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Research Article

A Continuous-Time Model for Valuing Foreign Exchange Options

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This paper makes use of stochastic calculus to develop a continuous-time model for valuing European options on foreign exchange (FX) when both domestic and foreign spot rates follow a generalized Wiener process. Using the dollar/euro exchange rate as input for parameter estimation and employing our FX option model as a yardstick, we find that the traditional Garman-Kohlhagen FX option model, which assumes constant spot rates, values incorrectly calls and puts for different values of the ratio of exchange rate to exercise price. Specifically, it undervalues calls when the ratio is between 0.70 and 1.08, and it overvalues calls when the ratio is between 1.18 and 1.30, whereas it overvalues puts when the ratio is between 0.70 and 0.82, and it undervalues puts when the ratio is between 0.86 and 1.30.

1. Introduction

In this paper, we make use of stochastic calculus [1–3] to develop a continuous-time model for valuing European options on foreign exchange when both domestic and foreign spot rates follow a generalized Wiener process.

Foreign exchange (FX) options have traded on the Philadelphia Stock Exchange since 1982. An FX option is an agreement between two parties in which one party pays a premium and obtains the right to buy or sell the stated amount of foreign exchange at a later date at the exercise price, where the exercise price is an exchange rate agreed upon at the initial time. Extending the Black-Scholes [4] pricing model for stock options and assuming that both domestic and foreign spot rates are constant during the life of the option, Garman and Kohlhagen [5] developed in 1983 a model for valuing European FX options.

However, the assumption of Garman and Kohlhagen (G-K) that the two spot rates are constant over the life of the option is inappropriate because they are, in actuality, evolving continuously and stochastically through time. In this study, we incorporate the stochastic character of the two spot rates into our FX option model. Specifically, we employ the following stochastic process, often referred to as the generalized Wiener process, to represent the evolution of the spot rate $r(t)$ and, accordingly, derive a continuous-time model for valuing European call and put FX options as

$$dr(t) = adt + bdW. \tag{1}$$

In (1), the generalized Wiener process has a drift rate of a, a variance rate of b^2, and W is a standard Wiener process whose increment dW has a normal distribution with zero mean and variance dt.

Remark 1. It is possible that the spot rate under (1) can become negative. But negative spot rate is not probable if the drift rate is positive. More importantly, most FX options traded on an exchange have an expiration of less than one year. Hence, if we use an initial positive value for $r(t)$ and suitable values for the drift rate and variance rate, the expected first-passage time of the spot rate to the origin can easily be made longer than one year.

Remark 2. Our FX option model essentially extends the traditional G-K model to incorporate the stochastic character of the two spot rates. Like the G-K model, our model is for valuing European FX options. To value American options, often we have to resort to numerical procedures because no analytic formulas are available. For some well-articulated

numerical procedures for valuing American options, see Hull [6] for pricing FX options with constant interest rates, Ho et al. [7] for pricing stock options with stochastic interest rates, and Zhang and Wang [8] for pricing bond options with a penalty method (see [9, 10]).

The rest of this paper is organized as follows. In Section 2, we derive an explicit formula for valuing European call and put FX options when both domestic and foreign spot rates evolve according to (1). In Section 3, we first estimate the various parameters of our FX option model based on the dollar/euro exchange rate; then, we compute the FX option prices for our model and the G-K model using the parameter estimates as inputs, and finally we examine the pricing biases in the G-K model employing our model as a yardstick. A short conclusion is given in Section 4. The derivation of the rather lengthy equation (27) in Section 2 is relegated to the Appendix.

2. Deriving a Formula for Call and Put FX Options

In this paper, we assume that the foreign exchange market is frictionless; that is, there are no trading costs, margin requirements, exchange rate controls, and taxes; trading takes place continuously; borrowing and short-selling are allowed; there exist pure discount bonds at which each currency can be borrowed or lent. To proceed, we adopt the following notation:

$S(t)$: the spot exchange rate (domestic currency price per unit of foreign currency) at time t,

$r_d(t)$: the domestic spot rate of interest at time t,

$r_f(t)$: the foreign spot rate of interest at time t,

$B_d(t,T)$: the domestic currency price of a pure discount bond at time t which pays one unit of domestic currency at time $t+T$,

$B_f(t,T)$: the foreign currency price of a pure discount bond at time t which pays one unit of foreign currency at time $t+T$,

$c(t,T)$: the domestic currency price at time t of a European call on one unit of foreign currency which expires at time $t+T$,

$p(t,T)$: the domestic currency price at time t of a European put on one unit of foreign currency which expires at time $t+T$,

X: the domestic currency exercise price of a European call or a put on foreign currency.

Using (1) for the domestic and foreign spot rates, their diffusion processes are expressed as follows:

$$\begin{aligned} dr_d(t) &= a_d dt + b_d dW_d, \\ dr_f(t) &= a_f dt + b_f dW_f, \end{aligned} \tag{2}$$

with $dW_d dW_f = \rho_{df} dt$. Using (2) and applying Ito's lemma [11–13], we have

$$\begin{aligned} dB_d &= \left[\frac{\partial B_d}{\partial t} + a_d \frac{\partial B_d}{\partial r_d} + \frac{1}{2} b_d^2 \frac{\partial^2 B_d}{\partial r_d^2}\right] dt + b_d \frac{\partial B_d}{\partial r_d} dW_d, \\ dB_f &= \left[\frac{\partial B_f}{\partial t} + a_f \frac{\partial B_f}{\partial r_f} + \frac{1}{2} b_f^2 \frac{\partial^2 B_f}{\partial r_f^2}\right] dt + b_f \frac{\partial B_f}{\partial r_f} dW_f. \end{aligned} \tag{3}$$

Letting

$$\begin{aligned} \mu_d &= \frac{1}{B_d}\left[\frac{\partial B_d}{\partial t} + a_d \frac{\partial B_d}{\partial r_d} + \frac{1}{2} b_d^2 \frac{\partial^2 B_d}{\partial r_d^2}\right], \\ \sigma_d &= \frac{-1}{B_d}\left[b_d \frac{\partial B_d}{\partial r_d}\right] = \frac{-b_d}{B_d}\frac{\partial B_d}{\partial r_d}, \\ \mu_f &= \frac{1}{B_f}\left[\frac{\partial B_f}{\partial t} + a_f \frac{\partial B_f}{\partial r_f} + \frac{1}{2} b_f^2 \frac{\partial^2 B_f}{\partial r_f^2}\right], \\ \sigma_f &= \frac{-1}{B_f}\left[b_f \frac{\partial B_f}{\partial r_f}\right] = \frac{-b_f}{B_f}\frac{\partial B_f}{\partial r_f}, \end{aligned} \tag{4}$$

and assuming the local expectations hypothesis holds for the term structure of interest rates (i.e., $\mu_d = r_d$ and $\mu_f = r_f$), we obtain

$$\frac{dB_d}{B_d} = r_d dt + \sigma_d dW_d, \tag{5}$$

$$\frac{dB_f}{B_f} = r_f dt + \sigma_f dW_f. \tag{6}$$

Solving (5) and (6) for B_d and B_f, we obtain explicit formulas for the prices of domestic and foreign pure discount bonds with time to maturity T as

$$\begin{aligned} B_d(t,T) &= \exp\left\{-r_d T - \frac{a_d T^2}{2} + \frac{b_d^2 T^3}{6}\right\}, \\ B_f(t,T) &= \exp\left\{-r_f T - \frac{a_f T^2}{2} + \frac{b_f^2 T^3}{6}\right\}. \end{aligned} \tag{7}$$

Note that $T = -(1/B_d)[\partial B_d/\partial r_d] = -(1/B_f)[\partial B_f/\partial r_f]$. Hence, we have $\sigma_d = -(b_d/B_d)[\partial B_d/\partial r_d] = b_d T$ and $\sigma_f = -(b_f/B_f)[\partial B_f/\partial r_f] = b_f T$.

Similar to the G-K FX option model, we assume that the spot exchange rate follows the geometric Wiener process

$$\frac{dS}{S} = \mu_s dt + \sigma_s dW_s, \tag{8}$$

where μ_s and σ_s are constant parameters, and W_s is a standard Wiener process. In addition, we assume $dW_s dW_d = \rho_{sd} dt$ and $dW_s dW_f = \rho_{sf} dt$.

Converting the price of a foreign pure discount bond into domestic currency price, we define a new variable $G = SB_f$ such that

$$\frac{dG}{G} = \mu_G dt + \sigma_G dW_G. \tag{9}$$

Applying Ito's lemma to the call function $c \equiv c(t,T) \equiv c(G,B_d,T;X)$ and the relation $dt = -dT$, we obtain

$$dc = \frac{\partial c}{\partial G}dG + \frac{\partial c}{\partial B_d}dB_d - \left[\frac{1}{2}\frac{\partial^2 c}{\partial G^2}\sigma_G^2 G^2 + \frac{1}{2}\frac{\partial^2 c}{\partial B_d^2}\sigma_d^2 B_d^2 + \frac{\partial^2 c}{\partial G \partial B_d}\rho_{Gd}\sigma_G\sigma_d G B_d - \frac{\partial c}{\partial T}\right]dT. \tag{10}$$

At this point, we set up a hedge consisting of three assets: G, B_d, and c. Let Q_G be the number of G, Q_d the number of B_d, and Q_c the number of c in the hedge. Let P_h be the value of the hedge portfolio. The hedge is set up in such a way that the value of this hedge portfolio is zero, that is, $P_h = Q_G G + Q_d B_d + Q_c c = 0$. Hence, the change in the value of this hedge portfolio is also zero, that is,

$$dP_h = Q_G dG + Q_d dB_d + Q_c dc = 0. \tag{11}$$

Remark 3. Our hedge is different from that of Black and Scholes [4]. In their case, they create their hedge such that the hedge portfolio is riskless. Hence, its return must equal the riskless rate or the spot rate. In our case, we create our hedge such that the value P_h of the hedge portfolio is zero (i.e., the aggregate investment is zero). Hence, we have $dP_h = 0$ in (11).

Substituting (5), (9), and (10) into (11) and grouping, (11) becomes

$$dP_h = Q_c\left[\frac{1}{2}\frac{\partial^2 c}{\partial G^2}\sigma_G^2 G^2 + \frac{1}{2}\frac{\partial^2 c}{\partial B_d^2}\sigma_d^2 B_d^2 + \frac{\partial^2 c}{\partial G \partial B_d}\rho_{Gd}\sigma_G\sigma_d G B_d - \frac{\partial c}{\partial T}\right]dT + \left[Q_c\frac{\partial c}{\partial G} + Q_G\right]dG + \left[Q_c\frac{\partial c}{\partial B_d} + Q_d\right]dB_d = 0. \tag{12}$$

Equation (12) suggests that $Q_c(\partial c/\partial G) + Q_G = 0$, $Q_c(\partial c/\partial B_d) + Q_d = 0$, and

$$\frac{1}{2}\frac{\partial^2 c}{\partial G^2}\sigma_G^2 G^2 + \frac{1}{2}\frac{\partial^2 c}{\partial B_d^2}\sigma_d^2 B_d^2 + \frac{\partial^2 c}{\partial G \partial B_d}\rho_{Gd}\sigma_G\sigma_d G B_d - \frac{\partial c}{\partial T} = 0. \tag{13}$$

Then the price of a call must satisfy (13) subject to two boundary conditions: (i) the call price is zero if $G = 0$; (ii) at its expiration, the call has a value of either zero if $G \le X$ or $G - X$ if $G > X$. In notations, the two boundary conditions are

$$c(G = 0, B_d, T; X) = 0, \qquad c(G, B_d = 1, 0; X) = \max(0, G - X). \tag{14}$$

The second-order partial differential equation in (13) has no well-known solution. Hence, to solve for c in (13), we transform (13) to a standard heat equation [14–16] of the form $u_t(x,t) = \alpha^2 u_{xx}(x,t)$, where α is some constant. Making use of the linear homogeneity of c in G and XB_d, we can carry out such transformation for (13). Consequently, we set $\theta \equiv \theta(G, B_d, T) = G/XB_d$.

Remark 4. A function $f \equiv f(x_1,\ldots,x_n)$ is said to be linear homogeneous or homogeneous of degree one in x_i, where $i = 1,\ldots,n$, if $f(\alpha x_1,\ldots,\alpha x_n) = \alpha f(x_1,\ldots,x_n)$, where α is some constant. In a competitive and perfect market, the fact that the value of a call is homogeneous of degree one in the asset and exercise pricemeans that the value of the call with exercise price X when the asset value is G will be exactly α times the value of a call on the same asset with exercise price X/α when the asset value is G/α. See [17–19] for more description on linear homogeneity.

Using Ito's lemma and (5) and (9), the total differential of θ is given by

$$d\theta = \left[\frac{\partial\theta}{\partial G}\mu_G G + \frac{\partial\theta}{\partial B_d}r_d B_d + \frac{\partial\theta}{\partial t} + \frac{1}{2}\frac{\partial^2\theta}{\partial G^2}\sigma_G^2 G^2 + \frac{1}{2}\frac{\partial^2\theta}{\partial B_d^2}\sigma_d^2 B_d^2 + \frac{\partial^2\theta}{\partial G\partial B_d}\rho_{Gd}\sigma_G\sigma_d G B_d\right]dt + \frac{\partial\theta}{\partial G}\sigma_G G dW_G + \frac{\partial\theta}{\partial B_d}\sigma_d B_d dW_d. \tag{15}$$

Substituting $\partial\theta/\partial G = 1/XB_d$, $\partial\theta/\partial B_d = -G/XB_d^2$, $\partial\theta/\partial t = 0$, $\partial^2\theta/\partial G^2 = 0$, $\partial^2\theta/\partial B_d^2 = G/XB_d^3$, and $\partial^2\theta/\partial G\partial B_d = -1/XB_d^2$ into (15) and simplifying, (15) becomes

$$\frac{d\theta}{\theta} = \mu_\theta dt + \sigma_\theta dW_\theta, \tag{16}$$

where $\mu_\theta = \mu_G - r_d + (\sigma_d^2/2) - \rho_{Gd}\sigma_G\sigma_d$ and $\sigma_\theta^2 = \sigma_G^2 + \sigma_d^2 - 2\rho_{Gd}\sigma_G\sigma_d$.

To solve (13) for $c(G,B_d,T;X)$ subject to the two boundary conditions in (14), we use another variable K such that $K \equiv K(\theta,T;X) = (c(G,B_d,T;X))/XB_d$. In words, K is the call price expressed in the same units as θ. Expressed in another way, $c(G,B_d,T;X) = XB_d K(\theta,T;X)$. Then $\partial^2 c/\partial G^2 = (1/XB_d)(\partial^2 K/\partial G^2)$, $\partial^2 c/\partial B_d^2 = (G^2/XB_d^3)(\partial^2 K/\partial G^2)$, $\partial^2 c/\partial G\partial B_d = -(G/XB_d^2)(\partial^2 K/\partial G^2)$, and $\partial c/\partial T = XB_d(\partial K/\partial T)$. Substituting them into (13) and simplifying, (13) becomes

$$\frac{1}{2}\left(\frac{G^2}{XB_d}\right)\left[\sigma_G^2 + \sigma_d^2 - 2\rho_{Gd}\sigma_G\sigma_d\right]\frac{\partial^2 K}{\partial G^2} - XB_d\frac{\partial K}{\partial T} = 0, \tag{17}$$

or

$$\frac{1}{2}\left(\frac{G}{XB_d}\right)^2\left[\sigma_G^2 + \sigma_d^2 - 2\rho_{Gd}\sigma_G\sigma_d\right]\frac{\partial^2 K}{\partial G^2} - \frac{\partial K}{\partial T} = 0. \tag{18}$$

Since $\theta^2 = (G/XB_d)^2$ and $\sigma_\theta^2 = \sigma_G^2 + \sigma_d^2 - 2\rho_{Gd}\sigma_G\sigma_d$, (18) becomes

$$\frac{1}{2}\sigma_\theta^2\theta^2\frac{\partial^2 K}{\partial\theta^2} - \frac{\partial K}{\partial T} = 0. \tag{19}$$

In other words, $K \equiv K(\theta, T; X)$ has to satisfy (19) subject to the following two boundary conditions: $K(\theta = 0, T; X) = 0$ and $K(\theta, 0; X) = \max(0, \theta - 1)$.

Remark 5. Given $\theta = G/XB_d$, $G = 0$ is equivalent to $\theta = 0$. Thus, we have the first boundary condition $K(\theta = 0, T; X) = 0$. At time $t+T$, $B_d(T, T) = 1$, and we have the second boundary condition $K(\theta, 0; X) = (c(G, B_d(T, T), 0; X))/(XB_d(T, T)) = (\max(0, G - X))/X = \max(0, (G/X) - 1) = \max(0, \theta - 1)$.

Setting up another variable $V \equiv \int_t^{t+T} \sigma_\theta^2(v)dv$ and then defining $(\theta, V) \equiv K(\theta, T)$, we have that $\partial^2 K/\partial\theta^2 = \partial^2 y/\partial\theta^2$ and $\partial K/\partial T = (\partial y/\partial V)(\partial V/\partial T) = (\partial y/\partial V)\sigma_\theta^2$. Substituting them into (19), we obtain

$$\frac{1}{2}\sigma_\theta^2\theta^2\frac{\partial^2 y}{\partial\theta^2} - \frac{\partial y}{\partial V}\sigma_\theta^2 = \sigma_\theta^2\left[\frac{1}{2}\theta^2\frac{\partial^2 y}{\partial\theta^2} - \frac{\partial y}{\partial V}\right] = 0, \tag{20}$$

or

$$\frac{1}{2}\theta^2\frac{\partial^2 y}{\partial\theta^2} - \frac{\partial y}{\partial V} = 0. \tag{21}$$

In other words, $y = y(\theta, V)$ has to satisfy (21) subject to the following two boundary conditions: $y(0, V) = 0$ and $y(\theta, 0) = \max(0, \theta - 1)$.

To solve (21) subject to the two boundary conditions, we transform (21) by using the change of variables $\delta \equiv \log\theta + (V/2)$ and $u(\delta, V) \equiv y(\theta, V)/\theta$. Then we have $y(\theta, V) = u(\log\theta + (1/2)\int_t^{t+T}\sigma_\theta^2(v)dv, \int_t^{t+T}\sigma_\theta^2(v)dv)\theta$, $\partial^2 y/\partial\theta^2 = (1/\theta)((\partial u/\partial\delta) + (\partial^2 u/\partial\delta^2))$, and $\partial y/\partial V = \theta((1/2)(\partial u/\partial\delta) + (\partial u/\partial V))$. Substituting them into (21) and simplifying, (21) becomes $\partial u/\partial V = (1/2)(\partial^2 u/\partial\delta^2)$. For the first boundary condition, we have that $u(\delta, V) = y(\theta, V)/\theta = (1/\theta)K(\theta, T) = (1/\theta)(c/XB_d) = (XB_d/G)(c/XB_d) = c/G$. As always, $0 \le c \le G$. Thus, the first boundary condition is $|u(\delta, V)| = (c/G) \le 1$. For the second boundary condition, we have that $1 - (1/\theta) = 1 - \exp[-\log\theta] = 1 - \exp[-\delta]$, because $\delta = \log\theta$ when $V = 0$. Hence, $u(\delta, V = 0) = (\theta - 1)/\theta = 1 - (1/\theta) = 1 - \exp[-\delta]$ if $1 \ge \exp[-\delta]$ and $u(\delta, V = 0) = 0$ if $1 < \exp[-\delta]$. In short, our boundary value problem is made up of a boundedness condition $|u(\delta, V)| \le 1$ and the following two conditions:

$$\frac{\partial u(\delta, V)}{\partial V} = \frac{1}{2}\frac{\partial^2 u(\delta, V)}{\partial\delta^2}, \tag{22}$$

$$u(\delta, 0) = \begin{cases} 1 - \exp[-\delta] & \text{if } 1 \ge \exp[-\delta] \\ 0 & \text{if } 1 < \exp[-\delta]. \end{cases} \tag{23}$$

Now (22) is in standard heat equation form and hence can be solved by the separation-of-variables method. Let $u(\delta, V) = f(\delta)g(V)$, where $f(\delta)$ is some function of δ and $g(V)$ is some function of V. Substituting $u(\delta, V) = f(\delta)g(V)$ into (22) and simplifying, we get

$$\frac{2}{g(V)}\frac{\partial g(V)}{\partial V} = \frac{1}{f(\delta)}\frac{\partial^2 f(\delta)}{\partial\delta^2}. \tag{24}$$

In other words, the right-hand side of (24) depends only on δ and the left-hand side depends only on V. Since δ and V are independent variables, we can equate the two sides of (24) to a constant $-k^2$, where $k > 0$.

Remark 6. We equate the two sides of (24) to $-k^2$ so that the two differential equations in (25) have continuous eigenvalues k^2.

Hence, by setting the two sides of (24) equal to $-k^2$, we obtain the following two ordinary differential equations:

$$\begin{aligned} \frac{\partial g(V)}{\partial V} + \frac{1}{2}k^2 g(V) &= 0, \\ \frac{\partial^2 f(\delta)}{\partial\delta^2} + k^2 f(\delta) &= 0. \end{aligned} \tag{25}$$

Rewriting (25), we have $g(V) = \exp(-(1/2)k^2 V)$ and $f(\delta) = A(\delta)\cos(k\delta) + B(\delta)\sin(k\delta)$. The generalized linear combination of functions $u(\delta, V) = f(\delta)g(V)$ becomes

$$\begin{aligned} u(\delta, V) = \int_0^\infty & [A(k)\cos(k\delta) + B(k)\sin(k\delta)] \\ & \times \exp\left(-\frac{1}{2}k^2 V\right)dk. \end{aligned} \tag{26}$$

By the Fourier integral theorem, the expression in (26) is legitimate if $A(k) = (1/\pi)\int_{-\infty}^{\infty} f(\omega)\cos(k\omega)d\omega$ and $B(k) = (1/\pi)\int_{-\infty}^{\infty} f(\omega)\sin(k\omega)d\omega$. Substituting $A(k)$ and $B(k)$ into (26), we obtain (see the Appendix for derivation of (27))

$$\begin{aligned} u(\delta, V) &= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} [\cos(k\omega)\cos(k\delta) + \sin(k\omega)\sin(k\delta)] \right. \\ &\qquad \left. \times f(\omega)\,d\omega\right\} \exp\left(-\frac{1}{2}k^2 V\right)dk \\ &= \frac{1}{\sqrt{2V\pi}}\int_{-\infty}^{\infty} f(\omega)\exp\left(-\frac{(\omega-\delta)^2}{2V}\right)d\omega. \end{aligned} \tag{27}$$

Setting $q = (\omega - \delta)/\sqrt{2V}$, we have $\omega = \delta + q\sqrt{2V}$ and $d\omega = \sqrt{2V}dq$. Substituting $q = (\omega - \delta)/\sqrt{2V}$ and (24) into (27), we get

$$\begin{aligned} u(\delta, V) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f\left(\delta + q\sqrt{2V}\right)\exp\left(-q^2\right)dq \\ &= \frac{1}{\sqrt{\pi}}\int_{-\delta/\sqrt{2V}}^{\infty}\left[1 - \exp\left(-\delta - q\sqrt{2V}\right)\right]\exp\left(-q^2\right)dq \\ &= \frac{1}{\sqrt{\pi}}\int_{-\delta/\sqrt{2V}}^{\infty}\exp\left(-q^2\right)dq \\ &\quad - \frac{1}{\sqrt{\pi}}\int_{-\delta/\sqrt{2V}}^{\infty}\exp\left(-\delta - q\sqrt{2V} - q^2\right)dq. \end{aligned} \tag{28}$$

In order to solve (28), we make another change of variables by setting $a = \sqrt{2}q$. Then the first term of (28) becomes

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{-\delta/\sqrt{2V}}^{\infty}\exp\left(-q^2\right)dq &= \frac{1}{\sqrt{2\pi}}\int_{-\delta/\sqrt{V}}^{\infty}\exp\left(-\frac{a^2}{2}\right)da \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\delta/\sqrt{V}}\exp\left(-\frac{a^2}{2}\right)da \\ &= N\left(\frac{\delta}{\sqrt{V}}\right) = N\left(a_1\right),\end{aligned} \tag{29}$$

where $a_1 = \delta/\sqrt{V}$ and $N(\cdot)$ is the cumulative probability distribution function for a standardized normal random variable; that is, $N(a_1)$ is the probability that such a variable will be less than a_1.

For the second term of (28), note that $-\delta - q\sqrt{2V} - q^2 = -\log\theta - (V/2) - a\sqrt{V} - (a^2/2) = \log(1/\theta) - (1/2)[a^2 + 2a\sqrt{V} + (\sqrt{V})^2] = \log(1/\theta) - (1/2)(a + \sqrt{V})^2$. Hence, the integrand of the second term is $\exp(-\delta - q\sqrt{2V} - q^2) = \exp[\log(1/\theta) - (1/2)(a + \sqrt{V})^2] = (1/\theta)\exp[-(1/2)(a + \sqrt{V})^2]$. Substituting it into the second term and simplifying, (28) becomes

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\int_{-\delta/\sqrt{2V}}^{\infty}\exp\left(-\delta - q\sqrt{2V} - q^2\right)dq \\ &\quad= \frac{1}{\theta\sqrt{2\pi}}\int_{-\infty}^{a_1}\exp\left[-\frac{1}{2}\left(a + \sqrt{V}\right)^2\right]da \\ &\quad= \frac{1}{\theta\sqrt{2\pi}}\int_{-\infty}^{a_2}\exp\left[-\frac{b^2}{2}\right]db = \frac{1}{\theta}N\left(a_2\right),\end{aligned} \tag{30}$$

where $b = a + \sqrt{V}$ and $a_2 = a_1 - \sqrt{V}$. Combining (29) and (30), we obtain

$$u(\delta, V) = N\left(a_1\right) - \frac{1}{\theta}N\left(a_2\right), \tag{31}$$

or

$$y(\theta, V) = \theta u(\delta, V) = \theta N\left(a_1\right) - N\left(a_2\right). \tag{32}$$

As stated earlier, $G = SB_f$, $\theta = G/XB_d$, and $c(G, B_d, T; X) = XB_dK(\theta, T; X) = XB_dy(\theta, V)$. By the linear homogeneity property of $c \equiv c(t, T) \equiv c(G, B_d, T; X)$, the price of a European call FX option is

$$c = XB_d\left[\theta N\left(a_1\right) - N\left(a_2\right)\right] = SB_fN\left(a_1\right) - XB_dN\left(a_2\right), \tag{33}$$

where $a_1 = \delta/\sqrt{V} = (\log\theta + (V/2))/\sqrt{V}$ and $a_2 = a_1 - \sqrt{V} = (\log\theta - (V/2))/\sqrt{V}$. In addition, V is given by

$$\begin{aligned}V &\equiv \int_t^{t+T}\sigma_\theta^2(v)\,dv \\ &= \int_t^{t+T}\left[\sigma_s^2 - 2\left(\rho_{sd}\sigma_sb_d - \rho_{sf}\sigma_sb_f\right)v \right. \\ &\qquad\left. + \left(b_d^2 + b_f^2 - 2\rho_{df}b_db_f\right)v^2\right]dv \\ &= \sigma_s^2T - \left(\rho_{sd}\sigma_sb_d - \rho_{sf}\sigma_sb_f\right)T^2 \\ &\quad + \frac{1}{3}\left(b_d^2 + b_f^2 - 2\rho_{df}b_db_f\right)T^3.\end{aligned} \tag{34}$$

According to put-call parity for European options, the price $p \equiv p(G, B_d, T; X)$ of a European put FX option is

$$p = c - SB_f + XB_d = XB_dN\left(-a_2\right) - SB_fN\left(-a_1\right). \tag{35}$$

Equations (33) through (35), as a whole, constitute our FX option model. If both domestic and foreign spot rates are constant (i.e., $a_d = a_f = b_d = b_f = 0$ in (2)), then the domestic and foreign bond prices in (7) become $B_d(t, T) = \exp(-r_dT)$ and $B_f(t, T) = \exp(-r_fT)$, and $V \equiv \int_t^{t+T}\sigma_\theta^2(v)dv = \sigma_s^2T$. Substituting them into (33) and (35), our FX model then reduces to the following G-K FX model for valuing European call and put options:

$$\begin{aligned}c_{GK} &= S\exp\left(-r_fT\right)N\left(d_1\right) - X\exp\left(-r_dT\right)N\left(d_2\right), \\ p_{GK} &= X\exp\left(-r_dT\right)N\left(-d_2\right) - S\exp\left(-r_fT\right)N\left(-d_1\right),\end{aligned} \tag{36}$$

where $d_1 = (\log(S/X) + (r_d - r_f + \sigma_s^2/2)T)/\sigma_s\sqrt{T}$ and $d_2 = d_1 - \sigma_s\sqrt{T}$.

3. FX Option Prices and Pricing Biases in the G-K Model

Retrieved from the Datastream database, three sets of daily data (a total of 2,869 observations from 4 January 1999 to 31 December 2009) are used for parameter estimation. One set is the dollar/euro exchange rate and the other two sets are the 3-month US. Treasury Bill Rate and the 1-month euro-currency rate; that is, we use the Treasury Bill Rate to represent the domestic spot rate r_d and the euro-currency rate to represent the foreign spot rate r_f. Accordingly, the estimates for the six parameters of our FX option model are as follows: $\widehat{\sigma_s} = 0.198428$, $\widehat{b_d} = 0.018534$, $\widehat{b_f} = 0.011398$, $\widehat{\rho_{sd}} = -0.347350$, $\widehat{\rho_{sf}} = -0.297520$, and $\widehat{\rho_{df}} = 0.524996$.

Remark 7. The euro was introduced as an accounting currency on 1 January 1999. Euro coins and banknotes have been in circulation since 1 January 2002.

Using the above parameter estimates as inputs, we compute the option prices for our FX model ((33)–(35)) and

Table 1: Prices for European call and put FX options when time to maturity $T = 1$ month.

S_t/X	Call_O	Call_{GK}	Bias_c	Put_O	Put_{GK}	Bias_p
0.70	0.000000	0.000000		0.617117	0.617122	−0.0008
0.72	0.000000	0.000000		0.560023	0.560026	−0.0005
0.74	0.000000	0.000000		0.506014	0.506017	−0.0006
0.76	0.000000	0.000000		0.454848	0.454851	−0.0007
0.78	0.000000	0.000000		0.406307	0.406308	−0.0002
0.80	0.000001	0.000001		0.360193	0.360194	−0.0003
0.82	0.000006	0.000006		0.316332	0.316333	−0.0003
0.84	0.000029	0.000029		0.274579	0.274578	0.0004
0.86	0.000115	0.000115		0.234832	0.234830	0.0009
0.88	0.000387	0.000385	0.5195	0.197081	0.197077	0.0020
0.90	0.001109	0.001105	0.3620	0.161470	0.161464	0.0037
0.92	0.002753	0.002746	0.2549	0.128360	0.128351	0.0070
0.94	0.006003	0.005992	0.1836	0.098336	0.098322	0.0142
0.96	0.011655	0.011639	0.1375	0.072100	0.072081	0.0264
0.98	0.020408	0.020390	0.0883	0.050267	0.050245	0.0438
1.00	0.032643	0.032624	0.0582	0.033139	0.033116	0.0695
1.02	0.048284	0.048267	0.0352	0.020568	0.020547	0.1022
1.04	0.066823	0.066811	0.0180	0.011981	0.011964	0.1421
1.06	0.087482	0.087475	0.0080	0.006538	0.006525	0.1992
1.08	0.109419	0.109416	0.0027	0.003339	0.003330	0.2703
1.10	0.131898	0.131898	0.0000	0.001596	0.001590	0.3774
1.12	0.154374	0.154377	−0.0019	0.000714	0.000711	0.4219
1.14	0.176497	0.176502	−0.0028	0.000300	0.000298	0.6711
1.16	0.198075	0.198082	−0.0035	0.000118	0.000117	0.8547
1.18	0.219023	0.219031	−0.0037	0.000044	0.000043	2.3256
1.20	0.239317	0.239324	−0.0029	0.000015	0.000015	
1.22	0.258962	0.258970	−0.0031	0.000005	0.000005	
1.24	0.277980	0.277988	−0.0029	0.000002	0.000002	
1.26	0.296397	0.296405	−0.0027	0.000000	0.000000	
1.28	0.314238	0.314247	−0.0029	0.000000	0.000000	
1.30	0.331531	0.331540	−0.0027	0.000000	0.000000	

S_t is the exchange rate at initial time t, X is the exercise price, Call_O and Put_O are call and put prices for our model, Call_{GK} and Put_{GK} are call and put prices for the G-K model, $\text{Bias}_c = 100(\text{Call}_O - \text{Call}_{GK})/\text{Call}_{GK}$, and $\text{Bias}_p = 100(\text{Put}_O - \text{Put}_{GK})/\text{Put}_{GK}$.

the G-K model (36) by setting the initial time $t = 1$ January 2010 when $r_d(t) = 0.0008$, $r_f(t) = 0.0049$, and $S_t = 1.4389$. In addition, employing our FX option model as a yardstick, we examine whether or not the G-K model values correctly FX call and put options for different values of S_t/X, where S_t is the exchange rate on 1 January 2010 and X is the exercise price. Given that S_t is fixed at 1.4389, we vary X such that S_t/X ranges from 0.70 to 1.30—a range large enough to include even extreme values of X not commonly used in practice. Tables 1, 2, 3, and 4 show the FX option prices and pricing biases in the G-K model when time to maturity T is 1, 3, 6, and 9 months, respectively.

We first focus on call options. For $T = 1, 3, 6$, and 9 months, call prices increase as S_t/X increases from 0.70 to 1.30 under both our model and the G-K model. For example, in Table 1 where $T = 1$ month, call price increases from 0.001109 to 0.032643 and to 0.131898 under our model, and from 0.001105 to 0.032624 and to 0.131898 under the G-K model as S_t/X increases from 0.90 to 1.00 and to 1.10, respectively. For each of the four Ts, the G-K model incorrectly values FX calls for different values of S_t/X. Specifically, it undervalues calls when S_t/X ranges from 0.70 to 1.08, and it overvalues calls when S_t/X ranges from 1.18 to 1.30. For example, in Table 2 where $T = 3$ months, call price is 0.000666 under our model and 0.000657 under the G-K model for $S_t/X = 0.80$, which amounts to a positive pricing bias of 1.3699%, whereas call price is 0.330616 under our model and 0.330678 under the G-K model for $S_t/X = 1.30$, which amounts to a negative pricing bias of −0.0187%.

The situation is completely opposite for put options. For $T = 1, 3, 6$, and 9 months, put prices decrease as S_t/X increases from 0.70 to 1.30 under both models. For example, in Table 3 where $T = 6$ months, put price decreases from 0.367578 to 0.082131 and to 0.008767 under our model, and from 0.367695 to 0.081844 and to 0.008639 under the G-K model as S_t/X increases from 0.80 to 1.00

Table 2: Prices for European call and put FX options when time to maturity $T = 3$ months.

S_t/X	Call_O	Call_{GK}	Bias_c	Put_O	Put_{GK}	Bias_p
0.70	0.000007	0.000007		0.617944	0.618028	−0.0136
0.72	0.000020	0.000020		0.560879	0.560954	−0.0134
0.74	0.000055	0.000054	1.8519	0.506920	0.506986	−0.0130
0.76	0.000137	0.000135	1.4815	0.455851	0.455907	−0.0123
0.78	0.000314	0.000309	1.6181	0.407499	0.407545	−0.0113
0.80	0.000666	0.000657	1.3699	0.361749	0.361783	−0.0094
0.82	0.001315	0.001299	1.2317	0.318545	0.318566	−0.0066
0.84	0.002432	0.002407	1.0386	0.277897	0.277902	−0.0018
0.86	0.004236	0.004200	0.8571	0.239879	0.239867	0.0050
0.88	0.006985	0.006935	0.7210	0.204615	0.204584	0.0151
0.90	0.010951	0.010888	0.5786	0.172258	0.172208	0.0290
0.92	0.016399	0.016322	0.4718	0.142962	0.142892	0.0490
0.94	0.023547	0.023459	0.3751	0.116845	0.116759	0.0737
0.96	0.032548	0.032452	0.2958	0.093967	0.093868	0.1055
0.98	0.043461	0.043362	0.2283	0.074302	0.074195	0.1442
1.00	0.056252	0.056155	0.1727	0.057739	0.057629	0.1909
1.02	0.070794	0.070703	0.1287	0.044077	0.043969	0.2456
1.04	0.086884	0.086803	0.0933	0.033048	0.032946	0.3096
1.06	0.104267	0.104200	0.0643	0.024335	0.024243	0.3795
1.08	0.122661	0.122610	0.0416	0.017600	0.017520	0.4566
1.10	0.141781	0.141745	0.0254	0.012505	0.012436	0.5548
1.12	0.161356	0.161337	0.0118	0.008730	0.008674	0.6456
1.14	0.181148	0.181143	0.0028	0.005991	0.005946	0.7568
1.16	0.200954	0.200963	−0.0045	0.004042	0.004007	0.8735
1.18	0.220612	0.220632	−0.0091	0.002683	0.002657	0.9785
1.20	0.239997	0.240028	−0.0129	0.001753	0.001733	1.1541
1.22	0.259022	0.259062	−0.0154	0.001128	0.001114	1.2567
1.24	0.277625	0.277672	−0.0169	0.000715	0.000705	1.4184
1.26	0.295770	0.295822	−0.0176	0.000446	0.000440	1.3636
1.28	0.313436	0.313493	−0.0182	0.000275	0.000270	1.8519
1.30	0.330616	0.330678	−0.0187	0.000167	0.000164	1.8293

S_t is the exchange rate at initial time t, X is the exercise price, Call_O and Put_O are call and put prices for our model, Call_{GK} and Put_{GK} are call and put prices for the G-K model, $\text{Bias}_c = 100(\text{Call}_O - \text{Call}_{GK})/\text{Call}_{GK}$, and $\text{Bias}_p = 100(\text{Put}_O - \text{Put}_{GK})/\text{Put}_{GK}$.

and to 1.20, respectively. For each of the four Ts, the G-K model incorrectly values FX puts for different values of S_t/X. Specifically, it overvalues puts when S_t/X ranges from 0.70 to 0.82, and it undervalues puts when S_t/X ranges from 0.86 to 1.30. For example, in Table 4 where $T = 9$ months, put price is 0.375783 under our model and 0.376021 under the G-K model for $S_t/X = 0.80$, which amounts to a negative pricing bias of −0.0633%, whereas put price is 0.006367 under our model and 0.006198 under the G-K model for $S_t/X = 1.30$, which amounts to a positive pricing bias of 2.7267%.

Foreign exchange options are actively traded on the Philadelphia Stock Exchange. The contract size of a euro-currency option is €62,500. For example, when $T = 9$ months and $S_t/X = 0.90$, call premium is (\$0.042077 × 62,500) = \$2,629.81 under our model and (\$0.041614 × 62,500) = \$2,600.88 under the G-K model—a difference of \$28.93 underpaid by a call option buyer; similarly, when $T = 9$ months and $S_t/X = 1.10$, put premium is (\$0.044294 × 62,500) = \$2,768.38 under our model and (\$0.043837 × 62,500) = \$2,739.81 under the G-K model—a difference of \$28.57 underpaid by a put option buyer. In other words, option sellers are at an obvious disadvantage if FX option valuation is based on the G-K model.

4. Conclusion

Prior research often assumes constant spot rates when valuing FX options. The traditional G-K FX option model assumes that both domestic and foreign spot rates remain unchanged over the life of the FX option. The fact of the matter is that both spot rates are changing continuously and stochastically through time. In this paper, we utilize stochastic calculus to develop a continuous-time model for valuing European call and put options on foreign exchange when both spot rates are assumed to follow a generalized Wiener process. Using the dollar/euro exchange rate as input for parameter estimation

Table 3: Prices for European call and put FX options when time to maturity $T = 6$ months.

S_t/X	Call_O	Call_{GK}	Bias_c	Put_O	Put_{GK}	Bias_p
0.70	0.000418	0.000404	3.4653	0.619406	0.619774	−0.0594
0.72	0.000760	0.000736	3.2609	0.562709	0.563030	−0.0570
0.74	0.001314	0.001278	2.8169	0.509307	0.509581	−0.0538
0.76	0.002172	0.002120	2.4528	0.459050	0.459273	−0.0486
0.78	0.003447	0.003374	2.1636	0.411830	0.412001	−0.0415
0.80	0.005265	0.005168	1.8769	0.367578	0.367695	−0.0318
0.82	0.007767	0.007643	1.6224	0.326258	0.326318	−0.0184
0.84	0.011095	0.010942	1.3983	0.287850	0.287854	−0.0014
0.86	0.015386	0.015204	1.1971	0.252347	0.252296	0.0202
0.88	0.020765	0.020555	1.0216	0.219740	0.219636	0.0474
0.90	0.027330	0.027095	0.8673	0.190008	0.189855	0.0806
0.92	0.035151	0.034897	0.7279	0.163110	0.162915	0.1197
0.94	0.044264	0.043997	0.6069	0.138981	0.138750	0.1665
0.96	0.054668	0.054394	0.5037	0.117527	0.117270	0.2192
0.98	0.066323	0.066051	0.4118	0.098627	0.098350	0.2816
1.00	0.079161	0.078898	0.3333	0.082131	0.081844	0.3507
1.02	0.093084	0.092837	0.2661	0.067869	0.067580	0.4276
1.04	0.107972	0.107748	0.2079	0.055657	0.055373	0.5129
1.06	0.123689	0.123494	0.1579	0.045298	0.045025	0.6063
1.08	0.140095	0.139932	0.1165	0.036592	0.036335	0.7073
1.10	0.157045	0.156917	0.0816	0.029344	0.029106	0.8177
1.12	0.174398	0.174306	0.0528	0.023362	0.023146	0.9332
1.14	0.192021	0.191967	0.0281	0.018470	0.018276	1.0615
1.16	0.209793	0.209776	0.0081	0.014503	0.014332	1.1931
1.18	0.227604	0.227623	−0.0083	0.011312	0.011163	1.3348
1.20	0.245361	0.245414	−0.0216	0.008767	0.008639	1.4817
1.22	0.262982	0.263067	−0.0323	0.006752	0.006643	1.6408
1.24	0.280401	0.280517	−0.0414	0.005169	0.005077	1.8121
1.26	0.297566	0.297709	−0.0480	0.003934	0.003857	1.9964
1.28	0.314434	0.314602	−0.0534	0.002977	0.002914	2.1620
1.30	0.330974	0.331165	−0.0577	0.002241	0.002189	2.3755

S_t is the exchange rate at initial time t, X is the exercise price, Call_O and Put_O are call and put prices for our model, Call_{GK} and Put_{GK} are call and put prices for the G-K model, $\text{Bias}_c = 100(\text{Call}_O - \text{Call}_{GK})/\text{Call}_{GK}$, and $\text{Bias}_p = 100(\text{Put}_O - \text{Put}_{GK})/\text{Put}_{GK}$.

and employing our FX option model as a yardstick, our numerical results show that the G-K model values incorrectly both call and put options for different values of S_t/X. Specifically, it undervalues calls when S_t/X is between 0.70 and 1.08, and it overvalues calls when S_t/X is between 1.18 and 1.30, whereas it overvalues puts when S_t/X is between 0.70 and 0.82, and it undervalues puts when S_t/X is between 0.86 and 1.30.

Appendix

To derive (27), we first prove the following:

$$\int_0^\infty \cos[k(\omega-\delta)]\exp\left[-\frac{1}{2}k^2V\right]dk = \sqrt{\frac{\pi}{2V}}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right]. \tag{A.1}$$

Proof. Since $\cos[k(\omega-\delta)] = (1/2)\exp[ik(\omega-\delta)] + (1/2)\exp[-ik(\omega-\delta)]$, we obtain

$$\begin{aligned}\cos[k(\omega-\delta)]\exp\left[-\frac{1}{2}k^2V\right] &= \frac{1}{2}\exp\left[-\frac{1}{2}k^2V + ik(\omega-\delta)\right] \\ &\quad + \frac{1}{2}\exp\left[-\frac{1}{2}k^2V - ik(\omega-\delta)\right],\end{aligned} \tag{A.2}$$

where $i = \sqrt{-1}$. Since $-(1/2)k^2V + ik(\omega-\delta) = -((\omega-\delta)^2/2V) - [(kV - i(\omega-\delta))/\sqrt{2V}]^2$, we have

$$\begin{aligned}\frac{1}{2}\int_0^\infty \exp\left[-\frac{1}{2}k^2V + ik(\omega-\delta)\right]dk &= \frac{1}{2}\int_0^\infty \exp\left[-\frac{(\omega-\delta)^2}{2V} - \left(\frac{kV - i(\omega-\delta)}{\sqrt{2V}}\right)^2\right]dk\end{aligned}$$

Table 4: Prices for European call and put FX options when time to maturity $T = 9$ months.

S_t/X	Call_O	Call_{GK}	Bias_c	Put_O	Put_{GK}	Bias_p
0.70	0.002017	0.001937	4.1301	0.621839	0.622653	−0.1307
0.72	0.003077	0.002967	3.7074	0.565920	0.566619	−0.1234
0.74	0.004544	0.004401	3.2493	0.513488	0.514072	−0.1136
0.76	0.006516	0.006333	2.8896	0.464397	0.464865	−0.1007
0.78	0.009091	0.008865	2.5494	0.418528	0.418880	−0.0840
0.80	0.012367	0.012096	2.2404	0.375783	0.376021	−0.0633
0.82	0.016438	0.016122	1.9601	0.336077	0.336204	−0.0378
0.84	0.021386	0.021026	1.7122	0.299333	0.299353	−0.0067
0.86	0.027279	0.026878	1.4919	0.265473	0.265392	0.0305
0.88	0.034166	0.033730	1.2926	0.234414	0.234242	0.0734
0.90	0.042077	0.041614	1.1126	0.206066	0.205811	0.1239
0.92	0.051021	0.050540	0.9517	0.180327	0.180002	0.1806
0.94	0.060985	0.060494	0.8117	0.157083	0.156699	0.2451
0.96	0.071935	0.071446	0.6844	0.136210	0.135780	0.3167
0.98	0.083822	0.083344	0.5735	0.117572	0.117107	0.3971
1.00	0.096577	0.096121	0.4744	0.101024	0.100537	0.4844
1.02	0.110123	0.109698	0.3874	0.086416	0.085917	0.5808
1.04	0.124372	0.123986	0.3113	0.073592	0.073092	0.6841
1.06	0.139229	0.138890	0.2441	0.062399	0.061907	0.7947
1.08	0.154598	0.154311	0.1860	0.052683	0.052205	0.9156
1.10	0.170382	0.170153	0.1346	0.044294	0.043837	1.0425
1.12	0.186487	0.186318	0.0907	0.037090	0.036658	1.1785
1.14	0.202823	0.202717	0.0523	0.030934	0.030531	1.3200
1.16	0.219306	0.219263	0.0196	0.025701	0.025328	1.4727
1.18	0.235858	0.235878	−0.0085	0.021273	0.020932	1.6291
1.20	0.252410	0.252493	−0.0329	0.017544	0.017235	1.7929
1.22	0.268900	0.269044	−0.0535	0.014419	0.014141	1.9659
1.24	0.285275	0.285477	−0.0708	0.011810	0.011562	2.1450
1.26	0.301487	0.301745	−0.0855	0.009642	0.009422	2.3350
1.28	0.317498	0.317809	−0.0979	0.007847	0.007654	2.5216
1.30	0.333276	0.333638	−0.1085	0.006367	0.006198	2.7267

S_t is the exchange rate at initial time t, X is the exercise price, Call_O and Put_O are call and put prices for our model, Call_{GK} and Put_{GK} are call and put prices for the G-K model, $\text{Bias}_c = 100(\text{Call}_O - \text{Call}_{GK})/\text{Call}_{GK}$, and $\text{Bias}_p = 100(\text{Put}_O - \text{Put}_{GK})/\text{Put}_{GK}$.

$$= \frac{1}{2}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right] \times \int_0^\infty \exp\left[-\left(\frac{kV - i(\omega-\delta)}{\sqrt{2V}}\right)^2\right] dk. \tag{A.3}$$

Letting $x = (kV - i(\omega-\delta))/\sqrt{2V}$, which implies $dk = \sqrt{(2/V)}dx$, we have

$$\frac{1}{2}\int_0^\infty \exp\left[-\frac{1}{2}k^2V + ik(\omega-\delta)\right] dk = \frac{1}{2}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right]\int_0^\infty \exp\left[-x^2\right]\sqrt{\frac{2}{V}}dx = \frac{1}{2}\sqrt{\frac{\pi}{2V}}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right]. \tag{A.4}$$

Similarly, since $-(1/2)k^2V - ik(\omega-\delta) = -((\omega-\delta)^2/2V) - [(kV + i(\omega-\delta))/\sqrt{2V}]^2$, we have

$$\frac{1}{2}\int_0^\infty \exp\left[-\frac{1}{2}k^2V - ik(\omega-\delta)\right] dk = \frac{1}{2}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right] \times \int_0^\infty \exp\left[-\left(\frac{kV + i(\omega-\delta)}{\sqrt{2V}}\right)^2\right] dk. \tag{A.5}$$

Letting $z = (kV + i(\omega - \delta))/\sqrt{2V}$, which implies $dk = \sqrt{(2/V)}dz$, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^{\infty} \exp\left[-\frac{1}{2}k^2V - ik(\omega-\delta)\right]dk \\
&\quad = \frac{1}{2}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right]\int_0^{\infty}\exp\left[-z^2\right]\sqrt{\frac{2}{V}}dz \qquad \text{(A.6)} \\
&\quad = \frac{1}{2}\sqrt{\frac{\pi}{2V}}\exp\left[-\frac{(\omega-\delta)^2}{2V}\right].
\end{aligned}$$

Hence, combining (A.4) and (A.6), we obtain (A.1). Now substituting $A(k) = (1/\pi)\int_{-\infty}^{\infty} f(\omega)\cos(k\omega)d\omega$ and $B(k) = (1/\pi)\int_{-\infty}^{\infty} f(\omega)\sin(k\omega)d\omega$ into (26), we obtain

$$\begin{aligned}
u(\delta, V) &= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty}[\cos(k\omega)\cos(k\delta) + \sin(k\omega)\sin(k\delta)]\right. \\
&\qquad \left.\times f(\omega)\,d\omega\right\}\exp\left(-\frac{1}{2}k^2V\right)dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\int_0^{\infty}\cos[k(\omega-\delta)] \\
&\qquad \times\exp\left(-\frac{1}{2}k^2V\right)dk\,d\omega.
\end{aligned} \qquad \text{(A.7)}$$

Substituting (A.1) into (A.7), we get (27) as follows:

$$\begin{aligned}
u(\delta, V) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\omega)\left[\sqrt{\frac{\pi}{2V}}\exp\left(-\frac{(\omega-\delta)^2}{2V}\right)\right]d\omega \\
&= \frac{1}{\sqrt{2V\pi}}\int_{-\infty}^{\infty} f(\omega)\exp\left(-\frac{(\omega-\delta)^2}{2V}\right)d\omega.
\end{aligned} \qquad \text{(A.8)}$$

□

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