Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 583710, 11 pages
http://dx.doi.org/10.1155/2013/583710

Research Article

A General Iterative Scheme Based on Regularization for Solving Equilibrium and Constrained Convex Minimization Problems

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Received 18 March 2013; Revised 25 May 2013; Accepted 28 June 2013

Academic Editor: Simeon Reich

The present paper is divided into two parts. First, we introduce implicit and explicit iterative schemes based on the regularization for solving equilibrium and constrained convex minimization problems. We establish results on the strong convergence of the sequences generated by the proposed schemes to a common solution of minimization and equilibrium problem. Such a point is also a solution of a variational inequality. In the second part, as applications, we apply the algorithm to solve split feasibility problem and equilibrium problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let ϕ be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Consider the equilibrium problem (EP) which is to find $z\in C$ such that

$$\phi(z,y)\geq 0,\quad \forall y\in C. \tag{1}$$

We denoted the set of solutions of EP by $\mathrm{EP}(\phi)$. Given a mapping $T:C\to H$, let $\phi(x,y)=\langle Tx,y-x\rangle$ for all $x,y\in C$; then $z\in\mathrm{EP}(\phi)$ if and only if $\langle Tz,y-z\rangle\geq 0$ for all $y\in C$; that is, z is a solution of the variational inequality. Numerous problems in physics, optimizations and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problem; see for instance see [1–8] and the references therein.

Some composite iterative algorithms were proposed by many authors for finding the common solution of equilibrium problem and fixed point problem. Next, we list some main results as follows.

With some appropriate assumptions, Ceng et al. [9] established the following iterative scheme: $x_1\in H$ and

$$\begin{gathered}\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\ x_{n+1}=\alpha_n u_n+(1-\alpha_n)Su_n,\quad \forall n\in\mathbb{N}.\end{gathered} \tag{2}$$

Under certain conditions, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\mathrm{EP}(\phi)\cap F(S)$.

For finding an element of $\mathrm{EP}(\phi)\cap F(S)$, S. Takahashi and W. Takahashi [10] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: $x_1\in H$ and

$$\begin{gathered}\phi(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\ x_{n+1}=\alpha_n h(x_n)+(1-\alpha_n)Su_n,\quad \forall n\in\mathbb{N}.\end{gathered} \tag{3}$$

Under suitable conditions, some strong convergence theorems are obtained.

In 2009, Liu [11] introduced two iterative schemes by the general iterative method for finding an element of $\mathrm{EP}(\phi)\cap F(S)$, where $S:C\to H$ is a k-strictly pseudocontraction

nonself mapping in the setting of a real Hilbert space. Let $\{x_n\}$ be a sequence generated by

$$\begin{gathered}\phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ y_n = \beta_n u_n + (1-\beta_n) S u_n,\\ x_n = \alpha_n \gamma h(x_n) + (I - \alpha_n B) y_n, \quad \forall n \in \mathbb{N},\end{gathered} \tag{4}$$

and $x_1 \in H$ arbitrarily,

$$\begin{gathered}\phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ y_n = \beta_n u_n + (1-\beta_n) S u_n,\\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n B) y_n, \quad \forall n \in \mathbb{N},\end{gathered} \tag{5}$$

where B is a strongly positive bounded linear operator on H. Under some assumptions, the strong convergence theorems are obtained.

In 2012, based on the concept of the shrinking projection method, Reich and Sabach [12] consider the following algorithm for finding the common solution of finite equilibrium problems in a reflexive Banach space

$$\begin{gathered}x_0 \in X,\\ Q_0^i = X, \quad i = 1, 2, \ldots, N,\\ y_n^i = \mathrm{Res}^f_{\lambda_n^i g_i}\left(x_n + e_n^i\right),\\ Q_{n+1}^i = \left\{z \in Q_n^i : \left\langle \nabla f\left(x_n + e_n^i\right) - \nabla f\left(y_n^i\right), z - y_n^i\right\rangle \leq 0\right\},\\ Q_{n+1} := \cap_{i=1}^N Q_{n+1}^i,\\ x_{n+1} = \mathrm{proj}_{Q_{n+1}^f}(x_0), \quad n = 0, 1, 2, \ldots.\end{gathered} \tag{6}$$

Under some consumption, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{Proj}_E^f(x_0)$.

The gradient-projection algorithm is a classical power method for solving constrained convex optimization problems and has been studied by many authors (see [13–26] and the reference therein). The method has recently been applied to solve split feasibility problems which find applications in image reconstructions and the intensity modulated radiation therapy (see [27–34]).

Consider the problem of minimizing f over the constraint set C (assuming C is a nonempty closed and convex subset of a real Hilbert space H). The main results we all know about the gradient projection are that if $f : H \to \mathbb{R}$ is a convex and continuously Fréchet differentiable functional, the gradient-projection algorithm generates a sequence $\{x_n\}_{n=0}^\infty$ determined by the gradient of f and the metric projection onto C. Under the condition that f has a Lipschitz continuous and strongly monotone gradient, the sequence $\{x_n\}_{n=0}^\infty$ can be strongly convergent to a minimizer of f in C. If the gradient of f is only assumed to be inverse strongly monotone, then $\{x_n\}_{n=0}^\infty$ can only be weakly convergent if H is infinite-dimensional.

Recently, Xu [35] gave an operator-oriented approach as an alternative to the gradient-projection method and to the relaxed gradient-projection algorithm, namely, an averaged mapping approach. He also presented two modifications of gradient-projection algorithms which are shown to have strong convergence.

On the other hand, regularization, in particular the traditional Tikhonov regularization, is usually used to solve ill-posed optimization problems [36]. The disadvantage is the weak convergence of the method RGPA for the regularization problem under some conditions.

The purpose of the paper is to study the iterative method for finding the common solution of an equilibrium problem and a constrained convex minimization problem. Based on the Viscosity method [18], we combine the RGPA and averaged mapping approaches to propose implicit and explicit composite iterative methods for finding the common element of the set of solutions of an equilibrium problem and the solution set of a constrained convex minimization problem and also to prove some strong convergence theorems.

2. Preliminaries

Throughout the paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, and C is a nonempty closed convex subset of H. The set of fixed points of a mapping T is denoted by $\mathrm{Fix}(T)$; that is, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. The fact that the sequence $\{x_n\}$ converges strongly to x is denoted by $x_n \to x$. The following definition and results are needed in the subsequent sections.

Recall that a mapping $V : H \to H$ is said to be L-Lipschitzian if

$$\|Vx - Vy\| \leq L\|x - y\|, \quad \forall x, y \in H, \tag{7}$$

where $L > 0$ is a constant. In particular, if $L \in [0, 1)$, then V is called a contraction on H; if $L = 1$, then V is called a nonexpansive mapping on H. V is called firmly nonexpansive if $2V - I$ is nonexpansive, or equivalently, $\langle x - y, Vx - Vy\rangle \geq \|Vx - Vy\|^2$, for all $x, y \in H$. Alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = (1/2)(I + S)$, where $S : H \to H$ is nonexpansive.

In 1978, Baillon et al. [37] defined the concept of averaged mapping which is used very frequently now.

Definition 1 (see [37]). A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha) I + \alpha S, \tag{8}$$

where α is a number in $(0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (8) holds, we say that T is α-averaged. Clearly, a firmly nonexpansive mapping (in particular, projection) is a $(1/2)$-averaged map.

Proposition 2 (see [28, 38]). *For given operators $S, T, V : H \to H$ one has the following.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

Recall that the metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \tag{9}$$

In 1984, Goebel and Reich [39] discussed the properties of the nearest point projection.

Lemma 3 (see [39]). *For given $x \in H$ one has the following:*

(i) *$z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \le 0, \quad \forall y \in C; \tag{10}$$

(ii) *$z = P_C x$ if and only if*

$$\|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C; \tag{11}$$

(iii)

$$\langle P_C x - P_C y, x - y \rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{12}$$

Consequently, P_C is nonexpansive and monotone.

Lemma 4. *The following inequality holds in a Hilbert space X*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in X. \tag{13}$$

Lemma 5 (see [40]). *In a Hilbert space H, one has*

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \quad \forall x, y \in H, \quad \lambda \in [0,1]. \tag{14}$$

Lemma 6 (Demiclosedness Principle [40]). *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$; if $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular if $y = 0$ then $x \in \mathrm{Fix}(T)$.*

Definition 7. A nonlinear operator T whose domain $D(T) \subseteq H$ and range $R(T) \subseteq H$ is said to be

(i) monotone if

$$\langle x - y, Tx - Ty \rangle \ge 0, \quad \forall x, y \in D(T); \tag{15}$$

(ii) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty \rangle \ge \beta \|x - y\|^2, \quad \forall x, y \in D(T); \tag{16}$$

(iii) ν-inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty \rangle \ge \nu \|Tx - Ty\|^2, \quad \forall x, y \in D(T). \tag{17}$$

Proposition 8 (see [28]). *Let $T : H \to H$ be an operator from H to itself.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Lemma 9 (see [18]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad n \ge 0, \tag{18}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

In order to solve the equilibrium problem for a bifunction $\phi : C \times C \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

$(A1)$ $\phi(x, x) = 0$, for all $x \in C$;

$(A2)$ ϕ is monotone; that is, $\phi(x, y) + \phi(y, x) \le 0$, for all $x, y \in C$;

$(A3)$ For all $x, y, z \in C$,

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \le \phi(x, y); \tag{19}$$

$(A4)$ for each fixed $x \in C$, the function $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

Let us recall the following lemmas which will be useful for our paper.

Lemma 10 (see [28]). *Let ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C. \tag{20}$$

Further, if $T_r x = \{z \in C : \phi(z, y) + (1/r)\langle y - z, z - x \rangle \ge 0, \forall y \in C\}$, then the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive; that is,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H; \tag{21}$$

(3) *$F(T_r) = EP(\phi)$;*

(4) *$EP(\phi)$ is closed and convex.*

3. Main Results

We now look at the constrained convex minimization problem:

$$\min_{x\in C} f(x), \tag{22}$$

where C is a closed and convex subset of a Hilbert space H and $f : C \to \mathbb{R}$ is a real-valued convex function. If f is Fréchet differentiable, then the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}_{n=0}^{\infty}$ according to the recursive formula

$$x_{n+1} = \mathrm{Proj}_C (I - \gamma\nabla f)(x_n), \quad n \geq 0, \tag{23}$$

or more generally,

$$x_{n+1} = \mathrm{Proj}_C (I - \gamma_n\nabla f)(x_n), \quad n \geq 0, \tag{24}$$

where, in both (23) and (24), the initial guess x_0 is taken from C arbitrarily and the parameters γ or γ_n are positive real numbers.

As a matter of fact, it is known that, if ∇f fails to be strongly monotone and is only $1/L$-ism; namely, there is constant $L > 0$ such that

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2, \quad x, y \in C, \tag{25}$$

under some assumption for γ or γ_n, then algorithms (23) and (24) can still converge in the weak topology.

Now, consider the regularized minimization problem

$$\min_{x\in C} f_\alpha(x) := f(x) + \frac{\alpha}{2}\|x\|^2, \tag{26}$$

where $\alpha > 0$ is the regularization parameter, and again f is convex with $1/L$-ism continuous gradient ∇f.

It is obvious that there exists a unique point $x_\alpha \in C$ such that x_α is the unique fixed point of the mapping

$$V_\alpha := \mathrm{Proj}_C (I - \gamma\nabla f_\alpha) = \mathrm{Proj}_C (I - \gamma(\nabla f + \alpha I)). \tag{27}$$

We can prove that $\{x_\alpha\} \rightharpoonup x^*$, where x^* is a solution of the constrained convex minimization problem.

Throughout the rest of this paper, assume that the minimization problem (22) is consistent and let U denote its solution set; we always assume that h is a contraction of C into H with coefficient $\rho \in (0, 1)$; let $\{G_{\beta_n}\}$ be a sequence of mappings defined as Lemma 3 and define a mapping $T_n : C \to C$ by

$$\begin{gathered} P_C\left(I - \gamma\nabla f_{\alpha_n}\right) = \lambda_n I + (1 - \lambda_n) T_n \\ \lambda_n := \frac{2 - \gamma(L + \alpha_n)}{4} \end{gathered} \tag{28}$$

Consider the following mapping Q_n on H defined by

$$Q_n x = \theta_n h(x) + (1 - \theta_n) T_n G_{\beta_n} x, \quad x \in H, n \in N, \tag{29}$$

where $\theta_n \in (0, 1)$; then by Lemmas 3 and 10

$$\begin{aligned} &\|Q_n x - Q_n y\| \\ &\quad \leq \theta_n\rho\|x - y\| + (1 - \theta_n)\|x - y\| = (1 - (1 - \rho)\theta_n). \end{aligned} \tag{30}$$

Since $0 < 1 - (1 - \rho)\theta_n < 1$, it follows that Q_n is a contraction. Therefore, by the Banach contraction principle, Q_n has a unique fixed point $x_n^h \in H$ such that

$$x_n^h = \theta_n h\left(x_n^h\right) + (1 - \theta_n) T_n G_{\beta_n} x_n^h. \tag{31}$$

For simplicity, we will write x_n for x_n^h provided that no confusion occurs. Next, we prove the convergence of $\{x_n\}$ while we claim the existence of the $q \in U \bigcap \mathrm{EP}(\phi)$ which solves the variational inequality

$$\langle (I - h) q, p - q\rangle \geq 0, \quad \forall p \in U \bigcap \mathrm{EP}(\phi). \tag{32}$$

3.1. Convergence of the Implicit Scheme

Proposition 11. *If $0 < \gamma < 2/L$, $0 < \alpha < 2/\gamma - L$, ∇f is $(1/L)$-ism, for all $x \in C$,*

$$\begin{gathered} \mathrm{Proj}_C (I - \gamma\nabla f_\alpha) = (1 - \mu_\alpha) I + \mu_\alpha T_\alpha, \\ \mathrm{Proj}_C (I - \gamma\nabla f) = (1 - \mu) I + \mu T, \end{gathered} \tag{33}$$

where $\mu_\alpha = (2 + \gamma(L + \alpha))/4$, $\mu = (2 + \gamma L)/4$, then

$$\|T_\alpha x - Tx\| \leq \alpha M(x), \tag{34}$$

where

$$M(x) = \gamma(5\|x\| + \|Tx\|). \tag{35}$$

Proof. One here

$$\begin{aligned} &\|\mathrm{Proj}_C (I - \gamma\nabla f_\alpha) x - \mathrm{Proj}_C (I - \gamma\nabla f_\alpha) x\| \\ &\quad = \|(\mu - \mu_\alpha) x + \mu_\alpha T_\alpha x - \mu Tx\| \\ &\quad \leq \|(I - \gamma\nabla f_\alpha) x - (I - \gamma\nabla f) x\| \\ &\quad = \gamma\|\nabla f_\alpha(x) - \nabla f(x)\| = \alpha\gamma\|x\|, \end{aligned} \tag{36}$$

then

$$\begin{gathered} \|\mu_\alpha (T_\alpha x) - \mu Tx\| \leq |\mu - \mu_\alpha|\|x\| + \alpha\gamma\|x\|, \\ \|T_\alpha x - Tx\| \leq \frac{\alpha\gamma(5\|x\| + \|Tx\|)}{2 + \gamma(L + \alpha)} \leq \alpha M(x), \end{gathered} \tag{37}$$

where, $M(x) = \gamma(5\|x\| + \|Tx\|)$. □

Theorem 12. *Let C be a nonempty closed convex subset of a real Hilbert space H and $h : C \to H$ a contraction with $\rho \in (0, 1)$, $U \cap EP(\phi) \neq \emptyset$, and ϕ a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Let $\{x_n\}$ be sequence generated by*

$$\begin{gathered} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_n = \theta_n h(x_n) + (1 - \theta_n) T_n(u_n), \end{gathered} \tag{38}$$

where, $P_c[I-\gamma\nabla f_{\alpha_n}] = \lambda_n I + (1-\lambda_n)T_n$, $0 < \gamma < 2/L$ *and* $\lambda_n = (2-\gamma(L+\alpha_n))/4$.

(i) $\{\beta_n\} \subset (0,+\infty)$ $\liminf_{n\to\infty}\beta_n > 0$;

(ii) $\{\theta_n\} \subset (0,1)$, $\lim_{n\to\infty}\theta_n = 0$;

(iii) $\alpha_n = o(\theta_n)$.

Then, $\{x_n\}$ *converges strongly to a point* $q \in U \cap EP(\phi)$ *which solves the variational inequality* (32).

Proof. Pick any $p \in U \cap \mathrm{EP}(\phi)$, $u_n = G_{\beta_n}x_n$, $p = G_{\beta_n}p$; then we have

$$\|u_n - p\| \le \|x_n - p\|, \tag{39}$$

(noting $Tp = p$)

$$\begin{aligned}
&\|x_n - p\| \\
&\quad = \|\theta_n h(x_n) + (1-\theta_n)T_n(u_n) - p\| \\
&\quad = \|\theta_n(h(x_n) - h(p)) + \theta_n(h(p) - p) \\
&\qquad + (1-\theta_n)(T_n(u_n) - p)\| \\
&\quad \le \theta_n\rho\|x_n - p\| + \theta_n\|h(p) - p\| \\
&\qquad + (1-\theta_n)[\|u_n - p\| + \|T_n(p) - T(p)\|] \\
&\quad \le (1-(1-\rho)\theta_n)\|x_n - p\| + \theta_n\|h(p) - p\| + \alpha_n M_1;
\end{aligned} \tag{40}$$

hence,

$$\|x_n - p\| \le \frac{1}{1-\rho}[\|h(p) - p\| + M_1]. \tag{41}$$

So, $\{x_n\}$ is bounded. Next, we claim that $\|x_n - u_n\| \to 0$.

Take $p \in U \cap \mathrm{EP}(\phi)$; by Lemma 3, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{\beta_n}x_n - T_{\beta_n}p\|^2 \le \langle x_n - p, u_n - p\rangle \\
&= \frac{1}{2}\left(\|x_n - p\|^2 + \|u_n - p\|^2 - \|u_n - x_n\|^2\right).
\end{aligned} \tag{42}$$

It follows that

$$\begin{aligned}
&\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2, \\
&\|x_n - p\|^2 \\
&\quad = \|\theta_n(h(x_n) - p) + (1-\theta_n)(T_n u_n - p)\|^2 \\
&\quad \le (1-\theta_n)^2\left[\|T_n u_n - T_n p + T_n p - Tp\|^2\right] \\
&\qquad + 2\theta_n\langle h(x_n) - p, x_n - p\rangle \\
&\quad \le (1-\theta_n)^2 \\
&\qquad \times \left[\|u_n - p\|^2 + 2\|u_n - p\| \right. \\
&\qquad\qquad \left. \times \|T_n p - Tp\| + \|T_n p - Tp\|^2\right] \\
&\qquad + 2\theta_n\langle h(x_n) - p, x_n - p\rangle \\
&\quad \le (1-\theta_n)^2 \\
&\qquad \times \left(\left[\|x_n - p\|^2 - \|x_n - u_n\|^2\right]\right. \\
&\qquad + 2\|u_n - p\|\|T_n p - Tp\| \\
&\qquad \left. + \|T_n p - Tp\|^2\right) + 2\theta_n\rho\|x_n - p\|^2 \\
&\qquad + 2\theta_n\langle h(p) - p, x_n - p\rangle.
\end{aligned} \tag{43}$$

So, $\|x_n - u_n\| \to 0$. Next, we show that $\|x_n - T_n x_n\| \to 0$,

$$\begin{aligned}
&\|x_n - T_n x_n\| \\
&\quad = \|x_n - T_n u_n + T_n u_n - T_n x_n\| \\
&\quad \le \theta_n\|h(x_n) - T_n u_n\| + \|u_n - x_n\| \longrightarrow 0. \\
&\|u_n - T_n u_n\| = \|u_n - x_n + x_n - T_n x_n + T_n x_n - T_n u_n\| \longrightarrow 0.
\end{aligned} \tag{44}$$

Observe that

$$\begin{aligned}
&\|P_C(I-\gamma\nabla f_{\alpha_n})u_n - u_n\| \\
&\quad = \|\lambda_n u_n + (1-\lambda_n)T_n u_n - u_n\| \\
&\quad = (1-\lambda_n)\|T_n u_n - u_n\| \le \|T_n u_n - u_n\|.
\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}
&\|P_C(I-\gamma\nabla f)u_n - u_n\| \\
&\quad \le \|P_C(I-\gamma\nabla f)u_n - P_C(I-\gamma\nabla f_{\alpha_n})u_n\| \\
&\qquad + \|P_C(I-\gamma\nabla f_{\alpha_n})u_n - u_n\| \\
&\quad \le \gamma\alpha_n\|u_n\| + \|T_n u_n - u_n\| \longrightarrow 0.
\end{aligned} \tag{46}$$

So,

$$\lim_{n\to\infty}\|P_C(I-\gamma\nabla f)u_n - u_n\| = 0. \tag{47}$$

Since $\{u_n\}$ is bounded, there exists $\{u_{n_i}\}$ such that $\{u_{n_i}\} \rightharpoonup q$. Since C is closed and convex, C is weakly closed. So, we have $q \in C$. Let us show that $q \in U$. Assume that $q \overline{\in} U$. Since $u_{n_i} \rightharpoonup q$ and $q \ne Tq$, it follows from the Opial's condition that

$$\begin{aligned}
&\liminf_{n\to\infty}\|u_{n_i} - q\| \\
&\quad < \liminf_{n\to\infty}\|u_{n_i} - Tq\| \\
&\quad \le \liminf_{n\to\infty}\left(\|u_{n_i} - Tu_{n_i}\| + \|Tu_{n_i} - Tq\|\right) \\
&\quad \le \liminf_{n\to\infty}\|u_{n_i} - q\|.
\end{aligned} \tag{48}$$

This is a contradiction. So, we get $q \in U$.

Next, we show that $q \in \mathrm{EP}(\phi)$. Since $u_n = G_{\beta_n}x_n$, for any $y \in C$, we obtain

$$\phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \ge 0. \tag{49}$$

From $(A2)$, we have

$$\frac{1}{\beta_n}\langle y-u_n,u_n-x_n\rangle \geq \phi(y,u_n). \tag{50}$$

Replacing n with n_i, we have

$$\left\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{\beta_{n_i}}\right\rangle \geq \phi(y,u_{n_i}). \tag{51}$$

Since $(u_{n_i}-x_{n_i})/\beta_{n_i}\to 0$ and $u_{n_i}\rightharpoonup q$, it follows from (A_4) that $0\geq\phi(y,q)$, for all $y\in C$. Let $z_t=ty+(1-t)q$ for all $t\in(0,1]$ and $y\in C$. Then, we have $z_t\in C$ and hence $\phi(z_t,q)\leq 0$. Thus, from $(A1)$ and $(A4)$ we have

$$0=\phi(z_t,z_t)\leq t\phi(z_t,y)+(1-t)\phi(z_t,q)\leq t\phi(z_t,y) \tag{52}$$

and hence $0\leq\phi(z_t,y)$. From (A_3), we have $0\leq\phi(q,y)$ for all $y\in C$ and hence $q\in \mathrm{EP}(\phi)$. Therefore, $q\in U\cap \mathrm{EP}(\phi)$.

On the other hand,

$$x_n-q=\theta_n(h(x_n)-q)+(1-\theta_n)(T_nu_n-q), \tag{53}$$

$$\begin{aligned}
&\|x_n-q\|^2\\
&\quad=\theta_n\langle h(x_n)-q,x_n-q\rangle+(1-\theta_n)\langle T_nu_n-q,x_n-q\rangle\\
&\quad=\theta_n\langle h(x_n)-h(q),x_n-q\rangle\\
&\qquad+(1-\theta_n)\langle T_nu_n-T_nq,x_n-q\rangle\\
&\qquad+(1-\theta_n)\langle T_nq-Tq,x_n-q\rangle+\theta_n\langle h(q)-q,x_n-q\rangle\\
&\quad\leq\theta_n\rho\|x_n-q\|^2+(1-\theta_n)\|x_n-q\|^2\\
&\qquad+\theta_n\langle h(q)-q,x_n-q\rangle+(1-\theta_n)\alpha_nM_2\|x_n-q\|;
\end{aligned} \tag{54}$$

then $x_{n_j}\to q$ if $x_{n_j}\rightharpoonup q$.

Next, we prove that q solves the VI (problem):

$$(I-h)(x_n)=-\frac{1}{\theta_n}\left(I-T_nQ_{\beta_n}\right)x_n+\left(I-T_nQ_{\beta_n}\right)x_n. \tag{55}$$

Note that

$$\begin{aligned}
&\langle(I-h)q,q-p\rangle\\
&\quad=\lim_{j\to\infty}\langle(I-h)\left(x_{n_j}\right),x_{n_j}-p\rangle\\
&\quad=\lim_{j\to\infty}\left[-\frac{1}{\theta_{n_j}}\left\langle\left(I-T_nQ_{\beta_{n_j}}\right)x_{n_j}\right.\right.\\
&\qquad\qquad\left.-\left(I-T_{n_j}Q_{\beta_{n_j}}\right)p,x_{n_j}-p\right\rangle\\
&\qquad-\frac{1}{\theta_{n_j}}\left\langle\left(I-T_{n_j}Q_{\beta_{n_j}}\right)p,x_{n_j}-p\right\rangle\\
&\qquad\left.+\left\langle\left(I-T_{n_j}Q_{\beta_{n_j}}\right)x_{n_j},x_{n_j}-p\right\rangle\right]\\
&\quad\leq 0.
\end{aligned} \tag{56}$$

□

3.2. Convergence of the Explicit Scheme

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H, $h:C\to H$ a contraction with $\rho\in(0,1)$, $U\cap EP(\phi)\neq\emptyset$, and ϕ a bifunction from $C\times C$ into $\mathbb{R}$ satisfying $(A1)$, $(A2)$, $(A3)$, and $(A4)$. Let $\{x_n\}$ be sequence generated by $x_1\in H$ and*

$$\begin{aligned}
&\phi(u_n,y)+\frac{1}{\beta_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C,\\
&x_{n+1}=\theta_nh(x_n)+(1-\theta_n)T_nu_n,
\end{aligned} \tag{57}$$

where $P_c[I-\gamma\nabla f_{\alpha_n}]=\lambda_nI+(1-\lambda_n)T_n$, $0<\gamma<2/L$ and $\lambda_n=(2-\gamma(L+\alpha_n))/4$, $u_n=G_{\beta_n}x_n$. Let $\{\alpha_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(i) $\{\theta_n\}\subset(0,1)$, $\lim_{n\to\infty}\theta_n=0$, $\sum_{n=1}^{\infty}\theta_n=\infty$, $\sum_{n=1}^{\infty}|\theta_{n+1}-\theta_n|<\infty$;

(ii) $\alpha_n=o(\theta_n)\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;

(iii) $\{\beta_n\}\in(0,+\infty)$, $\lim_{n\to\infty}\beta_n>0$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $q\in U\cap EP(\phi)$ which solves the variational inequality (32).

Proof. First we prove that $\{x_n\}$ is bounded.

Taking any $p\in U\cap \mathrm{EP}(\phi)$, we have

$$u_n=G_{\beta_n}x_n,\qquad p=G_{\beta_n}p. \tag{58}$$

So, $\|u_n-p\|\leq\|x_n-p\|$,

$$\begin{aligned}
&\|x_{n+1}-p\|\\
&\quad=\|\theta_nh(x_n)+(1-\theta_n)T_n(u_n)-p\|\\
&\quad=\|\theta_n(h(x_n)-h(p))+\theta_n(h(p)-p)\\
&\qquad+(1-\theta_n)(T_n(u_n)-p)\|\\
&\quad\leq\theta_n\rho\|x_n-p\|+\theta_n\|h(p)-p\|\\
&\qquad+(1-\theta_n)[\|x_n-p\|+\|T_n(p)-T(p)\|]\\
&\quad\leq(1-(1-\rho)\theta_n)\|x_n-p\|+\theta_n\|h(p)-p\|+\alpha_nM_3\\
&\quad\leq\max\left\{\|x_n-p\|;\frac{1}{1-\rho}[\|h(p)-p\|+M_3]\right\}.
\end{aligned} \tag{59}$$

So, $\{x_n\}$ is bounded.

Next we prove that $\|x_{n+1} - x_n\| \to 0$,

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \|[\theta_n h(x_n) + (1-\theta_n) T_n u_n] \\
&\qquad - [\theta_{n-1} h(x_{n-1}) + (1-\theta_{n-1}) T_{n-1} u_{n-1}]\| \\
&\quad = \|\theta_n (h(x_n) - h(x_{n-1})) + (1-\theta_n)(T_n u_n - T_n u_{n-1}) \\
&\qquad + (\theta_n - \theta_{n-1})(h(x_{n-1}) - T_n u_{n-1}) \\
&\qquad + (1-\theta_{n-1})(T_n u_{n-1} - T_{n-1} u_{n-1})\| \\
&\quad \le \theta_n \rho \|x_n - x_{n-1}\| \\
&\qquad + |\theta_n - \theta_{n-1}| [\|h(x_{n-1})\| + \|T_n u_{n-1}\|] \\
&\qquad + (1-\theta_n)\|u_n - u_{n-1}\| \\
&\qquad + (1-\theta_{n-1})\|T_n u_{n-1} - T_{n-1} u_{n-1}\|.
\end{aligned} \tag{60}$$

From $u_{n+1} = G_{\beta_{n+1}} x_{n+1}$ and $u_n = G_{\beta_n} x_n$, we note that

$$\phi(u_{n+1}, y) + \frac{1}{\beta_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \quad \forall y \in C, \tag{61}$$

$$\phi(u_n, y) + \frac{1}{\beta_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C. \tag{62}$$

Putting $y = u_n$ in (61) and $y = u_{n+1}$ in (62), we have

$$\begin{aligned}
&\phi(u_{n+1}, u_n) + \frac{1}{\beta_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \quad \forall y \in C, \\
&\phi(u_n, u_{n+1}) + \frac{1}{\beta_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C.
\end{aligned} \tag{63}$$

So, from $(A2)$, we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{\beta_n} - \frac{u_{n+1} - x_{n+1}}{\beta_{n+1}} \right\rangle \ge 0, \tag{64}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{\beta_n}{\beta_{n+1}} (u_{n+1} - x_{n+1}) \right\rangle \ge 0. \tag{65}$$

Since $\lim_{n\to\infty} \beta_n > 0$, without loss of generality, let us assume that there exists a real number a such that $\beta_n > a > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
&\|u_{n+1} - u_n\|^2 \\
&\quad \le \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\
&\quad \le \|u_{n+1} - u_n\| \\
&\qquad \times \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{\beta_n}{\beta_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\};
\end{aligned} \tag{66}$$

thus,

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{1}{a} |\beta_{n+1} - \beta_n| M_4, \tag{67}$$

where $M_4 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$,

$$\begin{aligned}
&\|T_n u_{n-1} - T_{n-1} u_{n-1}\| \\
&\quad = \left\| \frac{4P_C(I - \gamma\nabla f_{\alpha_n}) - [2 - \gamma(L + \alpha_n)] I}{2 + \gamma(L + \alpha_n)} u_{n-1} \right. \\
&\qquad \left. - \frac{4P_C(I - \gamma\nabla f_{\alpha_{n-1}}) - [2 - \gamma(L + \alpha_{n-1})] I}{2 + \gamma(L + \alpha_{n-1})} u_{n-1} \right\| \\
&\quad \le \left\| \frac{4P_C(I - \gamma\nabla f_{\alpha_n})}{2 + \gamma(L + \alpha_n)} u_{n-1} - \frac{4P_C(I - \gamma\nabla f_{\alpha_{n-1}})}{2 + \gamma(L + \alpha_{n-1})} u_{n-1} \right\| \\
&\qquad + \left\| \frac{-[2 - \gamma(L + \alpha_n)]}{2 + \gamma(L + \alpha_n)} u_{n-1} + \frac{[2 - \gamma(L + \alpha_{n-1})]}{2 + \gamma(L + \alpha_{n-1})} u_{n-1} \right\| \\
&\quad = \left\| \left(4[2 + \gamma(L + \alpha_{n-1})] P_C(I - \gamma\nabla f_{\alpha_n}) u_{n-1} \right. \right. \\
&\qquad \left. - 4[2 + \gamma(L + \alpha_n)] P_C(I - \gamma\nabla f_{\alpha_{n-1}}) u_{n-1} \right) \\
&\qquad \left. \times ([2 + \gamma(L + \alpha_n)][2 + \gamma(L + \alpha_{n-1})])^{-1} \right\| \\
&\qquad + \frac{4\gamma |\alpha_n - \alpha_{n-1}| \|u_{n-1}\|}{[2 + \gamma(L + \alpha_n)][2 + \gamma(L + \alpha_{n-1})]} \\
&\quad = \left\| \frac{4\gamma(\alpha_{n-1} - \alpha_n) P_C(I - \gamma\nabla f_{\alpha_n}) u_{n-1}}{[2 + \gamma(L + \alpha_n)][2 + \gamma(L + \alpha_{n-1})]} \right. \\
&\qquad + \left(4(2 + \gamma(L + \alpha_n)) \right. \\
&\qquad \times \left[P_C(I - \gamma\nabla f_{\alpha_n}) u_{n-1} \right. \\
&\qquad \left. \left. - P_C(I - \gamma\nabla f_{\alpha_{n-1}}) u_{n-1} \right] \right) \\
&\qquad \left. \times ([2 + \gamma(L + \alpha_n)][2 + \gamma(L + \alpha_{n-1})])^{-1} \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4\gamma\left|\alpha_n-\alpha_{n-1}\right|\left\|u_{n-1}\right\|}{\left[2+\gamma\left(L+\alpha_n\right)\right]\left[2+\gamma\left(L+\alpha_{n-1}\right)\right]}\\
&\leq\frac{4\gamma\left|\alpha_{n-1}-\alpha_n\right|\left\|P_C\left(I-\gamma\nabla f_{\alpha_n}\right)u_{n-1}\right\|}{\left[2+\gamma\left(L+\alpha_n\right)\right]\left[2+\gamma\left(L+\alpha_{n-1}\right)\right]}\\
&+\frac{4\gamma\left|\alpha_{n-1}-\alpha_n\right|\left[2+\gamma\left(L+\alpha_n\right)\right]\left\|u_{n-1}\right\|}{\left[2+\gamma\left(L+\alpha_n\right)\right]\left[2+\gamma\left(L+\alpha_{n-1}\right)\right]}\\
&+\frac{4\gamma\left|\alpha_n-\alpha_{n-1}\right|\left\|u_{n-1}\right\|}{\left[2+\gamma\left(L+\alpha_n\right)\right]\left[2+\gamma\left(L+\alpha_{n-1}\right)\right]}\\
&\leq\left|\alpha_{n-1}-\alpha_n\right|\left[\gamma\left\|P_C\left(I-\gamma\nabla f_{\alpha_n}\right)u_{n-1}\right\|+5\gamma\left\|u_{n-1}\right\|\right]\\
&\leq M_5\left|\alpha_{n-1}-\alpha_n\right|.
\end{aligned}
\tag{68}
$$

So

$$
\begin{aligned}
&\left\|x_{n+1}-x_n\right\|\\
&\quad\leq\left[1-\left(1-\rho\right)\theta_n\right]\left\|x_n-x_{n-1}\right\|\\
&\qquad+M_6\left[\left|\theta_n-\theta_{n-1}\right|+\left|\alpha_n-\alpha_{n-1}\right|+\left|\beta_n-\beta_{n-1}\right|\right].
\end{aligned}
\tag{69}
$$

So

$$
\begin{aligned}
&\lim\left\|x_{n+1}-x_n\right\|=0,\\
&\lim\left\|u_{n+1}-u_n\right\|=0.
\end{aligned}
\tag{70}
$$

Next, we prove that $\|x_n-T_nx_n\|\to 0$,

$$
\begin{aligned}
&\left\|x_{n+1}-p\right\|^2\\
&\quad=\left\|\theta_n\left(h\left(x_n\right)-p\right)+\left(1-\theta_n\right)\left(T_nu_n-p\right)\right\|^2\\
&\quad\leq\left(1-\theta_n\right)^2\left[\left\|T_nu_n-T_np+T_np-Tp\right\|^2\right]\\
&\qquad+2\theta_n\left\langle h\left(x_n\right)-p,x_n-p\right\rangle\\
&\quad\leq\left(1-\theta_n\right)^2\\
&\qquad\times\left[\left\|u_n-p\right\|^2+2\left\|u_n-p\right\|\right.\\
&\qquad\qquad\left.\times\left\|T_np-Tp\right\|+\left\|T_np-Tp\right\|^2\right]\\
&\qquad+2\theta_n\left\langle h\left(x_n\right)-p,x_n-p\right\rangle\\
&\quad\leq\left(1-\theta_n\right)^2\\
&\qquad\times\left(\left[\left\|x_n-p\right\|^2-\left\|x_n-u_n\right\|^2\right]\right.\\
&\qquad\qquad\left.+2\left\|u_n-p\right\|\left\|T_np-Tp\right\|+\left\|T_np-Tp\right\|^2\right)\\
&\qquad+2\theta_n\rho\left\|x_n-p\right\|^2+2\theta_n\left\langle h\left(p\right)-p,x_n-p\right\rangle.
\end{aligned}
\tag{71}
$$

So, $\|x_n-u_n\|\to 0$,

$$
\begin{aligned}
&\left\|x_n-T_nx_n\right\|\\
&\quad=\left\|x_n-x_{n+1}+x_{n+1}-T_nu_n+T_nu_n-T_nx_n\right\|\\
&\quad\leq\theta_n\left\|h\left(x_n\right)-T_nu_n\right\|+\left\|x_n-x_{n+1}\right\|+\left\|u_n-x_n\right\|\longrightarrow 0,\\
&\left\|u_n-T_nu_n\right\|=\left\|u_n-x_n+x_n-T_nx_n+T_nx_n-T_nu_n\right\|\longrightarrow 0.
\end{aligned}
\tag{72}
$$

Observe that

$$
\begin{aligned}
&\left\|P_C\left(I-\gamma\nabla f_{\alpha_n}\right)u_n-u_n\right\|\\
&\quad=\left\|\lambda_nu_n+\left(1-\lambda_n\right)T_nu_n-u_n\right\|\\
&\quad=\left(1-\lambda_n\right)\left\|T_nu_n-u_n\right\|\\
&\quad\leq\left\|T_nu_n-u_n\right\|.
\end{aligned}
\tag{73}
$$

Hence,

$$
\begin{aligned}
&\left\|P_C\left(I-\gamma\nabla f\right)u_n-u_n\right\|\\
&\quad\leq\left\|P_C\left(I-\gamma\nabla f\right)u_n-P_C\left(I-\gamma\nabla f_{\alpha_n}\right)u_n\right\|\\
&\qquad+\left\|P_C\left(I-\gamma\nabla f_{\alpha_n}\right)u_n-u_n\right\|\\
&\quad\leq\gamma\alpha_n\left\|u_n\right\|+\left\|T_nu_n-u_n\right\|\longrightarrow 0.
\end{aligned}
\tag{74}
$$

So,

$$
\lim_{n\to\infty}\left\|P_C\left(I-\gamma\nabla f\right)u_n-u_n\right\|=0.
\tag{75}
$$

Since $\{u_n\}$ is bounded, there exists $\{u_{n_i}\}$ such that $\{u_{n_i}\}\rightharpoonup q$. Since C is closed and convex, C is weakly closed. So, we have $q\in C$. Let us show that $q\in U$. Assume that $q\overline{\in}U$. Since $u_{n_i}\rightharpoonup q$ and $q\neq Tq$, it follows from the Opial's condition that

$$
\begin{aligned}
&\liminf_{n\to\infty}\left\|u_{n_i}-q\right\|\\
&\quad<\liminf_{n\to\infty}\left\|u_{n_i}-Tq\right\|\\
&\quad\leq\liminf_{n\to\infty}\left(\left\|u_{n_i}-Tu_{n_i}\right\|+\left\|Tu_{n_i}-Tq\right\|\right)\\
&\quad\leq\liminf_{n\to\infty}\left\|u_{n_i}-q\right\|.
\end{aligned}
\tag{76}
$$

This is a contradiction. So, we get $q\in U$.

Next We show that $q\in \mathrm{EP}(\phi)$. Since $u_n=G_{\beta_n}x_n$, for any $y\in C$, we obtain

$$
\phi\left(u_n,y\right)+\frac{1}{\beta_n}\left\langle y-u_n,u_n-x_n\right\rangle\geq 0.
\tag{77}
$$

From $(A2)$, we have

$$
\frac{1}{\beta_n}\left\langle y-u_n,u_n-x_n\right\rangle\geq\phi\left(y,u_n\right).
\tag{78}
$$

Replacing n with n_i, we have

$$
\left\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{\beta_{n_i}}\right\rangle\geq\phi\left(y,u_{n_i}\right).
\tag{79}
$$

Since $(u_{n_i} - x_{n_i})/\beta_{n_i} \to 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A_4) that $0 \geq \phi(y, q)$, for all $y \in C$. Let $z_t = ty + (1-t)q$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$ and hence $\phi(z_t, q) \leq 0$. Thus, from $(A1)$ and $(A4)$, we have

$$0 = \phi(z_t, z_t) \leq t\phi(z_t, y) + (1-t)\phi(z_t, q) \leq t\phi(z_t, y) \tag{80}$$

and hence $0 \leq \phi(z_t, y)$. From (A_3), we have $0 \leq \phi(q, y)$ for all $y \in C$ and hence $q \in \mathrm{EP}(\phi)$. Therefore, $q \in U \cap \mathrm{EP}(\phi)$.

We assume that; $x_{n_i} \rightharpoonup \tilde{x}$, then $\tilde{x} \in \mathrm{EP}(\phi) \cap U$,

$$\limsup_{n\to\infty} \langle (I-h)q, q - x_n\rangle = \langle (I-h)q, q - \tilde{x}\rangle \leq 0. \tag{81}$$

Finally, we prove that $x_n \to q$,

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \|\theta_n(h(x_n) - q) + (1-\theta_n)(T_n u_n - Tq)\|^2 \\
&\quad = \|\theta_n(h(x_n) - h(q)) \\
&\qquad + (1-\theta_n)(T_n u_n - Tq) + \theta_n(h(q) - q)\|^2 \\
&\quad \leq \|\theta_n(h(x_n) - h(q)) + (1-\theta_n)(T_n u_n - Tq)\|^2 \\
&\qquad + 2\theta_n\langle h(q) - q, x_{n+1} - q\rangle \\
&\quad \leq \theta_n \|(h(x_n) - h(q))\|^2 + (1-\theta_n) \\
&\qquad \times \|(T_n u_n - Tq)\|^2 + 2\theta_n \langle h(q) - q, x_{n+1} - q\rangle \\
&\quad \leq \theta_n \rho^2 \|x_n - q\|^2 + (1-\theta_n) \\
&\qquad \times \|(T_n u_n - T_n q + T_n q - Tq)\|^2 \\
&\qquad + 2\theta_n\langle h(q) - q, x_{n+1} - q\rangle \\
&\quad \leq \theta_n \rho^2 \|x_n - q\|^2 + (1-\theta_n)\left[\|x_n - q\| + \alpha_n M_7\right]^2 \\
&\qquad + 2\theta_n\langle h(q) - q, x_{n+1} - q\rangle \\
&\qquad \leq [1 - (1-\rho)\theta_n]\|x_n - q\|^2 \\
&\qquad\quad + (1-\theta_n)\left[2\alpha_n M_7 \|x_n - q\| + \alpha_n^2 M_7^2\right] \\
&\qquad\quad + 2\theta_n\langle h(q) - q, x_{n+1} - q\rangle \\
&\|x_{n+1} - q\|^2 = (1-\beta_n)\|x_n - q\|^2 + \beta_n \delta_n. \\
&\qquad \beta_n = (1-\rho)\theta_n, \\
&\delta_n = \frac{1}{1-\rho}\left[\frac{\alpha_n}{\theta_n}(1-\theta_n)2M_7\|x_n - q\| + (1-\theta_n)M_7^2\frac{\alpha_n^2}{\theta_n}\right. \\
&\qquad \left. + 2\langle h(q) - q, x_{n+1} - q\rangle\right],
\end{aligned} \tag{82}$$

by Lemma 9 and $\lim_{n\to\infty}\beta_n = 0, \sum_{n=0}^{\infty}\beta_n = \infty$; $\limsup_{n\to\infty}\delta_n \leq 0$, then $x_n \to q$. □

4. Application of the Iterative Method

Next, we give an application of Theorem 13 to the split feasibility problem (say SFP, for short) which was introduced by Censor and Elfving [27],

$$\text{find } x \in C, \text{ such that } Ax \in Q, \tag{83}$$

where C and Q are nonempty closed convex subsets of Hilbert space H_1 and H_2, respectively. $A : H_1 \to H_2$ is a bounded linear operator.

It is clear that x^* is a solution to the split feasibility problem (83) if and only if $x^* \in C$ and $Ax^* - P_Q Ax^* = 0$.

We define the proximity function f by

$$f(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2, \tag{84}$$

and consider the convex optimization problem

$$\min_{x\in C} f(x) = \min_{x\in C}\frac{1}{2}\|Ax - P_Q Ax\|^2. \tag{85}$$

Then, x^* solves the split feasibility problem (83) if and only if x^* solves the minimization (85) with the minimization equal to 0. Byrne [28] introduced the so-called CQ algorithm to solve the (SFP),

$$x_{n+1} = P_C(I - \mu A^*(I - P_Q)A)x_n, \quad n \geq 0, \tag{86}$$

where $0 < \mu < 2/\|A^*A\| = 2/\|A\|^2$.

He obtained that the sequence $\{x_n\}$ generated by (86) converges weakly to a solution of the (SFP).

Now we consider the regularization technique; let

$$f_\alpha(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{\alpha}{2}\|x\|^2. \tag{87}$$

Applying Theorem 13, we obtain the following result.

Theorem 14. *Assume that the split problem* (83) *is consistent. Let C be a nonempty closed convex subset of a real Hilbert space H, $h : C \to H$ a contraction with $\rho \in (0,1)$, $U \cap EP(\phi) \neq \emptyset$, and ϕ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* $(A1)$, $(A2)$, $(A3)$, *and* $(A4)$. *Let $\{x_n\}$ be sequence generated by $x_1 \in H$ and*

$$\begin{gathered}
\phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = \theta_n h(x_n) + (1-\theta_n)T_n u_n,
\end{gathered} \tag{88}$$

where

$$\begin{gathered}
P_C[I - \mu(A^*(I - P_Q)A + \alpha_n I)] = \lambda_n I + (1-\lambda_n)T_n, \\
\lambda_n = \frac{2 - \mu(\|A\|^2 + \alpha_n)}{4},
\end{gathered} \tag{89}$$

where $u_n = G_{\beta_n}x_n$; let $\{\alpha_n\}$, $\{\theta_n\}$ satisfy the following conditions:

(i) $\{\theta_n\} \subset (0,1)$, $\lim_{n\to\infty}\theta_n = 0, \sum_{n=1}^{\infty}\theta_n = \infty$, $\sum_{n=1}^{\infty}|\theta_{n+1} - \theta_n| < \infty$, $0 < \mu < 2/\|A^*A\| = 2/\|A\|^2$;

(ii) $\alpha_n = o(\theta_n)$ $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\{\beta_n\} \in (0, +\infty)$, $\lim_{n\to\infty}\beta_n > 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $q \in U \cap EP(\phi)$ *which solves the variational inequality* (32).

Proof. By the definition of the function f_α, we have

$$\nabla f_\alpha(x) = A^*(I - P_Q)Ax + \alpha x, \tag{90}$$

and ∇f_α is $1/(\|A\|^2 + \alpha)$-ism,

$$\|\nabla f_\alpha(x) - \nabla f_\alpha(y)\| \le \|A\|^2 + \alpha; \tag{91}$$

then, due to Theorem 13, we have the conclusion immediately. □

Acknowledgments

The author wishes to thank the referees for their helpful comments, which notably improved the presentation of this paper. This work was supported in part by The Fundamental Research Funds for the Central Universities (the Special Fund of Science in Civil Aviation University of China, no. 3122013 K004). of the Iterative Method

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