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Research Article

Approximate Multi-Jensen, Multi-Euler-Lagrange Additive and Quadratic Mappings in n-Banach Spaces

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We prove the generalized Hyers-Ulam stability of multi-Jensen, multi-Euler-Lagrange additive, and quadratic mappings in n-Banach spaces, using the socalled direct method. The corollaries from our main results correct some outcomes from Park (2011).

1. Introduction and Preliminaries

In 2005, Prager and Schwaiger (see [1] and also [2]) introduced the notion of multi-Jensen functions with the connection with generalized polynomials and obtained their general form. In 2008, (see [3]) they also proved the Hyers-Ulam stability of multi-Jensen equation, whereas Ciepliński (see [4, 5]) showed its generalized stability: in the spirit of Bourgin (see [6]) and Găvruţa (see [7]), and in the spirit of Aoki (see [8]) and Rassias (see [9]). Recently, some further results on the stability of multi-Jensen mappings were obtained in [10–14]. We refer the reader to [15–19] for more information on different aspects of stability of functional equations.

In this paper, we deal with the generalized Hyers-Ulam stability of multi-Jensen, multi-Euler-Lagrange additive, and quadratic mappings in n-Banach spaces. The corollaries from our main results correct some outcomes from [20]. The results of Sections 2 and 4 generalize those from [12].

The concept of 2-normed spaces was initially developed by Gähler [21, 22] in the middle of the 1960s, while that of n-normed spaces can be found in [23, 24]. Since then, many others have studied this concept and obtained various results (see [23, 25–27]).

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{R}$ represents the set of all real numbers. Moreover, we fix two positive integers k and n.

We recall some basic facts concerning n-normed spaces.

Definition 1. Let $n \in \mathbb{N}$ and let X be a real linear space with $\dim X \geq n$, and let $\|\cdot,\ldots,\cdot\| : X^n \to \mathbb{R}$ be a function satisfying the following properties:

(N1) $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent,

(N2) $\|x_1,\ldots,x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1,\ldots,x_n\| = |\alpha|\|x_1,\ldots,x_n\|$,

(N4) $\|x+y,x_2,\ldots,x_n\| \leq \|x,x_2,\ldots,x_n\| + \|y,x_2,\ldots,x_n\|$

for all $\alpha \in \mathbb{R}$ and $x,y,x_1,x_2,\ldots,x_n \in X$. Then the function $\|\cdot,\ldots,\cdot\|$ is called an n-norm on X, and the pair $(X,\|\cdot,\ldots,\cdot\|)$ is called an n-normed space.

A sequence $\{x_j\}_{j\in\mathbb{N}}$ in an n-normed space X is said to a converge to some $x \in X$ in the n-norm if

$$\lim_{j\to\infty}\left\|x_j - x, y_2,\ldots,y_n\right\| = 0, \tag{1}$$

for every $y_2,\ldots,y_n \in X$. Every convergent sequence has exactly one limit. If x is the limit of the sequence $\{x_j\}_{j\in\mathbb{N}}$, then we write $\lim_{j\to\infty}x_j = x$. For any convergent sequences $\{x_j\}_{j\in\mathbb{N}}$ and $\{y_j\}_{j\in\mathbb{N}}$ of elements of X, the sequence $\{x_j+y_j\}_{j\in\mathbb{N}}$ is convergent and

$$\lim_{j\to\infty}\left(x_j+y_j\right) = \lim_{j\to\infty}x_j + \lim_{j\to\infty}y_j. \tag{2}$$

If, moreover, $\{\alpha_j\}_{j\in\mathbb{N}}$ is a convergent sequence of real numbers, then the sequence $\{\alpha_j\cdot x_j\}_{j\in\mathbb{N}}$ is also convergent and

$$\lim_{j\to\infty}\left(\alpha_j\cdot x_j\right) = \lim_{j\to\infty}\alpha_j\cdot\lim_{j\to\infty}x_j. \tag{3}$$

A sequence $\{x_j\}_{j\in\mathbb{N}}$ in an n-normed space X is said to be a Cauchy sequence with respect to the n-norm if

$$\lim_{j,l\to\infty}\left\|x_j - x_l, y_2, \ldots, y_n\right\| = 0, \tag{4}$$

for every $y_2, \ldots, y_n \in X$. A linear n-normed space in which every Cauchy sequence is convergent is called an n-Banach space.

Example 2. For $x_1, \ldots, x_n \in \mathbb{R}^n$, the Euclidean n-norm $\|x_1, \ldots, x_n\|_E$ is defined by

$$\left\|x_1, \ldots, x_n\right\|_E = \left|\det\left(x_{ij}\right)\right| = \operatorname{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right), \tag{5}$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, \ldots, n$.

Example 3. The standard n-norm on X, a real inner product space of dimension $\dim X \geq n$, is as follows:

$$\left\|x_1, \ldots, x_n\right\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \cdots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}^{1/2}, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1, \ldots, x_n\|_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $\|x_1\| = \langle x_1, x_1\rangle^{1/2}$.

In what follows, we will also use the following lemma from [19].

Lemma 4. *Let X be an n-normed space. Then,*

(1) *for $x_i \in X$ $(i = 1, \ldots, n)$ and γ, a real number,*

$$\begin{aligned} &\left\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\right\| \\ &\quad = \left\|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\right\|, \end{aligned} \tag{7}$$

for all $1 \leq i \neq j \leq n$,

(2) $|\|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\|| \leq \|x - y, y_2, \ldots, y_n\|$ *for all $x, y, y_2, \ldots, y_n \in X$,*

(3) *if $\|x, y_2, \ldots, y_n\| = 0$ for all $y_2, \ldots, y_n \in X$, then $x = 0$,*

(4) *for a convergent sequence $\{x_j\}$ in X,*

$$\lim_{j\to\infty}\left\|x_j, y_2, \ldots, y_n\right\| = \left\|\lim_{j\to\infty} x_j, y_2, \ldots, y_n\right\|, \tag{8}$$

for all $y_2, \ldots, y_n \in X$.

2. Approximate Multi-Jensen Mappings

First, we prove the stability of the system of equations defining multi-Jensen mappings in n-Banach spaces. For a given mapping $f : V^k \to W$, we define the difference operators

$$\begin{aligned} &D_i f\left(x_1, \ldots, x_{k+1}\right) \\ &\quad := 2f\left(x_1, \ldots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \ldots, x_{k+1}\right) \\ &\qquad - f\left(x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k+1}\right) \\ &\qquad - f\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}\right), \\ &\qquad \left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}, \quad i \in \{1, \ldots, k\}. \end{aligned} \tag{9}$$

Theorem 5. *Let V be a commutative group uniquely divisible by 2, and, W be an n-Banach space. Assume also that for every $i \in \{1, \ldots, k\}$, $\varphi_i : V^{k+1} \to [0, \infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty}\frac{1}{3^{j+1}}\left[\varphi_i\left(3^j x_1, x_2, \ldots, x_{k+1}\right)\right. \\ &\qquad + \cdots + \varphi_i\left(x_1, \ldots, x_{i-2}, 3^j x_{i-1}, x_i, \ldots, x_{k+1}\right) \\ &\qquad + \varphi_i\left(x_1, \ldots, x_{i-1}, 3^j x_i, 3^j x_{i+1}, x_{i+2}, \ldots, x_{k+1}\right) \\ &\qquad + \varphi_i\left(x_1, \ldots, x_{i+1}, 3^j x_{i+2}, x_{i+3}, \ldots, x_{k+1}\right) \\ &\qquad \left. + \cdots + \varphi_i\left(x_1, \ldots, x_k, 3^j x_{k+1}\right)\right] < \infty, \\ &\qquad \left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}. \end{aligned} \tag{10}$$

If $f : V^k \to W$ is a function satisfying

$$\begin{aligned} &f\left(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k\right) = 0, \\ &\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\right) \in V^{k-1}, \quad i \in \{1, \ldots, k\}, \end{aligned} \tag{11}$$

$$\begin{aligned} &\left\|D_i f\left(x_1, \ldots, x_{k+1}\right), y_2, \ldots, y_n\right\| \leq \varphi_i\left(x_1, \ldots, x_{k+1}\right), \\ &\left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \quad y_2, \ldots, y_n \in W, \end{aligned} \tag{12}$$

then for every $i \in \{1, \ldots, k\}$, there exists a multi-Jensen mapping $F_i : V^k \to W$ for which

$$\begin{aligned} &\left\|f\left(x_1, \ldots, x_k\right) - F_i\left(x_1, \ldots, x_k\right), y_2, \ldots, y_n\right\| \\ &\leq \sum_{j=0}^{\infty}\frac{1}{3^{j+1}} \\ &\qquad \times\left[\varphi_i\left(x_1, \ldots, x_{i-1}, 3^j x_i, -3^j x_i, x_{i+1}, \ldots, x_k\right)\right. \\ &\qquad \left. + \varphi_i\left(x_1, \ldots, x_{i-1}, -3^j x_i, 3^{j+1} x_i, x_{i+1}, \ldots, x_k\right)\right], \\ &\qquad \left(x_1, \ldots, x_k\right) \in V^k, \quad y_2, \ldots, y_n \in W. \end{aligned} \tag{13}$$

For every $i \in \{1,\dots,k\}$, the function F_i is given by

$$F_i(x_1,\dots,x_k) := \lim_{j\to\infty} \frac{1}{3^j} f\left(x_1,\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_k\right), \quad (x_1,\dots,x_k) \in V^k. \tag{14}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$ and $i \in \{1,\dots,k\}$. By (12) and (11), we get

$$\begin{aligned} &\|f(x_1,\dots,x_k) + f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\| \\ &\quad \le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_k), \\ &\|2f(x_1,\dots,x_k) - f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_k) \\ &\quad - f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\| \\ &\quad \le \varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_k). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} &\|3f(x_1,\dots,x_k) - f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_k), \\ &\quad y_2,\dots,y_n\| \\ &\quad \le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_k) \\ &\qquad + \varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_k), \end{aligned} \tag{16}$$

and consequently for any nonnegative integers l and m such that $l < m$, we obtain

$$\begin{aligned} &\left\|\frac{1}{3^l} f\left(x_1,\dots,x_{i-1},3^l x_i,x_{i+1},\dots,x_k\right) \right. \\ &\quad \left. - \frac{1}{3^m} f\left(x_1,\dots,x_{i-1},3^m x_i,x_{i+1},\dots,x_k\right), y_2,\dots,y_n\right\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[\varphi_i\left(x_1,\dots,x_{i-1},3^j x_i,-3^j x_i,x_{i+1},\dots,x_k\right)\right. \\ &\qquad \left. + \varphi_i\left(x_1,\dots,x_{i-1},-3^j x_i,3^{j+1} x_i,x_{i+1},\dots,x_k\right)\right]. \end{aligned} \tag{17}$$

Therefore, from (10), it follows that $\{(1/3^j) f(x_1,\dots,x_{i-1}, 3^j x_i, x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $F_i : V^k \to W$ by (14). Putting $l = 0$, letting $m \to \infty$ in (17), and using Lemma 4 and (10), we see that (13) holds.

Finally, fix $x_i' \in V$, $j \in \mathbb{N}$, and note that according to (12), we have

$$\left\|\frac{1}{3^j} D_i f\left(x_1,\dots,x_{i-1},3^j x_i,3^j x_i',x_{i+1},\dots,x_k\right), y_2,\dots,y_n\right\| \le \frac{1}{3^j}\varphi_i\left(x_1,\dots,x_{i-1},3^j x_i,3^j x_i',x_{i+1},\dots,x_k\right). \tag{18}$$

Next, fix $s \in \{1,\dots,k\} \setminus \{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (12), it follows that

$$\begin{aligned} &\left\|\frac{1}{3^j} D_s f\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right.\right. \\ &\qquad \left.\left. 3^j x_i,x_{i+1},\dots,x_k\right), y_2,\dots,y_n\right\| \\ &\quad \le \frac{1}{3^j}\varphi_s\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,\right. \\ &\qquad \left. x_{i-1},3^j x_i,x_{i+1},\dots,x_k\right). \end{aligned} \tag{19}$$

Letting $j \to \infty$ in the above two inequalities and using (10) and Lemma 4, we see that the mapping F_i is multi-Jensen. □

Theorem 6. *Let V be a real linear space and, W be an n-Banach space. Assume also that for every $i \in \{1,\dots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty} 3^j \left[\varphi_i\left(\frac{x_1}{3^j},x_2,\dots,x_{k+1}\right)\right. \\ &\quad + \cdots + \varphi_i\left(x_1,\dots,x_{i-2},\frac{x_{i-1}}{3^j},x_i,\dots,x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},\frac{x_{i+1}}{3^j},x_{i+2},\dots,x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1,\dots,x_{i+1},\frac{x_{i+2}}{3^j},x_{i+3},\dots,x_{k+1}\right) \\ &\quad \left. + \cdots + \varphi_i\left(x_1,\dots,x_k,\frac{x_{k+1}}{3^j}\right)\right] < \infty, \\ &\qquad (x_1,\dots,x_{k+1}) \in V^{k+1}. \end{aligned} \tag{20}$$

If $f : V^k \to W$ is a function satisfying conditions (11) *and* (12), *then for every $i \in \{1,\dots,k\}$ there exists a multi-Jensen mapping $F_i : V^k \to W$ for which*

$$\begin{aligned} &\|f(x_1,\dots,x_k) - F_i(x_1,\dots,x_k), y_2,\dots,y_n\| \\ &\le \sum_{j=0}^{\infty} 3^j \left[\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^{j+1}},-\frac{x_i}{3^{j+1}},x_{i+1},\dots,x_k\right)\right. \\ &\qquad \left. + \varphi_i\left(x_1,\dots,x_{i-1},-\frac{x_i}{3^{j+1}},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right)\right], \\ &\qquad (x_1,\dots,x_k) \in V^k,\ y_2,\dots,y_n \in W. \end{aligned} \tag{21}$$

For every $i \in \{1,\dots,k\}$, the function F_i is given by

$$F_i(x_1,\dots,x_k) := \lim_{j\to\infty} 3^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right), \quad (x_1,\dots,x_k) \in V^k. \tag{22}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\dots,k\}$. By (12) and (11), we get

$$\begin{aligned}&\Big\|3^{j+1} f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^{j+1}},x_{i+1},\dots,x_k\right)\\&\quad -3^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\Big\|\\&\le 3^j\Big[\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^{j+1}},-\frac{x_i}{3^{j+1}},x_{i+1},\dots,x_k\right)\\&\quad +\varphi_i\left(x_1,\dots,x_{i-1},-\frac{x_i}{3^{j+1}},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right)\Big],\end{aligned}\tag{23}$$

and consequently for any non-negative integers l and m such that $l<m$, we obtain

$$\begin{aligned}&\Big\|3^l f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^l},x_{i+1},\dots,x_k\right)\\&\quad -3^m f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^m},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\Big\|\\&\le\sum_{j=l}^{m-1}3^j\Big[\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^{j+1}},-\frac{x_i}{3^{j+1}},x_{i+1},\dots,x_k\right)\\&\quad +\varphi_i\left(x_1,\dots,x_{i-1},-\frac{x_i}{3^{j+1}},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right)\Big].\end{aligned}\tag{24}$$

Therefore, from (20), it follows that $\{3^j f(x_1,\dots,x_{i-1},x_i/3^j,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $F_i : V^k \to W$ by (22). Putting $l=0$, letting $m\to\infty$ in (24), and using Lemma 4 and (20), we see that (21) holds.

Finally, fix $x_i' \in V$, and note that according to (12), we have

$$\begin{aligned}&\Big\|3^j D_i f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},\frac{x_i'}{3^j},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\Big\|\\&\quad\le 3^j\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},\frac{x_i'}{3^j},x_{i+1},\dots,x_k\right).\end{aligned}\tag{25}$$

Next, fix $s\in\{1,\dots,k\}\setminus\{i\}$, $x_s'\in V$, and assume that $s<i$ (the same arguments apply to the case where $s>i$). From (12), it follows that

$$\begin{aligned}&\Big\|3^j D_s f\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right),\\&\quad y_2,\dots,y_n\Big\|\\&\quad\le 3^j\varphi_s\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right).\end{aligned}\tag{26}$$

Letting $j\to\infty$ in the previous two inequalities and using (20) and Lemma 4, we see that the mapping F_i is multi-Jensen. □

As applications of Theorems 5 and 6 we get the following corollaries.

Corollary 7. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta\in[0,\infty)$ and $r\in(0,\infty)$ are such that $r\neq 1$. If $f:V^k\to W$ is a function satisfying* (11) *and*

$$\begin{aligned}&\|D_i f(x_1,\dots,x_{k+1}),y_2,\dots,y_n\|\\&\quad\le\theta\left[\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^r+\|x_{i+1}\|^r\right)\right.\\&\qquad\left.\times\|x_{i+2}\|^r\cdots\|x_{k+1}\|^r\right],\\&(x_1,\dots,x_{k+1})\in V^{k+1},\quad i\in\{1,\dots,k\},\\&\qquad y_2,\dots,y_n\in W,\end{aligned}\tag{27}$$

then for every $i\in\{1,\dots,k\}$ there exists a multi-Jensen mapping $F_i:V^k\to W$ for which

$$\begin{aligned}&\|f(x_1,\dots,x_k)-F_i(x_1,\dots,x_k),y_2,\dots,y_n\|\\&\quad\le\frac{\theta\|x_1\|^r\cdots\|x_k\|^r(3+3^r)}{|3-3^r|},\end{aligned}\tag{28}$$

for all $x_1,\dots,x_k\in V$, $y_2,\dots,y_n\in W$.

Corollary 8. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta\in[0,\infty)$ and $r,p,q\in(0,\infty)$ are such that $r,p+q\in(0,1)$ or $r,p+q\in(1,\infty)$. If $f:V^k\to W$ is a function satisfying* (11) *and*

$$\begin{aligned}&\|D_i f(x_1,\dots,x_{k+1}),y_2,\dots,y_n\|\\&\quad\le\theta\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^p\|x_{i+1}\|^q\right)\\&\qquad\times\|x_{i+2}\|^r\cdots\|x_{k+1}\|^r,\\&(x_1,\dots,x_{k+1})\in V^{k+1},\quad i\in\{1,\dots,k\},\\&\qquad y_2,\dots,y_n\in W,\end{aligned}\tag{29}$$

then for every $i\in\{1,\dots,k\}$, there exists a multi-Jensen mapping $F_i:V^k\to W$ for which

$$\begin{aligned}&\|f(x_1,\dots,x_k)-F_i(x_1,\dots,x_k),y_2,\dots,y_n\|\\&\quad\le\frac{\theta\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^{p+q}\|x_{i+1}\|^r\cdots\|x_k\|^r(1+3^q)}{|3-3^{p+q}|},\end{aligned}\tag{30}$$

for all $x_1,\dots,x_k\in V$, $y_2,\dots,y_n\in W$.

From Corollary 8, we obtain the following corollary which corrects Theorems 3.1 and 3.2 from [20].

Corollary 9. *Let V be a real normed linear space and W be an n-Banach space. Assume also that $\theta\in[0,\infty)$ and $p,q\in(0,\infty)$ are such that $p+q\neq 1$. If $f:V\to W$ is a function satisfying $f(0)=0$ and*

$$\begin{aligned}&\left\|2f\left(\frac{x_1+x_2}{2}\right)-f(x_1)-f(x_2),y\right\|\\&\quad\le\theta\|x_1\|^p\|x_2\|^q,\quad x_1,x_2\in V,\ y\in W,\end{aligned}\tag{31}$$

then there exists a Jensen mapping $F : V \to W$ for which

$$\|f(x) - F(x), y\| \le \frac{\theta\|x\|^{p+q}(1+3^q)}{|3-3^{p+q}|}, \quad x \in V,\ y \in W. \tag{32}$$

3. Approximate Multi-Euler-Lagrange Additive Mappings

In this section, we prove the stability of the system of equations defining multi-Euler-Lagrange additive mappings.

Throughout this section, let V be a real linear space and let W be an n-Banach space, and $a, b \in \mathbb{R} \setminus \{0\}$ are fixed with $\lambda := a + b \neq 0, \pm 1$.

A mapping $f : V^k \to W$ is called a multiEuler-Lagrange additive mapping as follows if it satisfies the Euler-Lagrange additive equations in each of their k arguments as follows:

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, ax_i + bx_i', x_{i+1}, \ldots, x_k) \\ &\quad + f(x_1, \ldots, x_{i-1}, bx_i + ax_i', x_{i+1}, \ldots, x_k) \\ &= (a+b)\left[f(x_1, \ldots, x_k) \right. \\ &\qquad \left. + f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k)\right], \end{aligned} \tag{33}$$

for all $i \in \{1, \ldots, k\}$ and all $x_1, \ldots, x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_k \in V$. If $a = b = 1$, then the multi-Euler-Lagrange additive mapping is multiadditive (see [28]). For a given mapping $f : V^k \to W$, we define the difference operators

$$\begin{aligned} &\widetilde{D}_i f(x_1, \ldots, x_{k+1}) \\ &\quad := f(x_1, \ldots, x_{i-1}, ax_i + bx_{i+1}, x_{i+2}, \ldots, x_{k+1}) \\ &\qquad + f(x_1, \ldots, x_{i-1}, bx_i + ax_{i+1}, x_{i+2}, \ldots, x_{k+1}) \\ &\qquad - (a+b)\left[f(x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k+1}) \right. \\ &\qquad\qquad \left. + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1})\right], \\ &\qquad (x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}. \end{aligned} \tag{34}$$

Theorem 10. *Assume that for every $i \in \{1, \ldots, k\}$, $\varphi_i : V^{k+1} \to [0, \infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{1}{|\lambda|^j}\left[\varphi_i(\lambda^j x_1, \ldots, x_{k+1}) \right. \\ &\quad + \cdots + \varphi_i(x_1, \ldots, x_{i-2}, \lambda^j x_{i-1}, x_i, \ldots, x_{k+1}) \\ &\quad + \varphi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_{i+1}, x_{i+2}, \ldots, x_{k+1}) \\ &\quad + \varphi_i(x_1, \ldots, x_{i+1}, \lambda^j x_{i+2}, x_{i+3}, \ldots, x_{k+1}) \\ &\quad \left. + \cdots + \varphi_i(x_1, \ldots, x_k, \lambda^j x_{k+1})\right] < \infty, \\ &\qquad (x_1, \ldots, x_{k+1}) \in V^{k+1}. \end{aligned} \tag{35}$$

If $f : V^k \to W$ is a function satisfying

$$\begin{aligned} &\left\|\widetilde{D}_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n\right\| \le \varphi_i(x_1, \ldots, x_{k+1}), \\ &(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \quad y_2, \ldots, y_n \in W, \end{aligned} \tag{36}$$

then for every $i \in \{1, \ldots, k\}$, there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\begin{aligned} &\|f(x_1, \ldots, x_k) - A_i(x_1, \ldots, x_k), y_2, \ldots, y_n\| \\ &\quad \le \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{|\lambda|^{j+1}} \\ &\qquad \times \varphi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k), \\ &\qquad (x_1, \ldots, x_k) \in V^k, \quad y_2, \ldots, y_n \in W. \end{aligned} \tag{37}$$

For every $i \in \{1, \ldots, k\}$, the function A_i is given by

$$\begin{aligned} &A_i(x_1, \ldots, x_k) \\ &\quad := \lim_{j \to \infty} \frac{1}{\lambda^j} f(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k), \\ &\qquad (x_1, \ldots, x_k) \in V^k. \end{aligned} \tag{38}$$

Proof. Fix $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$, $j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, k\}$. By (36), we get

$$\begin{aligned} &\left\|f(x_1, \ldots, x_k) - \frac{1}{\lambda} f(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_k), \right. \\ &\quad \left. y_2, \ldots, y_n\right\| \\ &\quad \le \frac{1}{|2\lambda|}\varphi_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_k), \end{aligned} \tag{39}$$

whence

$$\begin{aligned} &\left\|\frac{1}{\lambda^j} f(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k) \right. \\ &\quad \left. - \frac{1}{\lambda^{j+1}} f(x_1, \ldots, x_{i-1}, \lambda^{j+1} x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n\right\| \\ &\quad \le \frac{1}{2|\lambda|^{j+1}}\varphi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k). \end{aligned} \tag{40}$$

For any nonnegative integers l and m with $l < m$, using (40) we get

$$\begin{aligned} &\left\|\frac{1}{\lambda^l} f(x_1, \ldots, x_{i-1}, \lambda^l x_i, x_{i+1}, \ldots, x_k) \right. \\ &\quad \left. - \frac{1}{\lambda^m} f(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n\right\| \\ &\quad \le \sum_{j=l}^{m-1} \frac{1}{2|\lambda|^{j+1}} \\ &\qquad \times \varphi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k), \end{aligned} \tag{41}$$

which tends to zero as l tends to infinity. Therefore, from (35) it follows that $\{(1/\lambda^j) f(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in n-Banach space W and it thus converges. Hence, we can define $A_i : V^k \to W$ by

$$A_i(x_1,\dots,x_k) := \lim_{j\to\infty}\frac{1}{\lambda^j} f\left(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k\right). \tag{42}$$

Putting $l = 0$, letting $m \to \infty$ in (41), and using (35), we see that (37) holds.

Now, fix also $x_i' \in V$, and from (36), we have

$$\begin{aligned}&\left\|\frac{1}{\lambda^j}\widetilde{D}_i f\left(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i',x_{i+1},\dots,x_k\right),\right.\\&\left.y_2,\dots,y_n\right\|\\&\le \frac{1}{|\lambda|^j}\varphi_i\left(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i',x_{i+1},\dots,x_k\right).\end{aligned} \tag{43}$$

Next, fix $s \in \{1,\dots,k\}\setminus\{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (36) it follows that

$$\begin{aligned}&\left\|\frac{1}{\lambda^j}\widetilde{D}_s f\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right.\right.\\&\qquad\left.\left.\lambda^j x_i,x_{i+1},\dots,x_k\right),y_2,\dots,y_n\right\|\\&\le \frac{1}{|\lambda|^j}\varphi_s\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right.\\&\qquad\left.\lambda^j x_i,x_{i+1},\dots,x_k\right).\end{aligned} \tag{44}$$

Letting $j \to \infty$ in the above two inequalities and using (35) and Lemma 4, we see that the mapping A_i is multi-Euler-Lagrange additive.

Now, let us finally assume that $A_i' : V^k \to W$ is another multi-Euler-Lagrange additive mapping satisfying (37). Then we have

$$\begin{aligned}&\left\|A_i(x_1,\dots,x_k) - A_i'(x_1,\dots,x_k),y_2,\dots,y_n\right\|\\&= \lim_{m\to\infty}\frac{1}{|\lambda|^m}\\&\quad\times\left\|A_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\right.\\&\qquad\left.-A_i'(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),y_2,\dots,y_n\right\|\\&\le \lim_{m\to\infty}\frac{1}{|\lambda|^m}\\&\quad\times\left[\left\|A_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\right.\right.\\&\qquad-f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),y_2,\dots,y_n\|\\&\qquad+\|f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\\&\qquad\left.\left.-A_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),y_2,\dots,y_n\right\|\right]\\&\le \lim_{m\to\infty}\frac{1}{|\lambda|}\\&\qquad\times\sum_{j=0}^{\infty}\frac{1}{|\lambda|^{m+j}}\\&\qquad\quad\times\varphi_i\left(x_1,\dots,x_{i-1},\lambda^{m+j}x_i,\lambda^{m+j}x_i,x_{i+1},\dots,x_k\right)\\&= 0,\end{aligned} \tag{45}$$

and therefore $A_i = A_i'$. □

Theorem 11. *Assume that for every $i \in \{1,\dots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned}&\sum_{j=0}^{\infty}|\lambda|^j\left[\varphi_i\left(\frac{x_1}{\lambda^j},x_2,\dots,x_{k+1}\right)\right.\\&\quad+\cdots+\varphi_i\left(x_1,\dots,x_{i-2},\frac{x_{i-1}}{\lambda^j},x_i,\dots,x_{k+1}\right)\\&\quad+\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},\frac{x_{i+1}}{\lambda^j},x_{i+2},\dots,x_{k+1}\right)\\&\quad+\varphi_i\left(x_1,\dots,x_{i+1},\frac{x_{i+2}}{\lambda^j},x_{i+3},\dots,x_{k+1}\right)\\&\quad\left.+\cdots+\varphi_i\left(x_1,\dots,x_k,\frac{x_{k+1}}{\lambda^j}\right)\right]<\infty,\\&\qquad(x_1,\dots,x_{k+1})\in V^{k+1}.\end{aligned} \tag{46}$$

If $f : V^k \to W$ is a function satisfying (36), then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\begin{aligned}&\|f(x_1,\dots,x_k) - A_i(x_1,\dots,x_k),y_2,\dots,y_n\|\\&\le\frac{1}{2}\sum_{j=1}^{\infty}|\lambda|^{j-1}\\&\qquad\times\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},\frac{x_i}{\lambda^j},x_{i+1},\dots,x_k\right),\\&\qquad(x_1,\dots,x_k)\in V^k,\quad y_2,\dots,y_n\in W.\end{aligned} \tag{47}$$

For every $i \in \{1,\dots,k\}$, the function A_i is given by

$$\begin{aligned}&A_i(x_1,\dots,x_k)\\&:= \lim_{j\to\infty}\lambda^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\dots,x_k\right),\\&\qquad(x_1,\dots,x_k)\in V^k.\end{aligned} \tag{48}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\dots,k\}$. By (36) we get

$$\begin{aligned}&\left\|f(x_1,\dots,x_k)-\lambda f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda},x_{i+1},\dots,x_k\right),\right.\\&\left.y_2,\dots,y_n\right\|\\&\quad\le\frac{1}{2}\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda},\frac{x_i}{\lambda},x_{i+1},\dots,x_k\right).\end{aligned}\tag{49}$$

For any non-negative integers l and m with $0 \le l < m$, using (49), we get

$$\begin{aligned}&\left\|\lambda^l f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^l},x_{i+1},\dots,x_k\right)\right.\\&\quad\left.-\lambda^m f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^m},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\right\|\\&\quad\le\sum_{j=l}^{m-1}\frac{|\lambda|^j}{2}\\&\qquad\times\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^{j+1}},\frac{x_i}{\lambda^{j+1}},x_{i+1},\dots,x_k\right),\end{aligned}\tag{50}$$

which tends to zero as l tends to infinity. Therefore from (46), it follows that $\{\lambda^j f(x_1,\dots,x_{i-1},x_i/\lambda^j,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in n-Banach space W and it thus converges. Hence, we can define $A_i : V^k \to W$ by

$$A_i(x_1,\dots,x_k):=\lim_{j\to\infty}\lambda^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\dots,x_k\right).\tag{51}$$

Putting $l = 0$, letting $m \to \infty$ in (50), and using (46), we see that (47) holds. The further part of the proof is similar to the proof of Theorem 10. □

As applications of Theorems 10 and 11, we get the following corollaries.

Corollary 12. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta \in [0,\infty)$ and $r \in (0,\infty)$ are such that $r \ne 1$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned}&\left\|\widetilde{D}_i f(x_1,\dots,x_{k+1}),y_2,\dots,y_n\right\|\\&\quad\le\theta\left[\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^r+\|x_{i+1}\|^r\right)\right.\\&\qquad\left.\times\|x_{i+2}\|^r\cdots\|x_{k+1}\|^r\right],\\&(x_1,\dots,x_{k+1})\in V^{k+1},\quad i\in\{1,\dots,k\},\\&\qquad y_2,\dots,y_n\in W,\end{aligned}\tag{52}$$

then for every $i \in \{1,\dots,k\}$ there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\begin{aligned}&\left\|f(x_1,\dots,x_k)-A_i(x_1,\dots,x_k),y_2,\dots,y_n\right\|\\&\quad\le\frac{\theta\|x_1\|^r\cdots\|x_k\|^r}{\left||\lambda|-|\lambda|^r\right|},\end{aligned}\tag{53}$$

for all $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$.

Corollary 13. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta \in [0,\infty)$ and $r, p, q \in (0,\infty)$ are such that $r, p+q \in (0,1)$ or $r, p+q \in (1,\infty)$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned}&\left\|\widetilde{D}_i f(x_1,\dots,x_{k+1}),y_2,\dots,y_n\right\|\\&\quad\le\theta\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^p\|x_{i+1}\|^q\right)\\&\qquad\times\|x_{i+2}\|^r\cdots\|x_{k+1}\|^r,\\&(x_1,\dots,x_{k+1})\in V^{k+1},\quad i\in\{1,\dots,k\},\quad y_2,\dots,y_n\in W,\end{aligned}\tag{54}$$

then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\begin{aligned}&\left\|f(x_1,\dots,x_k)-A_i(x_1,\dots,x_k),y_2,\dots,y_n\right\|\\&\quad\le\frac{\theta\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^{p+q}\|x_{i+1}\|^r\cdots\|x_k\|^r}{2\left||\lambda|-|\lambda|^{p+q}\right|},\end{aligned}\tag{55}$$

for all $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$.

From Corollary 13 we obtain the following corollary which corrects Theorems 2.1 and 2.2 from [20].

Corollary 14. *Let V be a real normed linear space and W be an 2-Banach space. Assume also that $\theta \in [0,\infty)$ and $p, q \in (0,\infty)$ are such that $p+q \ne 1$. If $f : V \to W$ is a function satisfying*

$$\begin{aligned}&\left\|f(x_1+x_2)-f(x_1)-f(x_2),y\right\|\\&\quad\le\theta\|x_1\|^p\|x_2\|^q,\quad x_1,x_2\in V,\quad y\in W,\end{aligned}\tag{56}$$

then there exists a unique additive mapping $A : V \to W$ for which

$$\left\|f(x)-A(x),y\right\|\le\frac{\theta\|x\|^{p+q}}{|2-2^{p+q}|},\quad x\in V,\ y\in W.\tag{57}$$

4. Approximate Multi-Euler-Lagrange Quadratic Mappings

In this section, we prove the stability of the system of equations defining multi-Euler-Lagrange quadratic mappings.

Throughout this section, let V be a real linear space and let W be an n-Banach space, and $a, b \in \mathbb{R}\setminus\{0\}$ are fixed with $\lambda := a^2 + b^2 \ne 1$.

Rassias [29] introduced the notion of a generalized Euler-Lagrange-type quadratic mapping, and investigated its generalized stability.

A mapping $f : V^k \to W$ is called a multi-Euler-Lagrange quadratic mapping, if it satisfies the Euler-Lagrange quadratic equations in each of their k arguments:

$$\begin{aligned} &f\left(x_1,\ldots,x_{i-1},ax_i+bx_i',x_{i+1},\ldots,x_k\right)\\ &\quad+f\left(x_1,\ldots,x_{i-1},bx_i-ax_i',x_{i+1},\ldots,x_k\right)\\ &\quad=\left(a^2+b^2\right)\\ &\quad\times\left[f\left(x_1,\ldots,x_k\right)+f\left(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_k\right)\right], \end{aligned} \tag{58}$$

for all $i \in \{1,\ldots,k\}$ and all $x_1,\ldots,x_{i-1},x_i,x_i',x_{i+1},\ldots,x_k \in V$.

If $a = b = 1$, then the multi-Euler-Lagrange quadratic mapping is multiquadratic (see [30]). Letting $x_i = x_i' = 0$ in (58), we get $f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_k) = 0$. Putting $x_i' = 0$ in (58), we have

$$\begin{aligned} &f\left(x_1,\ldots,x_{i-1},ax_i,x_{i+1},\ldots,x_k\right)\\ &\quad+f\left(x_1,\ldots,x_{i-1},bx_i,x_{i+1},\ldots,x_k\right)\\ &\quad=\lambda f\left(x_1,\ldots,x_k\right). \end{aligned} \tag{59}$$

Replacing x_i by ax_i and x_i' by bx_i in (58), respectively, we obtain

$$\begin{aligned} &f\left(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_k\right)\\ &\quad=\lambda\left[f\left(x_1,\ldots,x_{i-1},ax_i,x_{i+1},\ldots,x_k\right)\right.\\ &\qquad\left.+f\left(x_1,\ldots,x_{i-1},bx_i,x_{i+1},\ldots,x_k\right)\right]. \end{aligned} \tag{60}$$

From (59) and (60), one gets

$$f\left(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_k\right)=\lambda^2 f\left(x_1,\ldots,x_k\right), \tag{61}$$

for all $i \in \{1,\ldots,k\}$ and all $x_1,\ldots,x_k \in V$.

For a given mapping $f : V^k \to W$, we define the difference operators

$$\begin{aligned} &\widehat{D}_i f\left(x_1,\ldots,x_{k+1}\right)\\ &:=f\left(x_1,\ldots,x_{i-1},ax_i+bx_{i+1},x_{i+2},\ldots,x_{k+1}\right)\\ &\quad+f\left(x_1,\ldots,x_{i-1},bx_i-ax_{i+1},x_{i+2},\ldots,x_{k+1}\right)-\left(a^2+b^2\right)\\ &\quad\times\left[f\left(x_1,\ldots,x_i,x_{i+2},\ldots,x_{k+1}\right)\right.\\ &\qquad\left.+f\left(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k\right)\right],\\ &\left(x_1,\ldots,x_{k+1}\right)\in V^{k+1},\quad i\in\{1,\ldots,k\}. \end{aligned} \tag{62}$$

Theorem 15. *Assume that for every $i \in \{1,\ldots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty}\frac{1}{\lambda^{2j}}\\ &\quad\times\left[\varphi_i\left(\lambda^j x_1,x_2,\ldots,x_{k+1}\right)\right.\\ &\qquad+\cdots+\varphi_i\left(x_1,\ldots,x_{i-2},\lambda^j x_{i-1},x_i,\ldots,x_{k+1}\right)\\ &\qquad+\varphi_i\left(x_1,\ldots,x_{i-1},\lambda^j x_i,\lambda^j x_{i+1},x_{i+2},\ldots,x_{k+1}\right)\\ &\qquad+\varphi_i\left(x_1,\ldots,x_{i+1},\lambda^j x_{i+2},x_{i+3},\ldots,x_{k+1}\right)\\ &\qquad\left.+\cdots+\varphi_i\left(x_1,\ldots,x_k,\lambda^j x_{k+1}\right)\right]<\infty,\\ &\qquad\left(x_1,\ldots,x_{k+1}\right)\in V^{k+1}. \end{aligned} \tag{63}$$

If $f : V^k \to W$ is a function satisfying condition (11) *and*

$$\begin{aligned} &\left\|\widehat{D}_i f\left(x_1,\ldots,x_{k+1}\right),y_2,\ldots,y_n\right\|\\ &\quad\le\varphi_i\left(x_1,\ldots,x_{k+1}\right),\\ &\qquad\left(x_1,\ldots,x_{k+1}\right)\in V^{k+1},\quad i\in\{1,\ldots,k\},\\ &\qquad y_2,\ldots,y_n\in W, \end{aligned} \tag{64}$$

then for every $i \in \{1,\ldots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\begin{aligned} &\left\|f\left(x_1,\ldots,x_k\right)-Q_i\left(x_1,\ldots,x_k\right),y_2,\ldots,y_n\right\|\\ &\le\sum_{j=0}^{\infty}\left[\frac{1}{\lambda^{2j+1}}\varphi_i\left(x_1,\ldots,x_{i-1},\lambda^j x_i,0,x_{i+1},\ldots,x_k\right)\right.\\ &\qquad+\frac{1}{\lambda^{2j+2}}\varphi_i\left(x_1,\ldots,x_{i-1},a\lambda^j x_i,\right.\\ &\qquad\left.\left.b\lambda^j x_i,x_{i+1},\ldots,x_k\right)\right],\\ &\left(x_1,\ldots,x_k\right)\in V^k,\qquad y_2,\ldots,y_n\in W. \end{aligned} \tag{65}$$

For every $i \in \{1,\ldots,k\}$, the function Q_i is given by

$$\begin{aligned} &Q_i\left(x_1,\ldots,x_k\right)\\ &\quad:=\lim_{j\to\infty}\frac{1}{\lambda^{2j}}\\ &\qquad\times f\left(x_1,\ldots,x_{i-1},\lambda^j x_i,x_{i+1},\ldots,x_k\right),\\ &\qquad\left(x_1,\ldots,x_k\right)\in V^k. \end{aligned} \tag{66}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\dots,k\}$. By (64), we get

$$\begin{aligned}
&\|f(x_1,\dots,x_{i-1},ax_i,x_{i+1},\dots,x_k)\\
&\quad + f(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k)\\
&\quad -\lambda f(x_1,\dots,x_k),y_2,\dots,y_n\|\\
&\qquad \le \varphi_i(x_1,\dots,x_i,0,x_{i+1},\dots,x_k),\\
&\|f(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k)\\
&\quad - \lambda f(x_1,\dots,x_{i-1},ax_i,x_{i+1},\dots,x_k)\\
&\quad - \lambda f(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k),\\
&\quad y_2,\dots,y_n\|\\
&\qquad \le \varphi_i(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_k).
\end{aligned} \tag{67}$$

From (67), we obtain

$$\begin{aligned}
&\left\|\frac{1}{\lambda^2}f(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k)\right.\\
&\quad \left. - f(x_1,\dots,x_k),y_2,\dots,y_n\right\|\\
&\qquad \le \frac{1}{\lambda}\varphi_i(x_1,\dots,x_i,0,x_{i+1},\dots,x_k)+\frac{1}{\lambda^2}\\
&\qquad\quad \times \varphi_i(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_k),
\end{aligned} \tag{68}$$

and consequently for any non-negative integers l and m such that $l<m$, we get

$$\begin{aligned}
&\left\|\frac{1}{\lambda^{2l}}f(x_1,\dots,x_{i-1},\lambda^l x_i,x_{i+1},\dots,x_k)-\frac{1}{\lambda^{2m}}\right.\\
&\quad \left.\times f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),y_2,\dots,y_n\right\|\\
&\le \sum_{j=l}^{m-1}\left\|\frac{1}{\lambda^{2(j+1)}}f(x_1,\dots,x_{i-1},\lambda^{j+1}x_i,x_{i+1},\dots,x_k)\right.\\
&\qquad -\frac{1}{\lambda^{2j}}f(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k),\\
&\qquad \left. y_2,\dots,y_n\right\|\\
&\le \sum_{j=l}^{m-1}\left[\frac{1}{\lambda^{2j+1}}\varphi_i(x_1,\dots,x_{i-1},\lambda^j x_i,0,x_{i+1},\dots,x_k)\right.\\
&\qquad +\frac{1}{\lambda^{2j+2}}\\
&\qquad \left.\times\varphi_i(x_1,\dots,x_{i-1},a\lambda^j x_i,b\lambda^j x_i,x_{i+1},\dots,x_k)\right].
\end{aligned} \tag{69}$$

Therefore from (63), it follows that $\{(1/\lambda^{2j})f(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $Q_i : V^k \to W$ by (66). Putting $l=0$, letting $m\to\infty$ in (69) and using Lemma 4 and (63), we see that (65) holds.

Now, fix also $x_i' \in V$ and note that according to (64), we have

$$\begin{aligned}
&\left\|\frac{1}{\lambda^{2j}}\widehat{D}_i f(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i',x_{i+1},\dots,x_k),\right.\\
&\quad \left. y_2,\dots,y_n\right\|\\
&\qquad \le \frac{1}{\lambda^{2j}}\varphi_i(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i',x_{i+1},\dots,x_k).
\end{aligned} \tag{70}$$

Next, fix $s\in\{1,\dots,k\}\setminus\{i\}$, $x_s'\in V$, and assume that $s<i$ (the same arguments apply to the case where $s>i$). From (64), it follows that

$$\begin{aligned}
&\left\|\frac{1}{\lambda^{2j}}\widehat{D}_s f(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right.\\
&\qquad \left.\lambda^j x_i,x_{i+1},\dots,x_k),y_2,\dots,y_n\right\|\\
&\quad \le \frac{1}{\lambda^{2j}}\varphi_s(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\\
&\qquad \lambda^j x_i,x_{i+1},\dots,x_k).
\end{aligned} \tag{71}$$

Letting $j\to\infty$ in the above two inequalities and using (63), and Lemma 4 we see that the mapping Q_i is multi-Euler-Lagrange quadratic.

Now, let us finally assume that $Q_i' : V^k \to W$ is another multi-Euler-Lagrange quadratic mapping satisfying (65) and note that according to (61) and using Lemma 4, and (63) we have

$$\begin{aligned}
&\|Q_i(x_1,\dots,x_k)-Q_i'(x_1,\dots,x_k),y_2,\dots,y_n\|\\
&= \lim_{m\to\infty}\frac{1}{\lambda^{2m}}\|Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\\
&\qquad - Q_i'(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),\\
&\qquad y_2,\dots,y_n\|\\
&\le \lim_{m\to\infty}\frac{1}{\lambda^{2m}}[\|Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\\
&\qquad - f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),\\
&\qquad y_2,\dots,y_n\|\\
&\qquad + \|f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k)\\
&\qquad - Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k),\\
&\qquad y_2,\dots,y_n\|]
\end{aligned}$$

$$\leq \lim_{m\to\infty}\sum_{j=0}^{\infty}\Big[\frac{1}{\lambda^{2(m+j)+1}} \times \varphi_i\left(x_1,\ldots,x_{i-1},\lambda^{m+j}x_i,0,x_{i+1},\ldots,x_k\right) + \frac{1}{\lambda^{2(m+j)+2}} \times \varphi_i\left(x_1,\ldots,x_{i-1},a\lambda^{m+j}x_i, b\lambda^{m+j}x_i,x_{i+1},\ldots,x_k\right)\Big] = 0. \tag{72}$$

Therefore, by Lemma 4, we can conclude that $Q_i = Q_i'$. □

Similar to Theorem 15, one can get the following.

Theorem 16. *Assume that for every $i \in \{1,\ldots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\sum_{j=0}^{\infty}\lambda^{2j}\Big[\varphi_i\left(\frac{x_1}{\lambda^j},x_2,\ldots,x_{k+1}\right) + \cdots + \varphi_i\left(x_1,\ldots,x_{i-2},\frac{x_{i-1}}{\lambda^j},x_i,\ldots,x_{k+1}\right) + \varphi_i\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^j},\frac{x_{i+1}}{\lambda^j},x_{i+2},\ldots,x_{k+1}\right) + \varphi_i\left(x_1,\ldots,x_{i+1},\frac{x_{i+2}}{\lambda^j},x_{i+3},\ldots,x_{k+1}\right) + \cdots + \varphi_i\left(x_1,\ldots,x_k,\frac{x_{k+1}}{\lambda^j}\right)\Big] < \infty, \quad (x_1,\ldots,x_{k+1}) \in V^{k+1}. \tag{73}$$

If $f : V^k \to W$ is a function satisfying condition (11) *and*

$$\left\|\widehat{D}_i f(x_1,\ldots,x_{k+1}),y_2,\ldots,y_n\right\| \leq \varphi_i(x_1,\ldots,x_{k+1}), \quad (x_1,\ldots,x_{k+1}) \in V^{k+1}, \quad i \in \{1,\ldots,k\}, \quad y_2,\ldots,y_n \in W, \tag{74}$$

then for every $i \in \{1,\ldots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\left\|f(x_1,\ldots,x_k) - Q_i(x_1,\ldots,x_k),y_2,\ldots,y_n\right\| \leq \sum_{j=0}^{\infty}\Big[\lambda^{2j+1} \times \varphi_i\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^{j+1}},0,x_{i+1},\ldots,x_k\right) + \lambda^{2j}\varphi_i\left(x_1,\ldots,x_{i-1},\frac{ax_i}{\lambda^{j+1}},\frac{bx_i}{\lambda^{j+1}},x_{i+1},\ldots,x_k\right)\Big], \quad (x_1,\ldots,x_k) \in V^k, \quad y_2,\ldots,y_n \in W. \tag{75}$$

For every $i \in \{1,\ldots,k\}$ the function Q_i is given by

$$Q_i(x_1,\ldots,x_k) := \lim_{j\to\infty}\lambda^{2j} f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\ldots,x_k\right), \quad (x_1,\ldots,x_k) \in V^k. \tag{76}$$

Proof. Fix $x_1,\ldots,x_k \in V$, $y_2,\ldots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\ldots,k\}$. By (74), we obtain

$$\left\|\lambda^2 f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda},x_{i+1},\ldots,x_k\right) - f(x_1,\ldots,x_k),y_2,\ldots,y_n\right\| \leq \lambda\varphi_i\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda},0,x_{i+1},\ldots,x_k\right), + \varphi_i\left(x_1,\ldots,x_{i-1},\frac{ax_i}{\lambda},\frac{bx_i}{\lambda},x_{i+1},\ldots,x_k\right), \tag{77}$$

and consequently for any non-negative integers l and m such that $l < m$, we get

$$\left\|\lambda^{2l} f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^l},x_{i+1},\ldots,x_k\right) - \lambda^{2m} f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^m},x_{i+1},\ldots,x_k\right),y_2,\ldots,y_n\right\| \leq \sum_{j=l}^{m-1}\left\|\lambda^{2(j+1)} f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^{j+1}},x_{i+1},\ldots,x_k\right) - \lambda^{2j} f\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\ldots,x_k\right),y_2,\ldots,y_n\right\| \leq \sum_{j=l}^{m-1}\Big[\lambda^{2j+1}\varphi_i\left(x_1,\ldots,x_{i-1},\frac{x_i}{\lambda^{j+1}},0,x_{i+1},\ldots,x_k\right) + \lambda^{2j}\varphi_i\left(x_1,\ldots,x_{i-1},\frac{ax_i}{\lambda^{j+1}},\frac{bx_i}{\lambda^{j+1}},x_{i+1},\ldots,x_k\right)\Big]. \tag{78}$$

Therefore, from (73), it follows that $\{\lambda^{2j} f(x_1,\ldots,x_{i-1},x_i/\lambda^j, x_{i+1},\ldots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $Q_i : V^k \to W$ by (76). Putting $l = 0$, letting $m \to \infty$ in (78), and using Lemma 4 and (73) we see that (75) holds. The further part of the proof is similar to the proof of Theorem 15. □

As applications of Theorems 15 and 16, we get the following corollaries.

Corollary 17. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta \in [0,\infty)$ and $r \in (0,\infty)$ are such that $r \neq 1$. If $f : V^k \to W$ is a function satisfying*

$$\left\|\widehat{D}_i f(x_1,\ldots,x_{k+1}),y_2,\ldots,y_n\right\| \leq \theta\left[\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^r + \|x_{i+1}\|^r\right) \times \|x_{i+2}\|^r\cdots\|x_{k+1}\|^r\right],$$

$$(x_1,\dots,x_{k+1}) \in V^{k+1}, \quad i \in \{1,\dots,k\}, \quad y_2,\dots,y_n \in W, \tag{79}$$

then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\begin{aligned}&\|f(x_1,\dots,x_k) - Q_i(x_1,\dots,x_k), y_2,\dots,y_n\| \\ &\quad \le \frac{\theta\|x_1\|^r\cdots\|x_k\|^r(\lambda+|a|^r+|b|^r)}{|\lambda^2-\lambda^r|},\end{aligned} \tag{80}$$

for all $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$.

Corollary 18. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta \in [0,\infty)$ and $r,p,q \in (0,\infty)$ are such that $r,p+q \in (0,2)$ or $r,p+q \in (2,\infty)$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned}&\|\widehat{D}_i f(x_1,\dots,x_{k+1}), y_2,\dots,y_n\| \\ &\quad \le \theta\|x_1\|^r\cdots\|x_{i-1}\|^r\left(\|x_i\|^p\|x_{i+1}\|^q\right) \\ &\qquad \times \|x_{i+2}\|^r\cdots\|x_{k+1}\|^r, \\ &(x_1,\dots,x_{k+1}) \in V^{k+1}, \quad i \in \{1,\dots,k\}, \\ &\qquad y_2,\dots,y_n \in W,\end{aligned} \tag{81}$$

then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\begin{aligned}&\|f(x_1,\dots,x_k) - Q_i(x_1,\dots,x_k), y_2,\dots,y_n\| \\ &\quad \le \frac{|a|^p|b|^q\theta\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^{p+q}\|x_{i+1}\|^r\cdots\|x_k\|^r}{|\lambda^2-\lambda^{p+q}|},\end{aligned} \tag{82}$$

for all $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$.

For $a = b = 1$, Corollary 18 yields the following corollary which corrects Theorems 4.1 and 4.2 from [20].

Corollary 19. *Let V be a real normed linear space and let W be an 2-Banach space. Assume also that $\theta \in [0,\infty)$ and $p,q \in (0,\infty)$ are such that $p+q \neq 2$. If $f : V \to W$ is a function satisfying*

$$\begin{aligned}&\|f(x_1+x_2) - f(x_1) - f(x_2), y\| \\ &\quad \le \theta\|x_1\|^p\|x_2\|^q, \quad x_1,x_2 \in V, \quad y \in W,\end{aligned} \tag{83}$$

then there exists a unique quadratic mapping $Q : V \to W$ for which

$$\|f(x) - Q(x), y\| \le \frac{\theta\|x\|^{p+q}}{|4-2^{p+q}|}, \quad x \in V, \ y \in W. \tag{84}$$

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