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Research Article

Determination of a Control Parameter for the Difference Schrödinger Equation

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The first order of accuracy difference scheme for the numerical solution of the boundary value problem for the differential equation with parameter p, $i(du(t)/dt) + Au(t) + iu(t) = f(t) + p$, $0 < t < T$, $u(0) = \varphi$, $u(T) = \psi$, in a Hilbert space H with self-adjoint positive definite operator A is constructed. The well-posedness of this difference scheme is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained.

1. Introduction: Difference Scheme

The theory and applications of well-posedness of inverse problems for partial differential equations have been studied extensively in a large cycle of papers (see, e.g., [1–24] and the references therein).

Our goal in this paper is to investigate Schrödinger equations with parameter. In the paper [25], the boundary value problem for the differential equation with parameter p

$$i\frac{du(t)}{dt} + Au(t) + iu(t) = f(t) + p, \quad 0 < t < T, \qquad u(0) = \varphi, \quad u(T) = \psi \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A was studied. The well-posedness of this problem was established. The stability inequalities for the solution of three determinations of control parameter problems for the Schrödinger equation were obtained. In the present paper, the first order of accuracy Rothe difference scheme

$$\begin{gathered} i\tau^{-1}(u_k - u_{k-1}) + Au_k + iu_k = \varphi_k + p, \quad \varphi_k = f(t_k), \\ t_k = k\tau, \quad 1 \le k \le N, \quad N\tau = T, \\ u_0 = \varphi, \quad u_N = \psi \end{gathered} \tag{2}$$

for the approximate solution of the boundary value problem (1) for the differential equation with parameter p is presented. It is easy to see that

$$\begin{gathered} u_k = v_k + (A + iI)^{-1}p, \\ p = (A + iI)(\psi - v_N), \end{gathered} \tag{3}$$

where $\{v_k\}_{k=0}^N$ is the solution of the following single-step difference scheme:

$$\begin{gathered} i\tau^{-1}(v_k - v_{k-1}) + Av_k + iv_k = \varphi_k, \quad \varphi_k = f(t_k), \\ t_k = k\tau, \quad 1 \le k \le N, \quad N\tau = T, \\ v_0 - v_N = \varphi - \psi. \end{gathered} \tag{4}$$

The theorem on well-posedness of difference problem (2) is proved. In practice, the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control parameter problems are obtained.

The paper is organized as follows. Section 1 is the introduction. In Section 2, the main theorem on stability of difference problem (2) is established. In Section 3, theorems on the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control

parameter problems are obtained. In Section 4, numerical results are given. Finally, Section 5 is the conclusion.

2. The Main Theorem on Stability

In this section, we will study the stability of difference scheme (2).

Let $[0,T]_\tau = \{t_k = k\tau, k = 1,\dots,N, N\tau = T\}$ be the uniform grid space with step size $\tau > 0$, where N is a fixed positive integer. Throughout the present paper, $\mathcal{F}([0,T]_\tau, H)$ denotes the linear space of grid functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Hilbert space H. Let $\mathcal{C}_\tau(H) = \mathcal{C}([0,T]_\tau, H)$ be the Banach space of bounded grid functions with the norm

$$\|\varphi^\tau\|_{\mathcal{C}_\tau(H)} = \max_{1\le k\le N}\|\varphi_k\|_H. \tag{5}$$

Let us start with a lemma we need below. We denote that $R = ((1+\tau)I - i\tau A)^{-1}$ is the step operator of problem (2).

Lemma 1. *Assume that A is a positive definite self-adjoint operator. The operator $I - R^N$ has an inverse $T_\tau = (I - R^N)^{-1}$ and the following estimate is satisfied:*

$$\|T_\tau\|_{H\to H} \le M(\delta). \tag{6}$$

Proof. The proof of estimate (6) is based on the triangle inequality and the estimate

$$\begin{aligned}\left\|\left(I - R^N\right)^{-1}\right\|_{H\to H} &\le \sup_{\delta\le\mu}\frac{1}{1 - \left|\left(1+\tau\left(1-i\mu\right)\right)^{-N}\right|}\\ &\le \frac{1}{1 - \left((1+\tau)^2 + \left(\tau\delta\right)^2\right)^{-N/2}} \le \mu(\delta).\end{aligned} \tag{7}$$

Now, let us obtain the formula for the solution of problem (2). It is clear that the first order of accuracy difference scheme

$$\begin{gathered} i\tau^{-1}\left(u_k - u_{k-1}\right) + Au_k + iu_k = p + \varphi_k, \quad \varphi_k = f\left(t_k\right),\\ t_k = k\tau, \quad 1\le k\le N, \quad N\tau = T, \quad u_0 = \varphi \end{gathered} \tag{8}$$

has a solution and the following formula

$$u_k = R^k\varphi - i\sum_{j=1}^{k} R^{k-j+1}\left(p + \varphi_j\right)\tau, \quad 1\le k\le N \tag{9}$$

is satisfied. Applying formula (9) and the boundary condition

$$u_N = \psi, \tag{10}$$

we can write

$$\psi = R^N\varphi - i\sum_{j=1}^{N} R^{N-j+1}\varphi_j\tau - i\sum_{j=1}^{N} R^{N-j+1}\tau p. \tag{11}$$

Since

$$\begin{aligned}-i\sum_{j=1}^{N} R^{N-j+1}\tau &= -i(I - iA)^{-1}(I - R)\sum_{j=1}^{N} R^{N-j}\\ &= -i(I - iA)^{-1}\left(I - R^N\right),\end{aligned} \tag{12}$$

we have that

$$\psi = R^N\varphi - i\sum_{j=1}^{N} R^{N-j+1}\varphi_j\tau - i(I - iA)^{-1}\left(I - R^N\right)p. \tag{13}$$

By Lemma 1, we get

$$\begin{aligned}p = T_\tau\Bigg(&(I - iA)\psi - (I - iA)R^N\varphi\\ &- \sum_{j=1}^{N}(I - iA)R^{N-j+1}\varphi_j\tau\Bigg).\end{aligned} \tag{14}$$

Using (9) and (14), we get

$$\begin{aligned}u_k = {}& R^k\varphi - i\sum_{j=1}^{k} R^{k-j+1}\varphi_j\tau\\ &+ \sum_{j=1}^{k} R^{k-j+1}\tau T_\tau\Bigg((I - iA)\psi - (I - iA)R^N\varphi\\ &- \sum_{j=1}^{N}(I - iA)R^{N-j+1}\varphi_j\tau\Bigg),\\ &1\le k\le N.\end{aligned} \tag{15}$$

Since

$$\begin{aligned}\sum_{j=1}^{k} R^{k-j+1}\tau &= (I - iA)^{-1}(I - R)\sum_{j=1}^{k} R^{k-j}\\ &= (I - iA)^{-1}\left(I - R^k\right),\end{aligned} \tag{16}$$

we have that

$$\begin{aligned}u_k = {}& R^k\varphi + \sum_{j=1}^{k} R^{k-j+1}\varphi_j\tau\\ &+ \left(I - R^k\right)T_\tau\left(\psi - R^N\varphi - \sum_{j=1}^{N} R^{N-j+1}\varphi_j\tau\right),\\ &1\le k\le N.\end{aligned} \tag{17}$$

Hence, difference scheme (2) is uniquely solvable and for the solution, formulas (14) and (17) hold. □

Theorem 2. *Suppose that the assumption of Lemma 1 holds. Let $\varphi, \psi \in D(A)$. Then, for the solution $(\{u_k\}_{k=1}^N, p)$ of difference scheme* (2) *in $C_\tau(H) \times H$, the estimates*

$$\|p\|_H \le M \left[\|A\varphi\|_H + \|A\psi\|_H + \|\varphi_1\|_H + \max_{2\le k\le N} \left\| \frac{\varphi_k - \varphi_{k-1}}{\tau} \right\|_H \right], \tag{18}$$

$$\left\| \{u_k\}_{k=1}^N \right\|_{C_\tau(H)} \le M \left[\|\varphi\|_H + \|\psi\|_H + \left\| \{\varphi_k\}_{k=1}^N \right\|_{C_\tau(H)} \right] \tag{19}$$

hold, where M is independent of τ, φ, ψ, and $\{\varphi_k\}_{k=1}^N$.

Proof. From formulas (9) and (14), it follows that

$$p = T_\tau \left[A\psi - R^N A\varphi - \varphi_N + R^N \varphi_1 - \sum_{j=2}^N R^{N-j+1} \left(\varphi_{j-1} - \varphi_j \right) \right]. \tag{20}$$

Using this formula, the triangle inequality, and estimate (6), we obtain

$$\begin{aligned} \|p\|_H &\le \|T_\tau\|_{H\to H} \left(\|A\psi\|_H + \left\|R^N\right\|_{H\to H} \|A\varphi\|_H + \sum_{j=2}^N \left\|R^{N-j+1}\right\|_{H\to H} \|\varphi_j - \varphi_{j-1}\|_H + \|\varphi_N\| + \left\|R^N\right\|_{H\to H} \|\varphi_1\|_H \right) \\ &\le M \left[\|A\varphi\|_H + \|A\psi\|_H + \left\| \{\varphi_k\}_{k=1}^N \right\|_{C_\tau^{(1)}(H)} \right]. \end{aligned} \tag{21}$$

Estimate (18) is proved. Using formula (17), the triangle inequality, and estimate (6), we obtain

$$\begin{aligned} \|u_k\|_H &\le \left[\left\|R^k\right\|_{H\to H} \|\varphi\|_H + \sum_{j=1}^k \left\|R^{k-j+1}\right\|_{H\to H} \|\varphi_j\|_H \tau + \left(1 + \left\|R^k\right\|_{H\to H}\right) \|T_\tau\|_{H\to H} \times \left(\|\psi\|_H + \left\|R^N\right\|_{H\to H} \|\varphi\|_H + \sum_{j=1}^N \left\|R^{N-j+1}\right\|_{H\to H} \|\varphi_j\|_H \tau \right) \right] \\ &\le M \left[\|\varphi\|_H + \|\psi\|_H + \left\| \{\varphi_k\}_{k=1}^N \right\|_{C_\tau(H)} \right] \end{aligned} \tag{22}$$

for any k. From that, it follows estimate (19). This completes the proof of Theorem 2. □

3. Applications

Now, we consider the simple applications of main Theorem 2.

First, the boundary value problem for the Schrödinger equation

$$\begin{gathered} iu_t - \left(a(x) u_x\right)_x + \delta u + iu = p(x) + f(t,x), \\ 0 < t < T, \quad 0 < x < 1, \\ u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad 0 \le x \le 1, \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad 0 \le t \le T \end{gathered} \tag{23}$$

is considered. Problem (23) has a unique smooth solution $(u(t,x), p(x))$ for the smooth functions $a(x) \ge a > 0$, $x \in (0,1)$, $\delta > 0$, $a(1) = a(0)$, $\varphi(x)$, $\psi(x)$ $(x \in [0,1])$, and $f(t,x)$ $(t \in (0,T)$, $x \in (0,1))$. This allows us to reduce the boundary value problem (23) to the boundary value problem (1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\left(a(x) u_x\right)_x + \delta u \tag{24}$$

with domain

$$\begin{gathered} D(A^x) = \{u(x) : u(x), u_x(x), \left(a(x) u_x\right)_x \in L_2[0,1], \\ u(1) = u(0), u_x(1) = u_x(0)\}. \end{gathered} \tag{25}$$

The discretization of problem (23) is carried out in two steps. In the first step, we define the grid space

$$[0,1]_h = \{x = x_n : x_n = nh, 0 \le n \le M, Mh = 1\}. \tag{26}$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,1]_h)$ of the grid functions

$$\varphi^h(x) = \{\varphi_n\}_1^{M-1} \tag{27}$$

defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,1]_h} |\varphi(x)|^2 h \right)^{1/2}. \tag{28}$$

To the differential operator A^x defined by formula (24), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-\left(a(x) \varphi_{\bar{x}}\right)_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{29}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the boundary value problem

$$\begin{gathered} i\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) + iu^h(t,x) = p^h(x) + f^h(t,x), \\ 0 < t < T, \quad x \in [0,1]_h, \\ u^h(0,x) = \varphi^h(x), \quad u^h(T,x) = \psi^h(x), \quad x \in [0,1]_h. \end{gathered} \tag{30}$$

In the second step, we replace (30) with the difference scheme (2)

$$i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)+iu_k^h(x)=p^h(x)+f_k^h(x),$$
$$f_k^h(x)=f^h(t_k,x),\quad t_k=k\tau,\quad N\tau=T,$$
$$1\le k\le N,\quad x\in[0,1]_h,$$
$$u^h(0,x)=\varphi^h(x),\quad u^h(T,x)=\psi^h(x),\quad x\in[0,1]_h. \tag{31}$$

Theorem 3. *The solution pairs* $(\{u_k^h(x)\}_0^N, p^h(x))$ *of problem* (31) *satisfy the stability estimates*

$$\left\|p^h\right\|_{L_{2h}}\le M_1\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|A_h^x\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^{(1)}(L_{2h})}\right],$$
$$\left\|\left\{u_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\le M_2\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right], \tag{32}$$

where M_1 *and* M_2 *do not depend on* φ^h, ψ^h, *and* f_k^h, $1\le k\le N$. *Here,* $C_\tau^{(1)}(L_{2h})$ *is the grid space of grid functions* $\{f_k^h\}_1^N$ *defined on* $[0,T]_\tau\times[0,1]_h$ *with norm*

$$\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^{(1)}(L_{2h})}=\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}+\sup_{1\le k<k+r\le N}\frac{\left\|f_{k+r}^h-f_k^h\right\|_{L_{2h}}}{r\tau},$$
$$\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}=\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}. \tag{33}$$

The proof of Theorem 3 is based on formulas for $p^h(x)$ and $\{u_k^h(x)\}_1^N$ and the symmetry property of operator A_h^x.

Second, let $\Omega=(x=(x_1,\dots,x_n):0<x_k<1,\ k=1,\dots,n)$ be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,T]\times\Omega$, the boundary value problem for the multidimensional Schrödinger equation

$$i\frac{\partial u(t,x)}{\partial t}-\sum_{r=1}^n\left(a_r(x)u_{x_r}\right)_{x_r}+iu=p(x)+f(t,x),$$
$$x=(x_1,\dots,x_n)\in\Omega,\quad 0<t<T, \tag{34}$$
$$u(0,x)=\varphi(x),\quad u(T,x)=\psi(x),\quad x\in\overline{\Omega},$$
$$u(t,x)=0,\quad x\in S,\ 0\le t\le T$$

is considered. Here, $a_r(x)\ge a>0$ $(x\in\Omega)$, $f(t,x)$ $(t\in(0,T)$, $x\in\Omega)$, and $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$ are given smooth functions.

We consider the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})}=\left(\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2dx_1\cdots dx_n\right)^{1/2}. \tag{35}$$

Problem (34) has a unique smooth solution $(u(t,x),p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$, and $f(t,x)$. This allows us to reduce the problem (34) to the boundary value problem (1) in the Hilbert space $H=L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by the formula

$$A^xu(x)=-\sum_{r=1}^n\left(a_r(x)u_{x_r}\right)_{x_r} \tag{36}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),\left(a_r(x)u_{x_r}\right)_{x_r}\in L_2\left(\overline{\Omega}\right),\right.$$
$$\left.1\le r\le n,\ u(x)=0,x\in S\right\}. \tag{37}$$

The discretization of problem (34) is carried out in two steps. In the first step, we define the grid space

$$\overline{\Omega}_h=\{x=x_r=(h_1j_1,\dots,h_nj_n),$$
$$j=(j_1,\dots,j_n),\ 0\le j_r\le N_r,$$
$$N_rh_r=1,\ r=1,\dots,n\},$$
$$\Omega_h=\overline{\Omega}_h\cap\Omega,\quad S_h=\overline{\Omega}_h\cap S \tag{38}$$

and introduce the Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ of the grid functions

$$\varphi^h(x)=\{\varphi(h_1j_1,\dots,h_nj_n)\} \tag{39}$$

defined on $\overline{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{x\in\Omega_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n\right)^{1/2}. \tag{40}$$

To the differential operator A^x defined by formula (36), we assign the difference operator A_h^x by the formula

$$A_h^xu^h=-\sum_{r=1}^n\left(\alpha_r(x)u_{x_r}^h\right)_{x_r,j_r}, \tag{41}$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x)=0$ for all $x\in S_h$. With the help of the difference operator A_h^x, we arrive to the following boundary value problem:

$$iu_t^h(t,x)+A_h^xu^h(t,x)+iu^h(t,x)=p^h(x)+f^h(t,x),$$
$$0<t<T,\quad x\in\Omega_h,$$
$$u^h(0,x)=\varphi^h(x),\quad u^h(T,x)=\psi^h(x),\quad x\in\Omega_h \tag{42}$$

for an infinite system of ordinary differential equations.

The first order of accuracy difference scheme for the solution of problem (42) is

$$\begin{gathered} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)+iu_k^h(x) \\ =p^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k,x), \quad t_k=k\tau, \quad N\tau=T, \\ 1\le k\le N, \quad x\in\Omega_h, \\ u^h(0,x)=\varphi^h(x), \quad u^h(T,x)=\psi^h(x), \quad x\in\Omega_h. \end{gathered} \tag{43}$$

Theorem 4. *The solution pairs* $(\{u_k^h(x)\}_0^N, p^h(x))$ *of problem* (43) *satisfy the stability estimates*

$$\begin{gathered} \left\|p^h\right\|_{L_{2h}}\le M_1\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}\right. \\ \left.+\left\|A_h^x\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^{(1)}(L_{2h})}\right], \\ \left\|\left\{u_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\le M_2\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right], \end{gathered} \tag{44}$$

where M_1 *and* M_2 *do not depend on* φ^h, ψ^h, *and* f_k^h, $1\le k\le N$. *Here,* $C_\tau^{(1)}(L_{2h})$ *is the grid space of grid functions* $\{f_k^h\}_1^N$ *defined on* $[0,T]_\tau\times\overline{\Omega_h}$ *with norm*

$$\begin{gathered} \left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^{(1)}(L_{2h})}=\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}+\sup_{1\le k<k+r\le N}\frac{\left\|f_{k+r}^h-f_k^h\right\|_{L_{2h}}}{r\tau}, \\ \left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}=\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}. \end{gathered} \tag{45}$$

The proof of Theorem 4 is based on Theorem 3 and the symmetry property of the operator A_h^x is defined by formula (34) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 5. *For the solutions of the elliptic difference problem [26]*

$$\begin{gathered} A_h^x u^h(x)=\omega^h(x), \quad x\in\Omega_h, \\ u^h(x)=0, \quad x\in S_h, \end{gathered} \tag{46}$$

the following coercivity inequality holds:

$$\sum_{r=1}^n\left\|u_{x_r x_{\bar r}}^h\right\|_{L_{2h}}\le M\left\|\omega^h\right\|_{L_{2h}}, \tag{47}$$

where M *does not depend on* h *and* ω^h.

Third, in $[0,T]\times\Omega$, the boundary value problem for the multidimensional Schrödinger equation

$$\begin{gathered} i\frac{\partial u(t,x)}{\partial t}-\sum_{r=1}^n\left(a_r(x)u_{x_r}\right)_{x_r}+\delta u+iu=p(x)+f(t,x), \\ x=(x_1,\dots,x_n)\in\Omega, \quad 0<t<T, \\ u(0,x)=\varphi(x), \quad u(T,x)=\psi(x), \quad x\in\overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial\vec n}=0, \quad x\in S, \quad 0\le t\le T \end{gathered} \tag{48}$$

with the Neumann condition is considered. Here, $\vec n$ is the normal vector to S, $\delta>0$, and $a_r(x)\ge a>0$ $(x\in\Omega)$, $f(t,x)$ $(t\in(0,T), x\in\Omega)$, and $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$ are given smooth functions.

Problem (48) has a unique smooth solution $(u(t,x), p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$, and $f(t,x)$. This allows us to reduce the problem (48) to the boundary value problem (1) in the Hilbert space $H=L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x)=-\sum_{r=1}^n\left(a_r(x)u_{x_r}\right)_{x_r}+\delta u \tag{49}$$

with domain

$$\begin{gathered} D(A^x)=\left\{u(x): u(x), u_{x_r}(x), \left(a_r(x)u_{x_r}\right)_{x_r}\in L_2\left(\overline{\Omega}\right),\right. \\ \left.1\le r\le n, \ \frac{\partial u(x)}{\partial\vec n}=0, \ x\in S\right\}. \end{gathered} \tag{50}$$

The discretization of problem (48) is carried out in two steps. In the first step, we define the difference operator A_h^x by the formula

$$A_h^x u^h=-\sum_{r=1}^n\left(\alpha_r(x)u_{x_r}^h\right)_{x_r,j_r}+\delta u^h, \tag{51}$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $D^h u^h(x)=0$ for all $x\in S_h$. Here, D^h is the approximation of the operator $\partial\cdot/\partial\vec n$. With the help of the difference operator A_h^x, we arrive to the following boundary value problem:

$$\begin{gathered} iu_t^h(t,x)+A_h^x u^h(t,x)+iu^h(t,x)=p^h(x)+f^h(t,x), \\ 0<t<T, \quad x\in\Omega_h, \\ u^h(0,x)=\varphi^h(x), \quad u^h(T,x)=\psi^h(x), \quad x\in\Omega_h \end{gathered} \tag{52}$$

for an infinite system of ordinary differential equations.

The first order of accuracy difference scheme for the solution of problem (52) is

$$\begin{gathered} i\frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)+iu_k^h(x)=p^h(x)+f_k^h(x), \\ f_k^h(x)=f^h(t_k,x), \quad t_k=k\tau, \quad N\tau=T, \\ 1\le k\le N, \quad x\in\Omega_h, \\ u^h(0,x)=\varphi^h(x), \quad u^h(T,x)=\psi^h(x), \quad x\in\Omega_h. \end{gathered} \tag{53}$$

Theorem 6. *The solution pairs* $(\{u_k^h(x)\}_0^N, p^h(x))$ *of problem* (53) *satisfy the stability estimates*

$$\begin{gathered} \left\|p^h\right\|_{L_{2h}}\le M_1\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}\right. \\ \left.+\left\|A_h^x\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau^{(1)}(L_{2h})}\right], \\ \left\|\left\{u_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\le M_2\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right], \end{gathered} \tag{54}$$

where M_1 *and* M_2 *do not depend on* φ^h, ψ^h, *and* f_k^h, $1\le k\le N$.

The proof of Theorem 6 is based on Theorem 2 and the symmetry property of the operator A_h^x is defined by formula (51) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 7. *For the solution of the elliptic difference problem [26]*

$$\begin{gathered} A_h^x u^h(x)=\omega^h(x), \quad x\in\Omega_h, \\ D^h u^h(x)=0, \quad x\in S_h, \end{gathered} \tag{55}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n}\left\|u_{x_r x_{\bar r}}^h\right\|_{L_{2h}}\le M\left\|\omega^h\right\|_{L_{2h}}, \tag{56}$$

where M *does not depend on* h *and* ω^h.

4. Numerical Results

In present section, for numerical analysis, the following boundary value problem

$$\begin{gathered} i\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}+iu(t,x)=p(x)+f(t,x), \\ x\in(0,\pi), \quad t\in(0,1), \\ u(0,x)=\sin x, \quad u(1,x)=e^{-1}\sin x, \quad x\in[0,\pi], \\ u(t,0)=u(t,\pi)=0, \quad t\in[0,1] \end{gathered} \tag{57}$$

is considered. The exact solution of problem (57) is $u(t,x)=e^{-t}\sin x$ and $p(x)=\sin x$.

The first order of accuracy difference scheme

$$\begin{gathered} i\frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+iu_n^k=\varphi_n^k+p(x_n), \\ 1\le k\le N, \quad 1\le n\le M-1, \\ \varphi_n^k=f(t_k,x_n)=\left(e^{-t_k}-1\right)\sin x_n, \\ t_k=k\tau, \quad 0\le k\le N, \quad N\tau=1, \\ x_n=nh, \quad 1\le n\le M-1, \quad Mh=\pi, \\ u_n^0=\sin(x_n), \quad u_n^N=e^{-1}\sin(x_n), \quad x_n=nh,\ 0\le n\le M, \\ u_0^k=u_M^k=0, \quad 0\le k\le N \end{gathered} \tag{58}$$

for the numerical solution of problem (57) is constructed.

For obtaining the values of $p(x_n)$ at the grid points, we will use the following equation:

$$\begin{gathered} p(x_n)=-e^{-1}\frac{\sin(x_{n+1})-2\sin(x_n)+\sin(x_{n-1})}{h^2} \\ +ie^{-1}\sin(x_n)+\frac{v_{n+1}^N-2v_n^N+v_{n-1}^N}{h^2}-iv_n^N, \\ x_n=nh, \quad 1\le n\le M-1, \end{gathered} \tag{59}$$

where v_s^k, $s=n\pm1$, and n is the solution of the first order of accuracy difference scheme

$$\begin{gathered} i\frac{v_n^k-v_n^{k-1}}{\tau}-\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}+iv_n^k=\varphi_n^k, \\ \varphi_n^k=f(t_k,x_n), \quad t_k=k\tau,\ 1\le k\le N,\ N\tau=1, \\ x_n=nh, \quad 1\le n\le M-1, \quad Mh=\pi, \\ v_n^N-v_n^0=\left(e^{-1}-1\right)\sin(x_n), \quad x_n=nh,\ 0\le n\le M, \\ v_0^k=v_M^k=0, \quad 0\le k\le N \end{gathered} \tag{60}$$

generated by difference scheme (58).

Using the difference scheme (60), we obtain $(N+1)\times(M+1)$ system of linear equations and we can write them in the matrix form as

$$\begin{gathered} Av_{n+1}+Bv_n+Cv_{n-1}=R\varphi_n, \quad 1\le n\le M-1, \\ v_0=v_M=\widetilde{0}, \end{gathered} \tag{61}$$

where

$$C = A = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & x & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & x & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & x & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & x \end{bmatrix}_{(N+1)\times(N+1)},$$
$$B = \begin{bmatrix} -1 & 0 & 0 & \cdot & 0 & 0 & 1 \\ y & z & 0 & \cdot & 0 & 0 & 0 \\ 0 & y & z & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & y & z & 0 \\ 0 & 0 & 0 & \cdot & 0 & y & z \end{bmatrix}_{(N+1)\times(N+1)}. \tag{62}$$

Here,

$$x = -\frac{1}{h^2}, \qquad y = -\frac{i}{\tau}, \qquad z = \frac{i}{\tau} + \frac{2}{h^2} + i,$$
$$v_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n+1, n, n-1,$$
$$\varphi_n = \begin{bmatrix} \left(e^{-1} - 1\right)\sin x_n \\ \varphi_n^1 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}. \tag{63}$$

So, we have the second-order difference equation with respect to n with matrix coefficients. Using the modified Gauss elimination method, we can obtain v_n^k, $0 \le k \le N$, $0 \le n \le M$.

For the solution of the matrix equations, we seek the solution of the form

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1,$$
$$v_M = \widetilde{0}, \tag{64}$$

where α_j and β_j, $j = 1, \ldots, M$, are calculated as

$$\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1}(A),$$
$$\beta_{n+1} = \left(B + C\alpha_n\right)^{-1}\left(D\varphi_n - C\beta_n\right), \tag{65}$$

where α_1 is $(N+1)\times(N+1)$ and β_1 is $(N+1)\times 1$ zero matrix.

Then, using (59), values of $p(x_n)$ at grid points are obtained. Replacing $p(x_n)$ in (58), we get $(N+1)\times(M+1)$ system of linear equations and it can be written in the matrix form

$$A_2 u_{n+1} + B_2 u_n + C_2 u_{n-1} = R\theta_n, \quad 1 \le n \le M-1,$$
$$1u_0 = u_M = \widetilde{0}, \tag{66}$$

Table 1: Error analysis for the exact solution $u(t, x)$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
1st order of accuracy d.s.	0.0024	0.0012	6.0463×10^{-4}

where

$$C_2 = C, \qquad A_2 = A,$$
$$B_2 = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ y & z & 0 & \cdot & 0 & 0 & 0 \\ 0 & y & z & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & y & z & 0 \\ 0 & 0 & 0 & \cdot & 0 & y & z \end{bmatrix}_{(N+1)\times(N+1)}. \tag{67}$$

Here,

$$y = -\frac{i}{\tau}, \qquad z = \frac{i}{\tau} + \frac{2}{h^2} + i,$$
$$u_s = \begin{bmatrix} u_s^0 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n+1, n, n-1,$$
$$\theta_n = \begin{bmatrix} \sin x_n \\ \varphi_n^1 + p(x_n) \\ \vdots \\ \varphi_n^{N-1} + p(x_n) \\ \varphi_n^N + p(x_n) \end{bmatrix}_{(N+1)\times 1}. \tag{68}$$

Using the modified Gauss elimination method again, we can obtain u_n^k, $0 \le k \le N$, $0 \le n \le M$.

We will give the results of the numerical analysis. The numerical solutions are recorded for different values of N and M and u_n^k represents the numerical solutions of the difference scheme at (t_k, x_n). Table 1 is constructed for $N = M = 20$, 40, and 80, respectively and the errors are computed by the following formula:

$$E = \max_{1\le k\le N}\left\{\sum_{n=1}^{M}\left|u(t_k, x_n) - u_n^k\right|^2 h\right\}^{1/2}. \tag{69}$$

For their comparison, Table 2 is constructed when errors are computed by

$$E = \max_{\substack{1\le k\le N \\ 1\le n\le M}}\left|u(t_k, x_n) - u_n^k\right|. \tag{70}$$

Table 3 is constructed for the error of $p(x)$ at the nodes in maximum norm.

5. Conclusion

In the present study, the well-posedness of difference problem for the approximate solution of determination of a control

Table 2: Error analysis for the exact solution $u(t, x)$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
1st order of accuracy d.s.	0.0019	9.5692×10^{-4}	4.8241×10^{-4}

Table 3: Error analysis for $p(x)$.

Method	$N = 20$	$N = 40$	$N = 80$
1st order of accuracy d.s.	0.0145	0.0072	0.0036

parameter for the Schrödinger equation is established. In practice, the stability inequalities for the solution of difference schemes of the approximate solution of three different types of control parameter problems are obtained. The well-posedness of the boundary value problem (1) is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained. Moreover, applying the result of the monograph [15], the high order of accuracy single-step difference schemes for the numerical solution of the boundary value problem (1) can be presented. Of course, the stability inequalities for the solution of these difference schemes have been established without any assumptions about the grid steps.

Conflict of Interests

The authors declare that they have no conflict of interests.

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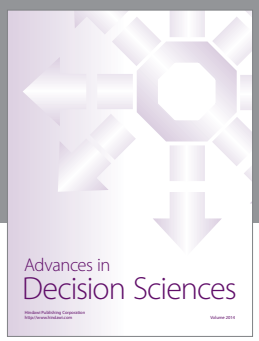

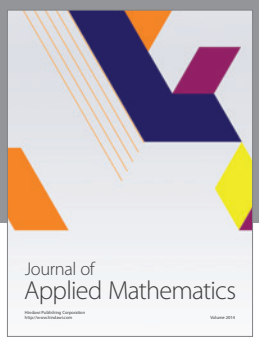

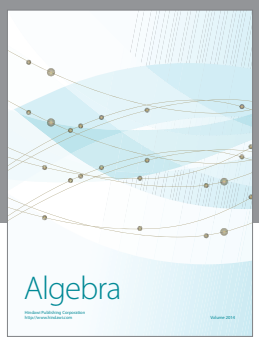

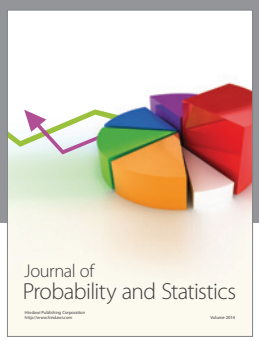

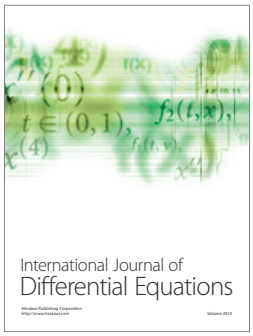

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