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Research Article

Solutions and Conservation Laws of a (2+1)-Dimensional Boussinesq Equation

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We study a nonlinear evolution partial differential equation, namely, the (2+1)-dimensional Boussinesq equation. For the first time Lie symmetry method together with simplest equation method is used to find the exact solutions of the (2+1)-dimensional Boussinesq equation. Furthermore, the new conservation theorem due to Ibragimov will be utilized to construct the conservation laws of the (2+1)-dimensional Boussinesq equation.

1. Introduction

Nonlinear evolution equations (NLEEs) are broadly used as models to represent physical phenomena in numerous fields of sciences, especially in biology, solid state physics, plasma physics, plasma waves, and fluid mechanics. It is therefore of paramount importance that exact solutions of such NLEEs are obtained. However, finding exact solutions of NLEEs is an onerous exercise and only in certain distinctive cases one can explicitly write down their solutions. Nevertheless, in the last few decades important progress has been made and many powerful and effective methods for obtaining exact solutions of NLEEs have been suggested in the literature. Some of the important methods found in the literature include the Darboux transformation method [1], the inverse scattering transform method [2], Hirota's bilinear method [3], Jacobi elliptic function expansion method [4], the sine-cosine method [5], the auxiliary ordinary differential equation method [6], Lie symmetry analysis [7–11], the F-expansion method [12], and the exp-function expansion method [13].

In this paper we consider one such NLEE, namely, the (2+1)-dimensional Boussinesq equation given by

$$u_{tt} - u_{xx} - u_{yy} - \alpha\left(u^2\right)_{xx} - u_{xxxx} = 0, \tag{1}$$

which describes the propagation of gravity waves on the surface of water; in particular it describes the head-on collision of an oblique wave. The (2+1)-dimensional Boussinesq equation (1) combines the two-way propagation of the classical Boussinesq equation with the dependence on a second spatial variable, as that occurs in the two-dimensional Kadomstev-Petviashvili (KP) equation. This equation provides a description of head-on collision of oblique waves and it possesses some interesting properties. The unknown function describes the elevation of the free surface of the fluid. Moreover, (1) involves the two dissipative terms u_{xx} and u_{yy} in addition to the fourth-order spatial derivative u_{xxxx} that represents the dispersion phenomenon. Unlike the standard Boussinesq equation, which is completely integrable, that admits multiple solitons solutions, (1) is not integrable and gives two soliton solutions at most. It is to be noted that if we delete the dissipative term u_{yy} from (1), we obtain the standard Boussinesq equation. The standard Boussinesq equation arises in many physical applications such as nonlinear lattice waves and iron sound waves in plasma and in vibrations in a nonlinear string. It is used in many physical applications such as the percolation of water in porous subsurface of a horizontal layer of material. See also [14].

In [15] the authors used a generalized transformation in homogeneous balance method and found some explicit solitary wave solutions of the (2+1)-dimensional Boussinesq equation. Applied homotopy perturbation method was used in [16] to construct numerical solutions of (1). Extended

ansätz method was employed in [17] to derive exact periodic solitary wave solutions. Recently, the Hirota bilinear method was used in [18] to obtain two soliton solutions.

Lie group method is one of the most effective methods to find solutions of nonlinear partial differential equations (PDEs). Originally, developed by Sophus Lie (1842–1899) in the latter half of the nineteenth century, this method is based upon the study of the invariance under one parameter Lie group of point transformations and it is highly algorithmic [7–9].

In the study of PDEs, conservation laws play a vital role in their solution process and the reduction of PDEs. Conservation laws have been broadly used in studying the existence, uniqueness, and stability of solutions of nonlinear PDEs (see, e.g., [19–21]). They have also been used in the development of numerical methods (see, e.g., [22, 23]). Exact solutions (by exploiting a double reduction method) of some classical partial differential equations have been obtained using conserved vectors associated with the Lie point symmetries [24–26].

In this paper, for the first time, Lie group analysis in conjunction with the simplest equation method [27, 28] is employed to obtain some exact solutions of (1). In addition to this, conservation laws will be derived for (1) using the new conservation theorem due to Ibragimov [29].

2. Solutions of (1)

In this section we obtain exact solutions of (1) using Lie group analysis along with the simplest equation method.

2.1. Exact Solutions Using Lie Point Symmetries. In this subsection we first calculate the Lie point symmetries of (1) and later use the translation symmetries to construct the exact solutions.

2.1.1. Lie Point Symmetries. The symmetry group of the (2+1)-dimensional Boussinesq equation (1) will be generated by the vector field of the form

$$R = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \tag{2}$$

where ξ^i, $i = 1, 2, 3$ and η depend on x, y, t and u. Applying the fourth prolongation $\mathrm{pr}^{(4)}R$ to (1) we obtain an overdetermined system of linear partial differential equations. Solving this resultant system one obtains the following five Lie point symmetries:

$$\begin{gathered} R_1 = \frac{\partial}{\partial x}, \qquad R_2 = \frac{\partial}{\partial t}, \qquad R_3 = \frac{\partial}{\partial y}, \\ R_4 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \\ R_5 = -2\alpha t\frac{\partial}{\partial t} - \alpha x\frac{\partial}{\partial x} - 2\alpha y\frac{\partial}{\partial y} + (1 + 2\alpha u)\frac{\partial}{\partial u}. \end{gathered} \tag{3}$$

We now utilize the symmetry $R = R_1 + R_2 + cR_3$, where c is a constant, and reduce the Boussinesq equation (1) to a PDE in two independent variables. Solving the associated Lagrange system for R, we obtain the following three invariants:

$$f = y - ct, \qquad g = t - x, \qquad \theta = u. \tag{4}$$

Now treating θ as the new dependent variable and f and g as new independent variables, the Boussinesq equation (1) transforms to

$$\left(1 - c^2\right)\theta_{ff} + 2c\theta_{fg} + 2\alpha\theta_g^2 + 2\alpha\theta\theta_{gg} + \theta_{gggg} = 0, \tag{5}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (5) and transform it to an ordinary differential equation (ODE). The equation (5) has the following three symmetries:

$$\begin{gathered} \Gamma_1 = \frac{\partial}{\partial g}, \qquad \Gamma_2 = \frac{\partial}{\partial f}, \\ \Gamma_3 = \left(2\alpha f - 2\alpha f c^2\right)\frac{\partial}{\partial f} + \left(\alpha c f - c^2\alpha g + \alpha g\right)\frac{\partial}{\partial g} \\ + \left(c^2 + 2c^2\alpha\theta - 2\alpha\theta\right)\frac{\partial}{\partial \theta}. \end{gathered} \tag{6}$$

The combination of the first two translational symmetries, $\Gamma = \Gamma_1 + \nu\Gamma_2$, where ν is a constant, yields the two invariants

$$z = f - \nu g, \qquad \psi = \theta, \tag{7}$$

which gives rise to a group-invariant solution $\psi = \psi(z)$, and consequently using these invariants, (5) is transformed into the fourth-order nonlinear ODE

$$2\alpha\nu^2\psi'^2 + \left(1 - c^2 - 2c\nu\right)\psi'' + 2\alpha\nu^2\psi\psi'' + \nu^4\psi'''' = 0. \tag{8}$$

Integrating the above equation four times and taking the constants of integration to be zero (because we are looking for soliton solutions) and reverting back to the original variables, we obtain the following group-invariant solutions of the Boussinesq equation (1):

$$u(x, y, t) = \frac{A_1}{A_2}\operatorname{sech}^2\left[\frac{\sqrt{A_1}}{2}(B \pm z)\right], \tag{9}$$

where B is a constant of integration and

$$\begin{gathered} A_1 = \frac{c^2 + 2\nu c - 1}{\nu^4}, \qquad A_2 = \frac{2\alpha}{3\nu^2}, \\ z = \nu x + y - (c + \nu)t. \end{gathered} \tag{10}$$

It is worth noting that the obtained solitary wave solution $u(x, y, t)$ is a regular soliton in the form of a bell-shaped soliton.

2.2. Exact Solutions of (1) *Using Simplest Equation Method.* In this section we employ the simplest equation method [27, 28] to solve the nonlinear ODE (8). This will then give us the exact solutions for our Boussinesq equation (1). The simplest equations that we will use in our work are the Bernoulli and Riccati equations.

Here we first present the simplest equation method and consider the solutions of (8) in the form

$$F(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{11}$$

where $G(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure, and $A_0, \ldots, A_M$ are parameters to be determined.

The Bernoulli equation

$$G'(z) = aG(z) + bG^2(z), \tag{12}$$

where a and b are arbitrary constants has the general solution given by

$$G(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \tag{13}$$

and represents a solitary wave solution.

The Riccati equation considered in this work is

$$G'(z) = bG^2(z) + aG(z) + d, \tag{14}$$

where a, b, and d are arbitrary constants. Its solutions are

$$\begin{aligned} G(z) &= -\frac{a}{2b} - \frac{\theta}{2b}\tanh\left[\frac{1}{2}\theta(z+C)\right], \\ G(z) &= -\frac{a}{2b} - \frac{\theta}{2b}\tanh\left(\frac{1}{2}\theta z\right) \\ &\quad + \frac{\operatorname{sech}(\theta z/2)}{C\cosh(\theta z/2) - (2b/\theta)\sinh(\theta z/2)}, \end{aligned} \tag{15}$$

with $\theta^2 = a^2 - 4bd$ and C is an arbitrary constant of integration.

2.2.1. Solutions of (1) *Using the Bernoulli Equation as the Simplest Equation.* The balancing procedure gives $M = 2$ so the solutions of (8) are of the form

$$F(z) = A_0 + A_1 G + A_2 G^2. \tag{16}$$

Inserting (16) into (8) and using the Bernoulli equation (12) and, thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of six equations in terms of A_0, A_1, and A_2, namely,

$$\begin{aligned} &-120\nu^4 A_2 b^4 - 20\alpha\nu^2 A_2^2 b^2 = 0, \\ &-336\nu^4 A_2 ab^3 - 36\alpha\nu^2 A_2^2 ab - 24\nu^4 A_1 b^4 \\ &\quad - 24\alpha\nu^2 A_1 A_2 b^2 = 0, \\ &-A_1 a^2 + 2\nu A_1 a^2 c - \nu A_1 a^4 + A_1 a^2 c^2 \\ &\quad - 2\alpha\nu^2 A_1 A_2 a^2 = 0, \\ &-16\alpha\nu^2 A_2^2 a^2 + 12\nu A_2 b^2 c - 6\alpha\nu^2 A_1^2 b^2 \\ &\quad - 60\nu^4 A_1 ab^3 - 12\alpha\nu^2 A_0 A_1 b^2 - 330\nu^4 A_2 a^2 b^2 \\ &\quad - 42\alpha\nu^2 A_1 ab A_2 + 6A_2 b^2 c^2 - 6A_2 b^2 = 0, \\ &-15\nu^4 A_1 a^3 b + 8\nu A_2 a^2 c - 3A_1 ab - 4A_2 a^2 \\ &\quad + 4A_2 a^2 c^2 - 4\alpha\nu^2 A_1^2 a^2 - 6\alpha\nu^2 A_0 A_1 ab - 16\nu^4 A_2 a^4 \\ &\quad + 6A_1 abc\nu - 8\alpha\nu^2 A_0 A_2 a^2 + 3A_1 abc^2 = 0, \\ &-18\alpha\nu^2 A_1 A_2 a^2 - 10\alpha\nu^2 A_1^2 ab - 4\alpha\nu^2 A_0 A_1 b^2 \\ &\quad + 10A_2 abc^2 + 4\nu A_1 b^2 c + 20\nu A_2 abc - 2A_1 b^2 \\ &\quad + 2A_1 b^2 c^2 - 20\alpha\nu^2 A_0 A_2 ab - 10A_2 ab \\ &\quad - 130\nu^4 A_2 a^3 b - 50\nu^4 A_1 a^2 b^2 = 0. \end{aligned} \tag{17}$$

With the aid of Mathematica, solving the above system of algebraic equations, one possible solution for A_0, A_1, and A_2 is

$$\begin{aligned} A_0 &= \frac{-\left(1 - c^2 - 2c\nu + a^2\nu^4\right)}{2\alpha\nu^2}, \\ A_1 &= \frac{-6ab\nu^2}{\alpha}, \\ A_2 &= \frac{-6b^2\nu^2}{\alpha}. \end{aligned} \tag{18}$$

Thus, reverting back to the original variables, a solution of (1) is

$$\begin{aligned} &u(t,x,y) \\ &\quad = A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\ &\quad + A_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \end{aligned} \tag{19}$$

where $z = \nu x + y - (c+\nu)t$ and C is an arbitrary constant of integration, which represents a solitary wave solution.

2.2.2. Solutions of (1) *Using the Riccati Equation as the Simplest Equation.* The balancing procedure yields $M = 2$ so the solutions of (8) take the form

$$F(z) = A_0 + A_1 G + A_2 G^2. \tag{20}$$

Inserting (20) into (8) and making use of the Riccati equation (14), we obtain algebraic system of equations in terms of A_0, A_1, and A_2 by equating the coefficients of powers of G^i to zero. The resulting algebraic equations are

$$\begin{aligned} &-120\nu^4 A_2 b^4 - 20\alpha\nu^2 A_2^2 b^2 = 0, \\ &-36\alpha\nu^2 A_2^2 ab - 336\nu^4 A_2 ab^3 - 24\nu^4 A_1 b^4 \\ &\quad - 24\alpha\nu^2 A_1 A_2 b^2 = 0, \end{aligned}$$

$$
\begin{aligned}
&-32\alpha\nu^2A_2^2bd-6\alpha\nu^2A_1^2b^2-240\nu^4A_2b^3d\\
&\quad+6A_2b^2c^2+12A_2b^2c\nu-6A_2b^2\\
&\quad-12\alpha\nu^2A_0A_1b^2-42\alpha\nu^2A_1A_2ab-16\alpha\nu^2A_2^2a^2\\
&\quad-60\nu^4A_1ab^3-330A_2a^2b^2=0,\\
&-16\nu^4A_2bd^3-14\nu^4A_2a^2d^2+2A_1acd\nu\\
&\quad-A_1ad+4A_2cd^2\nu+A_1ac^2d-8\nu^4A_1abd^2\\
&\quad+2A_2c^2d^2-\nu^4A_1a^3d-2\alpha\nu^2A_0A_1ad\\
&\quad-4\alpha\nu^2A_0A_2d^2-2\alpha\nu^2A_1^2d^2-2A_2d^2=0,\\
&2A_1b^2c^2-28\alpha\nu^2A_2^2ad-20\alpha\nu^2A_0A_2ab\\
&\quad-36\alpha\nu^2A_1A_2bd-18\alpha\nu^2A_1A_2a^2-10\alpha\nu^2A_1^2ab\\
&\quad-10A_2ab+10A_2abc^2+4A_1b^2c\nu-40\nu^4A_1b^3d\\
&\quad-4\alpha\nu^2A_0A_1b^2-50\nu^4A_1a^2b^2+20A_2abc\nu-2A_1b^2\\
&\quad-130\nu^4A_2a^3b-440\nu^4A_2ab^2d=0,\\
&2A_1a^2c\nu-6A_2ad-\nu^4A_1a^4\\
&\quad+12A_2acd\nu-6\alpha\nu^2A_1^2ad+6A_2ac^2d\\
&\quad+A_1a^2c^2-12\alpha\nu^2A_1A_2d^2-4\alpha\nu^2A_0A_1bd\\
&\quad-120\nu^4A_2abd^2+4A_1bcd\nu-2\alpha\nu^2A_0A_1a^2\\
&\quad-12\alpha\nu^2A_0A_2ad-16\nu^4A_1b^2d^2-A_1a^2\\
&\quad-30\nu^4A_2a^3d-2A_1bd+2A_1bc^2d\\
&\quad-22\nu^4A_1a^2bd=0,\\
&-8\alpha\nu^2A_0A_2a^2+3A_1abc^2-8A_2bd\\
&\quad+6A_1abc\nu-3A_1ab-6\alpha\nu^2A_0A_1ab\\
&\quad-136\nu^4A_2b^2d^2-4A_2a^2-12\alpha\nu^2A_2^2d^2\\
&\quad+8A_2a^2c\nu-16\alpha\nu^2A_0A_2bd-232\nu^4A_2a^2bd\\
&\quad-8\alpha\nu^2A_1^2bd+16A_2bcd\nu-15\nu^4A_1a^3b\\
&\quad-16\nu^4A_2a^4-60\nu^4A_1ab^2d+8A_2bc^2d\\
&\quad+4A_2a^2c^2-30\alpha\nu^2A_1A_2ad-4\alpha\nu^2A_1^2a^2=0.
\end{aligned}
\tag{21}
$$

Solving the above equations, we get

$$
\begin{aligned}
A_0&=\frac{-8bd\nu^4-a^2\nu^4+c^2+2c\nu-1}{2\nu^2\alpha},\\
A_1&=\frac{-6ab\nu^2}{\alpha},\qquad A_2=\frac{-6b^2\nu^2}{\alpha},
\end{aligned}
\tag{22}
$$

and consequently, the solutions of (1) are

$$
\begin{aligned}
&u(t,x,y)\\
&\quad=A_0+A_1\left\{-\frac{a}{2b}-\frac{\theta}{2b}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&\qquad+A_2\left\{-\frac{a}{2b}-\frac{\theta}{2b}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,\\
&u(t,x,y)\\
&\quad=A_0+A_1\left\{-\frac{a}{2b}-\frac{\theta}{2b}\tanh\left(\frac{1}{2}\theta z\right)\right.\\
&\qquad\left.+\frac{\operatorname{sech}(\theta z/2)}{C\cosh(\theta z/2)-(2b/\theta)\sinh(\theta z/2)}\right\}\\
&\qquad+A_2\left\{-\frac{a}{2b}-\frac{\theta}{2b}\tanh\left(\frac{1}{2}\theta z\right)\right.\\
&\qquad\left.+\frac{\operatorname{sech}(\theta z/2)}{C\cosh(\theta z/2)-(2b/\theta)\sinh(\theta z/2)}\right\}^2,
\end{aligned}
\tag{23}
$$

where $z=\nu x+y-(c+\nu)t$ and C is an arbitrary constant of integration.

The solution (16) is a solitary wave solution in the form of a kink solution.

3. Conservation Laws for (1)

In this section we obtain conservation laws for the (2+1)-dimensional Boussinesq equation (1) using Ibragimov theorem [29], but first we give some definitions and notations which we will utilize later.

3.1. Fundamental Operators and Their Relationship. Let us consider a kth-order system of PDEs of n independent variables $x=(x^1,x^2,\ldots,x^n)$ and m dependent variables $u=(u^1,u^2,\ldots,u^m)$, namely,

$$E_\alpha\left(x,u,u_{(1)},\ldots,u_{(k)}\right)=0,\quad \alpha=1,\ldots,m. \tag{24}$$

Here $u_{(1)},u_{(2)},\ldots,u_{(k)}$ denote the collections of all first, second, …, kth-order partial derivatives; that is, $u_i^\alpha=D_i(u^\alpha)$, $u_{ij}^\alpha=D_jD_i(u^\alpha),\ldots$, respectively, with the total derivative operator with respect to x^i defined by

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha}+\cdots,\quad i=1,\ldots,n. \tag{25}$$

The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2\cdots i_s}}, \quad \alpha = 1,\ldots,m, \tag{26}$$

and the Lie-Bäcklund operator is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathscr{A}, \tag{27}$$

where $\mathscr{A}$ is the space of differential functions. The operator (27) can be written as

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}\zeta^\alpha_{i_1 i_2\cdots i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2\cdots i_s}}, \tag{28}$$

where

$$\begin{aligned} \zeta^\alpha_i &= D_i\left(W^\alpha\right) + \xi^j u^\alpha_{ij}, \\ \zeta^\alpha_{i_1\cdots i_s} &= D_{i_1}\cdots D_{i_s}\left(W^\alpha\right) + \xi^j u^\alpha_{ji_1\cdots i_s}, \quad s > 1. \end{aligned} \tag{29}$$

Here W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{30}$$

We can write the Lie-Bäcklund operator (28) in characteristic form as

$$\begin{aligned} X = {} & \xi^i D_i + W^\alpha\frac{\partial}{\partial u^\alpha} \\ & + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}\left(W^\alpha\right)\frac{\partial}{\partial u^\alpha_{i_1 i_2\cdots i_s}}. \end{aligned} \tag{31}$$

The Noether operators associated with a Lie-Bäcklund symmetry operator X are defined as

$$\begin{aligned} N^i = {} & \xi^i + W^\alpha\frac{\delta}{\delta u^\alpha_i} \\ & + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}\left(W^\alpha\right)\frac{\delta}{\delta u^\alpha_{i i_1 i_2\cdots i_s}}, \quad i = 1,\ldots,n, \end{aligned} \tag{32}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (26) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{i j_1 j_2\cdots j_s}}, \quad i = 1,\ldots,n, \quad \alpha = 1,\ldots,m, \tag{33}$$

and the Euler-Lagrange, Lie-Bäcklund, and Noether operators are connected by the operator identity

$$X + D_i\left(\xi^i\right) = W^\alpha\frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{34}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathscr{A}$, $j = 1,\ldots,n$, is a conserved vector of (24) if T^i satisfies

$$D_i T^i|_{(11)} = 0, \tag{35}$$

which defines a local conservation law of system (24).

The system of adjoint equations to (24) is defined by

$$E^*_\alpha\left(x, u, v, \ldots, u_{(k)}, v_{(k)}\right) = 0, \quad \alpha = 1,\ldots,m, \tag{36}$$

where

$$E^*_\alpha\left(x, u, v, \ldots, u_{(k)}, v_{(k)}\right) = \frac{\delta\left(v^\beta E_\beta\right)}{\delta u^\alpha}, \quad \alpha = 1,\ldots,m, \quad v = v(x) \tag{37}$$

and $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables.

The system of (24) is known as self-adjoint if the substitution of $v = u$ into the system of adjoint equations (36) yields the same system (24).

Let us now assume the system of (24) admits the symmetry generator

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{38}$$

Then the system of adjoint equations (36) admits the operator

$$\begin{aligned} Y &= \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta^\alpha_*\frac{\partial}{\partial v^\alpha}, \\ \eta^\alpha_* &= -\left[\lambda^\alpha_\beta v^\beta + v^\alpha D_i\left(\xi^i\right)\right], \end{aligned} \tag{39}$$

where the operator (39) is an extension of (38) to the variable v^α and the λ^α_β are obtainable from

$$X\left(E_\alpha\right) = \lambda^\beta_\alpha E_\beta. \tag{40}$$

We now state the following theorem.

Theorem 1 (see [29]). *Every Lie point, Lie-Bäcklund, and nonlocal symmetry* (38) *admitted by the system of* (24) *gives rise to a conservation law for the system consisting of* (24) *and the adjoint equation* (36), *where the components T^i of the conserved vector $T = (T^1, \ldots, T^n)$ are determined by*

$$T^i = \xi^i L + W^\alpha\frac{\delta L}{\delta u^\alpha_i} + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}\left(W^\alpha\right)\frac{\delta L}{\delta u^\alpha_{i i_1 i_2\cdots i_s}}, \quad i = 1,\ldots,n, \tag{41}$$

with Lagrangian given by

$$L = v^\alpha E_\alpha\left(x, u, \ldots, u_{(k)}\right). \tag{42}$$

3.2. Construction of Conservation Laws for (1) . In this subsection, we obtain conservation laws of (2+1)-dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx} = 0. \tag{43}$$

Recall that (43) admits the following five Lie point symmetry generators:

$$R_1 = \frac{\partial}{\partial x}, \qquad R_2 = \frac{\partial}{\partial t}, \qquad R_3 = \frac{\partial}{\partial y},$$
$$R_4 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \tag{44}$$
$$R_5 = -2\alpha t\frac{\partial}{\partial t} - x\alpha\frac{\partial}{\partial x} - 2\alpha y\frac{\partial}{\partial y} + (1 + 2\alpha u)\frac{\partial}{\partial u}.$$

We now find five conserved vectors corresponding to each of these five Lie point symmetries.

The adjoint equation of (43), by invoking (37), is

$$\begin{aligned} E^*&\left(t, x, u, v, \ldots, u_{xxxx}, v_{xxxx}\right) \\ &= \frac{\delta}{\delta u}\left[v\left(u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2,\right.\right. \\ &\qquad \left.\left. -2\alpha u u_{xx} - u_{xxxx}\right)\right] = 0, \end{aligned} \tag{45}$$

where $v = v(t, x, y)$ is a new dependent variable and (45) gives

$$v_{tt} - v_{xx} - v_{yy} - 2\alpha u v_{xx} - v_{xxxx} = 0. \tag{46}$$

It is obvious from the adjoint equation (46) that (43) is not self-adjoint. By recalling (42), we get the following Lagrangian for the system of (43) and (46):

$$L = v\left(u_{tt} - u_{xx} - u_{yy} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx}\right). \tag{47}$$

(i) We first consider the Lie point symmetry generator $R_1 = \partial/\partial x$. It can be verified from (39) that the operator Y_1 is the same as R_1 and the Lie characteristic function is $W = -u_x$. Thus, by using (41), the components T^i, $i = 1, 2, 3$, of the conserved vector $T = (T^1, T^2, T^3)$ are given by

$$\begin{aligned} T^1 &= u_x v_t - v u_{tx}, \\ T^2 &= v u_{tt} - v u_{yy} - u_x v_x - 2\alpha u u_x v_x \\ &\quad - u_x v_{xxx} + v_{xx} u_{xx} - v_x u_{xxx}, \\ T^3 &= -u_x v_y + v u_{xy}. \end{aligned} \tag{48}$$

Remark 2. The conserved vector T contains the arbitrary solution v of the adjoint equation (46) and hence gives an infinite number of conservation laws.

The same remark applies to all the following four cases.

(ii) Now for the second symmetry generator $R_2 = \partial/\partial t$, we have $W = -u_t$. Hence, by invoking (41), the symmetry generator R_2 gives rise to the following components of the conserved vector:

$$\begin{aligned} T^1 &= -v u_{xx} - v u_{yy} - 2\alpha v u_x^2 - 2\alpha u v u_{xx} - v u_{xxxx} + u_t v_t, \\ T^2 &= -u_t v_x + 2\alpha v u_t u_x - 2\alpha u u_t v_x - u_t v_{xxx} \\ &\quad + v u_{tx} + 2\alpha u v u_{tx} + v_{xx} u_{tx} - v_x u_{txx} + v u_{txxx}, \\ T^3 &= -v_y u_t + v u_{ty}. \end{aligned} \tag{49}$$

(iii) The third symmetry generator, $R_3 = \partial/\partial y$, gives $W = -u_y$ and the corresponding components of the conserved vector are

$$\begin{aligned} T^1 &= v_t u_y - v u_{ty}, \\ T^2 &= -u_y v_x + 2\alpha v u_y u_x - 2\alpha u u_y v_x - u_y v_{xxx} + v u_{xy} \\ &\quad + 2\alpha u v u_{xy} + v_{xx} u_{xy} - v_x u_{xxy} + v u_{xxxy}, \\ T^3 &= v u_{tt} - v u_{xx} - 2\alpha v u_x^2 - 2\alpha u v u_{xx} - v u_{xxxx} - u_y v_y. \end{aligned} \tag{50}$$

(iv) For the symmetry generator $R_4 = y\partial/\partial t + t\partial/\partial y$ the components of the conserved vector, as before, are given by

$$\begin{aligned} T^1 &= -y v u_{xx} - y v u_{yy} - 2\alpha y v u_x^2 \\ &\quad - 2\alpha y u v u_{xx} - y v u_{xxxx} + y u_t v_t \\ &\quad + t u_y v_t - v u_y - t v u_{ty}, \\ T^2 &= -y u_t v_x + 2\alpha y v u_t u_x - 2\alpha y u u_t v_x \\ &\quad - y u_t v_{xxx} - t v_x u_y + 2\alpha t v u_y u_x - 2\alpha t u v_x u_y \\ &\quad - t u_y v_{xxx} + y v u_{tx} + 2\alpha y u v u_{tx} + y u_{tx} v_{xx} \\ &\quad + t v u_{xy} + 2\alpha t u v u_{xy} + t u_{xy} v_{xx} - y v_x u_{txx} \\ &\quad - t v_x u_{xxy} + y v u_{txxx} + t v u_{xxxy}, \\ T^3 &= t v u_{tt} - t v u_{xx} - 2\alpha t v u_x^2 \\ &\quad - 2\alpha t u v u_{xx} - t v u_{xxxx} - y v_y u_t \\ &\quad - t v_y u_y + v u_t + y v u_{ty}. \end{aligned} \tag{51}$$

(v) Finally, for the symmetry generator

$$R_5 = -2\alpha t\frac{\partial}{\partial t} - x\alpha\frac{\partial}{\partial x} - 2\alpha y\frac{\partial}{\partial y} + (1 + 2\alpha u)\frac{\partial}{\partial u} \tag{52}$$

the value of Y_5 is different from R_5 and is given by

$$Y_5 = -2\alpha t\frac{\partial}{\partial t} - x\alpha\frac{\partial}{\partial x} - 2\alpha y\frac{\partial}{\partial y} + (1 + 2\alpha u)\frac{\partial}{\partial u} - v\alpha\frac{\partial}{\partial v}. \tag{53}$$

In this case the Lie characteristic function is $W = 1 + 2\alpha u + 2\alpha t u_t + x\alpha u_x + 2\alpha y u_y$. So using (41), one can obtain the conserved vector T whose components are given by

$$\begin{aligned}
T^1 &= 2\alpha t v u_{xx} + 2\alpha t v u_{yy} + 4\alpha^2 t v u_x^2 \\
&\quad + 4\alpha^2 t v u u_{xx} + 2\alpha t v u_{xxxx} - v_t \\
&\quad - 2\alpha u v_t - 2\alpha t u_t v_t - x\alpha u_x v_t - 2\alpha y u_y v_t \\
&\quad + 4\alpha v u_t + x\alpha v u_{tx} + 2\alpha y v u_{ty}, \\
T^2 &= -x\alpha v u_{tt} + x\alpha v u_{yy} + 2\alpha x u v u_{xx} \\
&\quad + v_x + 4\alpha u v_x + v_{xxx} + 4\alpha^2 u^2 v_x \\
&\quad + 2\alpha u v_{xxx} - 8\alpha t v u_x u_t + 2\alpha t u_t v_x \\
&\quad + 4\alpha^2 t v u_t u_x + 4\alpha^2 t u u_t v_x + 2\alpha t u_t v_{xxx} \\
&\quad + x\alpha u_x v_x + 2\alpha^2 x u u_x v_x + x\alpha u_x v_{xxx} \\
&\quad - 8\alpha^2 y v u_x u_y + 2\alpha y u_y v_x + 4\alpha^2 y v u_x u_y \\
&\quad + 4\alpha^2 y u u_y v_x + 2\alpha y u_y v_{xxx} - 2\alpha t v u_{tx} \\
&\quad - 5\alpha v u_x - 2\alpha y v u_{xy} - 4\alpha^2 t u v u_{tx} \\
&\quad - 10\alpha^2 u v u_x - 2\alpha^2 x u v u_{xx} - 4\alpha^2 y u v u_{xy} \\
&\quad - 3\alpha u_x v_{xx} - 2\alpha t u_{tx} v_{xx} - x\alpha u_{xx} v_{xx} \\
&\quad - 2\alpha y u_{xy} v_{xx} + 2\alpha t v_x u_{txx} + 4\alpha v_x u_{xx} \\
&\quad + x\alpha v_x u_{xxx} + 2\alpha y v_x u_{xxy} - 2\alpha t v u_{txxx} \\
&\quad - 5\alpha v u_{xxx} - 2\alpha y v u_{xxxy}, \\
T^3 &= -2\alpha y v u_{tt} + 2\alpha y v u_{xx} + 4\alpha^2 y v u_x^2 \\
&\quad + 4\alpha^2 y v u u_{xx} + 2\alpha y v u_{xxxx} + v_y \\
&\quad + 2\alpha u v_y + 2\alpha t u_t v_y + x\alpha u_x v_y + 2\alpha y u_y v_y \\
&\quad - 4\alpha v u_y - x\alpha v u_{xy} - 2\alpha t v u_{ty}.
\end{aligned} \tag{54}$$

4. Conclusions

In this paper, for the first time, Lie symmetries as well as the simplest equation method were used to obtain exact solutions of the (2+1)-dimensional Boussinesq equation (1). The solutions obtained were solitary waves and nontopological soliton. It is obvious from the analysis we conducted that the (2+1)-dimensional equation gives rise to a variety of solitary wave solutions that ranges from kink to soliton solutions. The obtained kink and soliton solutions are regular solitons given in the form of hyperbolic tan or the sech^2 form. Moreover, the conservation laws for the (2+1)-dimensional Boussinesq equation were also derived by using the new conservation theorem due to Ibragimov [29].

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