{M-1} + 9C_{M-1},$$
$$G_1 = I\theta_{M-2} - C_{M-2}\gamma_{M-3}, \qquad G_2 = I\theta_{M-1}. \tag{60}$$

We can rewrite difference scheme (53) in the matrix form

$$\begin{gathered} A_n w_{n+1} + E_n w_n + C_n w_{n-1} = I\eta_n^k, \quad n = 1, \ldots, M-1, \\ -3w_0 + 4w_1 - w_2 = 0, \\ -3w_M + 4w_{M-1} - w_{M-2} = 0, \end{gathered} \tag{61}$$

where A_n, E_n, C_n are defined by (42), (49), (43), and (44) and η_n is defined by

$$\eta_n = \begin{bmatrix} \eta_n^0 \\ \vdots \\ \eta_n^N \end{bmatrix},$$
$$\begin{aligned} \eta_n^0 &= (\exp(-\lambda) + \lambda + 1)\cos(x_n) \\ &\quad + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1}, \\ \eta_n^N &= (\exp(-\lambda) + \lambda + 1)\cos(x_n) + \left(\frac{\lambda}{\tau} - l - 1\right) v_n^l \\ &\quad - \left(\frac{\lambda}{\tau} - l\right) v_n^{l+1}, \quad n = 0, \ldots, M, \\ \eta_n^k &= -p_n, \quad k = 1, \ldots, N-1, \quad n = 1, \ldots, M-1. \end{aligned} \tag{62}$$

Now, we give the results of the numerical realization of finite difference method for (34) by using MATLAB programs. The numerical solutions are recorded for $T = 2$ and different values of $N = M$. Grid functions v_n^k, u_n^k represent the numerical solutions of difference schemes for auxiliary nonlocal problem (35) and inverse problem (34) at (t_k, x_n), respectively. Grid function p_n calculated by (40) represents numerical solution at x_n for unknown function p. The errors are computed by the norms

$$\begin{aligned} Ev_M^N &= \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|v(t_k, x_n) - v_n^k\right|^2 h\right)^{1/2}, \\ Eu_M^N &= \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k, x_n) - u_n^k\right|^2 h\right)^{1/2}, \\ Ep_M &= \left(\sum_{n=1}^{M-1}\left|p(x_n) - p_n\right|^2 h\right)^{1/2}. \end{aligned} \tag{63}$$

Table 1: Error analysis for nonlocal problem.

	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
Difference scheme (38)	0.30522	0.14933	0.073953	0.036814
Difference scheme (52)	0.024714	0.0031054	4.36×10^{-4}	7.52×10^{-5}

Table 2: Error analysis for p.

	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
Difference scheme (38), (40)	0.57878	0.33755	0.20387	0.12905
Difference scheme (52), (40)	0.058201	0.010646	0.0020228	4.03×10^{-4}

Table 3: Error analysis for u.

	$N = M = 20$	$N = M = 40$	$N = M = 80$	$N = M = 160$
Difference scheme (38), (40), (39)	0.088225	0.038586	0.01815	0.008818
Difference scheme (52), (40), (53)	0.017034	0.0020225	2.47×10^{-4}	3.08×10^{-5}

Tables 1–3 present the error between the exact solution and numerical solutions derived by corresponding difference schemes. The results are recorded for $N = M = 20, 40, 80$ and 160, respectively. The tables show that the second order of accuracy difference scheme is more accurate than the first order of accuracy difference scheme for both auxiliary nonlocal and inverse problems. Table 1 contains error between the exact and approximate solutions v of auxiliary nonlocal boundary value problem (35). Table 2 includes error between the exact and approximate solutions p of inverse problem (34). Table 3 represents error between the exact solution u of inverse problem (34) and approximate solution which is derived by the first and second orders accuracy of difference schemes.

5. Conclusion

In this paper, inverse problem for multidimensional elliptic equation with Dirichlet-Neumann conditions is considered. The stability and coercive stability estimates for solution of this problem are established. First and second order of accuracy difference schemes are presented for approximate solutions of inverse problem. Theorems on the stability and coercive stability inequalities for difference schemes are proved. The theoretical statements for the solution of these difference schemes are supported by the results of numerical example in a two-dimensional case. As it can be seen from Tables 1–3, second order of accuracy difference

scheme is more accurate compared with the first order of accuracy difference scheme. Moreover, applying the result of the monograph [29] the high order of accuracy difference schemes for the numerical solution of the boundary value problem (2) can be presented.

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