Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 619068, 5 pages
http://dx.doi.org/10.1155/2013/619068

Research Article

The Local Strong Solutions and Global Weak Solutions for a Nonlinear Equation

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Received 31 March 2013; Accepted 26 April 2013

Academic Editor: Shaoyong Lai

The existence and uniqueness of local strong solutions for a nonlinear equation are investigated in the Sobolev space $C([0,T);H^s(R))\cap C^1([0,T);H^{s-1}(R))$ provided that the initial value lies in $H^s(R)$ with $s>3/2$. Meanwhile, we prove the existence of global weak solutions in $L^\infty([0,\infty);L^2(R))$ for the equation.

1. Introduction

Coclite and Karlsen [1] investigated the well posedness in classes of discontinuous functions for the generalized Degasperis-Procesi equation:

$$\begin{aligned} u_t - u_{txx} + 4h'(u)u_x &\\ = h'''(u)u_x^3 + 3h''(u)u_xu_{xx} + h'(u)u_{xxx}, & \end{aligned} \tag{1}$$

which is subject to the condition

$$\left|h'(u)\right| \le c|u|, \qquad |h(u)| \le c|u|^2, \tag{2}$$

or

$$\left|h'(u)\right| \le c, \qquad |h(u)| \le c|u|, \tag{3}$$

where c is a positive constant. The existence and L^1 stability of entropy weak solutions belonging to the class $L^1(R)\cap BV(R)$ are established for (1) in paper [1].

In this work, we study the following model:

$$\begin{aligned} u_t - u_{txx} + mh'(u)u_x &\\ = h'''(u)u_x^3 + 3h''(u)u_xu_{xx} + h'(u)u_{xxx}, & \end{aligned} \tag{4}$$

where m is a positive constant and $h(u)\in C^3$. If $m=4$ and $h(u)=u^2/2$, (4) reduces to the classical Degasperis-Procesi model (see [2–13]). Here, we notice that assumptions (2) and (3) do not include the case $h(u)=u^3$. In this paper, we will study the case $h(u)=u^3$, and m is an arbitrary positive constant.

In fact, the Cauchy problem of (4) in the case $h(u)=u^3$ is equivalent to the following system:

$$\begin{gathered} u_t - u_{txx} + 3mu^2u_x = 6u_x^3 + 18uu_xu_{xx} + 3u^2u_{xxx}, \\ u(0,x) = u_0(x). \end{gathered} \tag{5}$$

Using the operator $(1-\partial_x^2)^{-1}$ to multiply the first equation of the problem (5), we obtain

$$\begin{gathered} u_t + 3u^2u_x + (m-1)\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^3\right) = 0, \\ u(0,x) = u_0(x). \end{gathered} \tag{6}$$

It is shown in this work that there exists a unique local strong solution in the Sobolev space $C([0,T);H^s(R))\cap C^1([0,T);H^{s-1}(R))$ by assuming that the initial value $u_0(x)$ belongs to $H^s(R)$ with $s>3/2$. In addition, we prove the existence of global weak solutions in $L^\infty([0,\infty);L^2(R))$ for the system (6).

This paper is organized as follows. Section 2 investigates the existence and uniqueness of local strong solutions. The result about global weak solution is given in Section 3.

2. Local Existence

In this section, we will use the Kato theorem in [14] for abstract differential equation to establish the existence of local strong solution for the problem (6). Let us consider the following problem:

$$\frac{dv}{dt}+H(v)v=g(v), \qquad t\geq 0, \qquad v(0)=v_0. \tag{7}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q: Y \to X$ be a topological isomorphism. Let $L(Y,X)$ be the space of all bounded linear operators from Y to X. In the case of $X = Y$, we denote this space by $L(X)$. We illustrate the following conditions in which σ_1, σ_2, σ_3, and σ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $H(y)\in L(Y,X)$ for $y\in X$ with

$$\left\|(H(y)-H(z))w\right\|_X \leq \sigma_1\left\|y-z\right\|_X\|w\|_Y, \quad y,z,w\in Y, \tag{8}$$

and $H(y) \in G(X,1,\beta)$ (i.e., $H(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QH(y)Q^{-1} = H(y)+A(y)$, where $A(y)\in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left\|(A(y)-A(z))w\right\|_X \leq \sigma_2\left\|y-z\right\|_Y\|w\|_X, \quad y,z\in Y,\ w\in X. \tag{9}$$

(iii) $g: Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\left\|g(y)-g(z)\right\|_Y \leq \sigma_3\left\|y-z\right\|_Y, \quad y,z\in Y, \tag{10}$$

$$\left\|g(y)-g(z)\right\|_X \leq \sigma_4\left\|y-z\right\|_X, \quad y,z\in X. \tag{11}$$

Kato Theorem (see [14]). *Assume that conditions (i), (ii), and (iii) hold. If $v_0\in Y$, there is a maximal $T>0$ depending only on $\|v_0\|_Y$ and a unique solution v to the problem* (7) *such that*

$$v=v(\cdot,v_0)\in C([0,T);Y)\cap C^1([0,T);X). \tag{12}$$

Moreover, the map $v_0\to v(\cdot,v_0)$ is a continuous map from Y to the following space:

$$C([0,T);Y)\cap C^1([0,T);X). \tag{13}$$

In order to apply the Kato theorem to establish the local well posedness for the problem (6), we let $H(u)=3u^2\partial_x$, $Y=H^s(R)$, $X=H^{s-1}(R)$, $\Lambda=(1-\partial_x^2)^{1/2}$, $g(u)=(m-1)\Lambda^{-2}\partial_x(u^3)$, and $Q=\Lambda^s$. We know that Q is an isomorphism of H^s onto H^{s-1}. Now, we cite the following Lemmas.

Lemma 1. *The operator $A(u)=u^2\partial_x$ with $u\in H^s(R)$, $s>3/2$ belongs to $G(H^{s-1}(R),1,\beta)$.*

Lemma 2. *Assume that $H(u)=3u^2\partial_x$ with $u\in H^s(R)$ and $s>3/2$. Then, $H(u)\in L(H^s(R),H^{s-1}(R))$ for all $u\in H^s(R)$. Moreover,*

$$\begin{aligned}\|(H(u)-H(z))w\|_{H^{s-1}} &\leq \sigma_1\|u-z\|_{H^{s-1}}\|w\|_{H^s},\\ &\qquad u,z,w\in H^s(R).\end{aligned} \tag{14}$$

Lemma 3. *For $s>3/2$, $u,z\in H^s(R)$ and $w\in H^{s-1}$, it holds that $A(u)=[\Lambda^s,3u^2\partial_x]\Lambda^{-s}\in L(H^{s-1})$ for $u\in H^s$ and*

$$\|(A(u)-A(z))w\|_{H^{s-1}} \leq \sigma_2\|u-z\|_{H^s}\|w\|_{H^{s-1}}. \tag{15}$$

The above three Lemmas can be found in Ni and Zhou [15].

Lemma 4. *Let $u,z\in H^s$ with $s>3/2$ and $g(u)=(m-1)\Lambda^{-2}\partial_x(u^3)$. Then, g is bounded on bounded sets in H^s and satisfies*

$$\begin{aligned}\left\|g(u)-g(z)\right\|_{H^s} &\leq \sigma_3\|u-z\|_{H^s},\\ \left\|g(u)-g(z)\right\|_{H^{s-1}} &\leq \sigma_4\|u-z\|_{H^{s-1}}.\end{aligned} \tag{16}$$

Proof. For $s_0>1/2$, we know that $\|uv\|_{H^{s_0}(R)}\leq c\|u\|_{H^{s_0}(R)}\|v\|_{H^{s_0}(R)}$. Consequently, we have

$$\begin{aligned}\|g(u)-g(z)\|_{H^s} &\leq c\left\|u^3-z^3\right\|_{H^{s-1}}\\ &\leq c\|u-z\|_{H^{s-1}}\left(\|u\|^2_{H^{s-1}}+\|v\|^2_{H^{s-1}}\right)\\ &\leq \sigma_3\|u-z\|_{H^s},\\ \left\|g(u)-g(z)\right\|_{H^{s-1}} &\leq c\left\|u^3-z^3\right\|_{H^{s-2}}\\ &\leq c\left\|u^3-z^3\right\|_{H^{s-1}}\\ &\leq c\|u-z\|_{H^{s-1}}\left(\|u\|^2_{H^{s-1}}+\|v\|^2_{H^{s-1}}\right)\\ &\leq \sigma_4\|u-z\|_{H^{s-1}}.\end{aligned} \tag{17}$$

□

Using the Kato Theorem, Lemmas 1–4, we immediately obtain the local well-posedness theorem.

Theorem 5. *Assume that $u_0\in H^s(R)$ with $s>3/2$. Then, there exists a $T>0$ such that the system* (5) *or the problem* (6) *has a unique solution $u(t,x)$ satisfying*

$$u(t,x)\in C\left([0,T);H^s(R)\right)\cap C^1\left([0,T);H^{s-1}(R)\right). \tag{18}$$

3. Weak Solutions

In this section, our aim is to establish the existence of global weak solutions for the system (6). Firstly, we prove that the solution of the problem (5) is bounded in the space $L^2(R)$ and $L^\infty(R)$.

Lemma 6. *The solution of the problem* (5) *with $m>0$ satisfies*

$$\int_R K_1K\,dx=\int_R\frac{1+\xi^2}{m+\xi^2}\left|\hat{u}(\xi)\right|^2d\xi=\int_R\frac{1+\xi^2}{m+\xi^2}\left|\hat{u}_0(\xi)\right|^2d\xi, \tag{19}$$

where $K_1=u-\partial_{xx}^2u$, and $K=(m-\partial_{xx}^2)^{-1}u$. Moreover, there exist two constants $c_1>0$ and $c_2>0$ depending only on m such that

$$c_1\left\|u_0\right\|_{L^2(R)}\leq c_1\|u\|_{L^2(R)}\leq c_2\left\|u_0\right\|_{L^2(R)}. \tag{20}$$

Proof. Setting $K_1 = u - \partial_{xx}^2 u$ and $K = (m - \partial_{xx}^2)^{-1} u$ and using the first equation of the problem (5), we obtain $u = my - y_{xx}$ and

$$\begin{aligned}
&\frac{d}{dt}\int_R K_1 K\,dx \\
&= \int_R \frac{\partial K_1}{\partial t} K\,dx + \int_R K_1 \frac{\partial K}{\partial t} dx = 2\int_R \frac{\partial K_1}{\partial t} K\,dx \\
&= 2\int_R \left[-3mu^2 u_x + 6u_x^3 + 18u_x u_{xx} + 3u^2 u_{xxx}\right] K\,dx \\
&= 2\int_R \left[-m\partial_x\left(u^3\right) + \left(u^3\right)_{xxx}\right] K\,dx \\
&= \int_R \left(mu^3\right) K_x - u^3 K_{xxx} dx \\
&= \int_R \left[mu^3\right] K_x - u^3\left(mK_x - u_x\right) dx \\
&= \int_R u^3 u_x dx, \\
&= 0.
\end{aligned} \tag{21}$$

Using the Parseval identity and (21), we obtain (19) and (20). □

From Theorem 5, we know that for any $u_0 \in H^s(R)$ with $s > 3/2$, there exists a maximal $T = T(u_0) > 0$ and a unique strong solution u to the problem (6) such that

$$u \in C\left([0,T); H^s(R)\right) \cap C^1\left([0,T); H^{s-1}(R)\right). \tag{22}$$

Firstly, we study the following differential equation:

$$\begin{aligned}
p_t &= 3u^2(t,p), \quad t \in [0,T), \\
p(0,x) &= x.
\end{aligned} \tag{23}$$

Lemma 7. *Let $u_0 \in H^s$, $s > 3$, and let $T > 0$ be the maximal existence time of the solution to the problem (6). Then, the problem (23) has a unique solution $p \in C^1([0,T) \times R, R)$. Moreover, the map $p(t,\cdot)$ is an increasing diffeomorphism of R with $p_x(t,x) > 0$ for $(t,x) \in [0,T) \times R$.*

Proof. Using Theorem 5, we obtain $u \in C^1([0,T); H^{s-1}(R))$ and $H^{s-1} \in C^1(R)$. Therefore, we know that functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in space, and C^1 in time. Using the existence and uniqueness theorem for ordinary differential equations derives that the problem (23) has a unique solution $p \in C^1([0,T) \times R, R)$.

Differentiating (23) with respect to x gives rise to the following:

$$\begin{aligned}
\frac{d}{dt}p_x &= 6uu_x(t,p)\,p_x, \quad t \in [0,T), \\
p_x(0,x) &= 1,
\end{aligned} \tag{24}$$

from which we obtain

$$p_x(t,x) = \exp\left(\int_0^t 6uu_x\left(\tau, p(\tau,x)\right) d\tau\right). \tag{25}$$

For every $T' < T$, using the Sobolev imbedding theorem yields that

$$\sup_{(\tau,x)\in[0,T')\times R} \left|u_x(\tau,x)\right| < \infty. \tag{26}$$

It is inferred that there exists a constant $K_0 > 0$ such that $p_x(t,x) \geq e^{-K_0 t}$ for $(t,x) \in [0,T) \times R$. It completes the proof. □

Lemma 8. *Assume that $u_0 \in H^s(R), s > 3/2$. Let T be the maximal existence time of the solution u to the problem (6). Then, it has*

$$\|u(t,x)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{ct} \quad \forall t \in [0,T], \tag{27}$$

where $c > 0$ is a constant independent of t.

Proof. Let $\xi(x) = (1/2)e^{-|x|}$, we have $(1 - \partial_x^2)^{-1} g = \xi \star f$ for all $g \in L^2(R)$ and $u = \xi \star K_1(t,x)$. Using a simple density argument presented in [7], it suffices to consider $s = 3$ to prove this lemma. Let T be the maximal existence time of the solution u to the problem (6) with the initial value $u_0 \in H^3(R)$ such that $u \in C([0,T), H^3(R)) \cap C^1([0,T), H^2(R))$. From (6), we have

$$u_t + 3u^2 u_x = -(m-1)\,\xi \star \left(3u^2 u_x\right). \tag{28}$$

Since

$$\begin{aligned}
-\xi \star \left(3u^2 u_x\right) &= -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\eta|} 3u^2 u_\eta d\eta \\
&= -\frac{3}{2}\int_{-\infty}^{x} e^{-x+\eta} u^2 u_\eta d\eta - \frac{3}{2}\int_x^{+\infty} e^{x-\eta} u^2 u_\eta d\eta \\
&= \frac{1}{2}\int_{\infty}^{x} e^{-|x-\eta|} u^3 d\eta - \frac{1}{2}\int_x^{\infty} e^{-|x-\eta|} u^3 d\eta, \\
\frac{du\left(t,p(t,x)\right)}{dt} &= u_t\left(t,p(t,x)\right) + u_x\left(t,p(t,x)\right)\frac{dp(t,x)}{dt} \\
&= \left(u_t + 3u^2 u_x\right)\left(t,p(t,x)\right),
\end{aligned} \tag{29}$$

from (29), we have

$$\begin{aligned}
\frac{du\left(t,p(t,x)\right)}{dt} = {} & \frac{m-1}{2}\int_{-\infty}^{p(t,x)} e^{-|p(t,x)-\eta|} u^3 d\eta \\
& - \frac{m-1}{2}\int_{p(t,x)}^{\infty} e^{-|p(t,x)-\eta|} u^3 d\eta.
\end{aligned} \tag{30}$$

Using Lemma 6 and (30) derives that

$$\begin{aligned}\left|\frac{du(t,p(t,x))}{dt}\right| &\le \frac{|m-1|}{2}\int_{-\infty}^{\infty} e^{-|p(t,x)-\eta|}u^2 d\eta \\ &\le \frac{|m-1|}{2}\int_{-\infty}^{\infty} u^3 d\eta \\ &\le \frac{|m-1|}{2}\|u\|_{L^2}^2\|u\|_{L^\infty} \\ &\le c\|u_0\|_{L^2(R)}\|u\|_{L^\infty} \\ &\le c\|u\|_{L^\infty},\end{aligned} \tag{31}$$

where c is a positive constant independent of t. Using (31) results in the following:

$$-c\int_0^t \|u\|_{L^\infty(R)}dt + u_0 \le u(t,p(t,x)) \le c\int_0^t \|u\|_{L^\infty(R)}dt + u_0. \tag{32}$$

Therefore,

$$\begin{aligned}|u(t,p(t,x))| &\le \|u(t,p(t,x))\|_{L^\infty} \\ &\le \|u_0\|_{L^\infty} + c\int_0^t \|u\|_{L^\infty(R)}dt.\end{aligned} \tag{33}$$

Using the Sobolev embedding theorem to ensure the uniform boundedness of $u_x(s,\eta)$ for $(s,\eta)\in[0,t]\times R$ with $t\in[0,T')$, from Lemma 7, for every $t\in[0,T')$, we get a constant $C(t)$ such that

$$e^{-C(t)} \le p_x(t,x) \le e^{C(t)}, \quad x\in R. \tag{34}$$

We deduce from (34) that the function $p(t,\cdot)$ is strictly increasing on R with $\lim_{x\to\pm\infty}p(t,x) = \pm\infty$ as long as $t\in[0,T')$. It follows from (33) that

$$\|u(t,x)\|_{L^\infty} = \|u(t,p(t,x))\|_{L^\infty} \le \|u_0\|_{L^\infty} + c\int_0^t \|u\|_{L^\infty(R)}dt. \tag{35}$$

Using the Gronwall inequality and (35) derives that (27) holds. □

For a real number s with $s>0$, suppose that the function $u_0(x)$ is in $H^s(R)$, and let $u_{\varepsilon 0}$ be the convolution $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$ of the function $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ and u_0 such that the Fourier transform $\widehat{\phi}$ of ϕ satisfies $\widehat{\phi}\in C_0^\infty$, $\widehat{\phi(\xi)}\ge 0$, and $\widehat{\phi(\xi)} = 1$ for any $\xi\in(-1,1)$. Then, we have $u_{\varepsilon 0}(x)\in C^\infty$. It follows from Theorem 5 that for each ε satisfying $0<\varepsilon<1/2$, the Cauchy problem,

$$\begin{gathered}u_t - u_{txx} + 3mu^2u_x = 6u_x^3 + 18uu_xu_{xx} + 3u^2u_{xxx}, \\ u(0,x) = u_{\varepsilon 0}(x),\end{gathered} \tag{36}$$

has a unique solution $u_\varepsilon(t,x)\in C^\infty([0,T);H^\infty)$. Using Lemmas 6 and 8, for every $t\in[0,T)$, we obtain

$$\begin{gathered}\|u_\varepsilon(t,x)\|_{L^2(R)} \le c\|u_\varepsilon(0,x)\|_{L^2(R)} \le c\|u_0\|_{L^2(R)}, \\ \|u_\varepsilon(t,x)\|_{L^\infty} \le \|u_\varepsilon(0,x)\|_{L^\infty}e^{ct} \le c\|u_0\|_{L^\infty}e^{ct}.\end{gathered} \tag{37}$$

Sending $t\to T$, we know that inequalities (37) are still valid. This means that for $t\in[0,\infty)$, (37) hold.

Now, we state the concepts of weak solutions.

Definition 9 (weak solution). We call a function $u: R_+\times R\to R$ a weak solution of the Cauchy problem (5) provided that

(i) $u\in L^\infty(R_+;L^2(R))$;

(ii) $u_t - u_{txx} + 3mu^2u_x = 6u_x^3 + 18uu_xu_{xx} + 3u^2u_{xxx}$ in $D'([0,\infty)\times R)$, that is, for all $\varphi\in C_0^\infty([0,\infty)\times R)$ there holds the following identity:

$$\begin{aligned}&\int_{R^+}\int_R \left(u(\varphi_t - \varphi_{txx}) + mu^3\varphi_x - u^3\varphi_{xxx}\right)dx\,dt \\ &\quad + \int_R u_0(x)\varphi(0,x)\,dx = 0.\end{aligned} \tag{38}$$

Theorem 10. *Let $u_0(x)\in L^2(R)$. Then, there exists a weak solution $u(t,x)\in L^\infty([0,\infty);L^2(R))$ to the problem* (5).

Proof. Consider the problem (36). For an arbitrary $T>0$, choosing a subsequence $\varepsilon_n\to 0$, from (37), we know that u_{ε_n} is bounded in L^∞ and $\|u_{\varepsilon_n}\|_{L^2(R)}$ is uniformly bounded in $L^2(R)$. Therefore, we obtain that $u_{\varepsilon_n}^3$ is bounded in $L^2(R)$. Therefore, there exist subsequences $\{u_{\varepsilon_n}\}$ and $\{u_{\varepsilon_n}^3\}$, still denoted by $\{u_{\varepsilon_n}\}$ and $\{u_{\varepsilon_n}^3\}$, are weakly convergent to v in $L^2(R)$. Noticing (38) completes the proof. □

Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities (JBK120504).

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