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Research Article

Viscosity Approximation Methods for Two Accretive Operators in Banach Spaces

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We introduced a viscosity iterative scheme for approximating the common zero of two accretive operators in a strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Some strong convergence theorems are proved, which improve and extend the results of Ceng et al. (2009) and some others.

1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided that $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C, Tx = x\}$. Throughout this paper, we assume that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, for all $x, y \in C$. Let $\Sigma_C = \{f : C \to C \mid f$ is a contraction with constant $\alpha\}$. The normalized duality mapping J from E into 2^{E^*} is given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|(x + y)/2\| < 1$, for all $x \neq y \in E$ with $\|x\| = \|y\| = 1$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$, for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x\| = \|y\| = 1$ and $\lim_{n\to\infty}(\|x_n + y_n\|/2) = 1$.

The norm of E is said to be Gâteaux differentiable if

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for each x, y in its unit sphere $U = \{x \in E, \|x\| = 1\}$. Such an E is called a smooth Banach space. The norm is said to be uniformly Gâteaux differentiable if, for each $y \in U$, the limit is attained uniformly for $x \in U$. It is well known that E is smooth if and only if the duality mapping J is single valued and that, if E has a uniformly Gâteaux differentiable norm, J is uniformly norm to weak* continuous on each bounded subset of E (cf. [1]).

Let D be a subset of C. Then $Q : C \to D$ is called a retraction from C onto D if $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \ge 0$ whenever $Qx + t(x - Qx) \in C$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is known that $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds (cf. [2, page 48]):

$$\langle x - Qx, J(z - Qx)\rangle \le 0, \quad x \in C,\ z \in D. \tag{2}$$

Recall that an operator A with $D(A)$ and $R(A)$ in E is said to be accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists a $j \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j\rangle \ge 0$. An accretive operator A is m-accretive if $R(I + \lambda A) = E$, for all $\lambda > 0$. Denote by $A^{-1}0$ the set of zeros of A; that is, $A^{-1}0 = \{x \in D(A),\ Ax = 0\}$.

Denote by J_r $(r > 0)$ the resolvent of A; that is, $J_r = (I + rA)^{-1}$. It is well known that $F(J_r) = A^{-1}0$, for all $r > 0$. And if $D(A)$ is convex, then J_r is a nonexpansive mapping from E to

$D(A)$. If E is a Hilbert space, then A is a maximal monotone operator if and only if A is an m-accretive operator.

Recently, the approximation of zeros of accretive operators has been studied extensively (see, e.g., [3–9]). Specially, Ceng et al. [10] studied the following composite iterative scheme in uniformly smooth Banach spaces:

$$\begin{aligned} y_n &= \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n y_n + (1-\beta_n) J_{r_n} y_n, \end{aligned} \tag{3}$$

where $u \in \overline{D(A)}$ is an arbitrary (but fixed) element. They proved that $\{x_n\}$ generated by (3) converges strongly to a zero of m-accretive operator A under certain appropriate conditions.

Very recently, Chen et al. [11] considered the following viscosity iterative scheme in a reflexive Banach space having a weakly sequentially continuous duality mapping:

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) y_n, \end{aligned} \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Under some conditions, they showed that $\{x_n\}$ generated by (4) converges strongly to a zero of m-accretive operator A.

In this paper, motivated by [10–14], we will consider the following so-called composite viscosity iterative scheme for finding a common zero of two m-accretive operators:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n) S_{r_n} x_n, \\ x_{n+1} &= \beta_n f(y_n) + (1-\beta_n) y_n, \end{aligned} \tag{5}$$

where A and B are m-accretive operators, $S_{r_n} = (1-\lambda)J_r^A + \lambda J_r^B$ such that $F(S_r) = A^{-1}0 \cap B^{-1}0 \neq \emptyset$, $C = \overline{D(A)} = \overline{D(B)}$, $f \in \Sigma_C$, and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. Under some conditions, we will prove that $\{x_n\}$ generated by (5) converges strongly to a common zero of A and B in a strictly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm, which improve the corresponding results in [10–13].

Lemma 1 (see [10]). *In a Banach space E, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E, \tag{6}$$

where $j(x+y) \in J(x+y)$.

Lemma 2 (see [10, 13]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \sigma_n\gamma_n, \quad \forall n \ge 0, \tag{7}$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ such that

(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty}\sigma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 3 (the resolvent identity [10]). *For $\lambda > 0$, $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right). \tag{8}$$

Lemma 4 (see [3, Theorem 4.1, page 287]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Sigma_C$. Then $\{z_t\}$ defined by the following*

$$z_t = tf(z_t) + (1-t)Tz_t, \quad z_t \in C, \tag{9}$$

converges strongly to a point in $\mathrm{Fix}(T)$. *If, moreover, one defines $Q : \Sigma_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0} z_t, \quad f \in \Sigma_C, \tag{10}$$

then $Q(f)$ solves the variational inequality

$$\begin{aligned} \langle (I-f)Q(f), J(Q(f)-p)\rangle &\le 0, \\ f \in \Sigma_C, \quad p &\in F(T). \end{aligned} \tag{11}$$

Recall that a mapping $g : C \to C$ is said to be weakly contractive [15, 16] if

$$\|g(x) - g(y)\| \le \|x-y\| - \psi(\|x-y\|), \quad \forall x, y \in C, \tag{12}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0,+\infty)$ and $\psi(0) = 0$. As a special case, if $\psi(t) = (1-k)t$ for $t \in [0,+\infty)$, where $k \in (0,1)$, then the weakly contractive mapping g is a contraction with constant k. Rhodes [17] obtained the following result for weakly contractive mapping (see also [16]).

Lemma 5 (see [17, Theorem 2]). *Let (X,d) be a complete metric space and g a weakly contractive mapping on X. Then g has a unique fixed point p in X.*

Lemma 6. *Let $\{s_n\}$ and $\{\gamma_n\}$ be two sequences of nonnegative real numbers and $\{\lambda_n\}$ a sequence of positive numbers satisfying the conditions*

(i) $\sum_{n=1}^{\infty}\lambda_n = +\infty$,

(ii) $\lim_{n\to\infty}(\gamma_n/\lambda_n) = 0$.

Let the recursive inequality

$$s_{n+1} \le s_n - \lambda_n\psi(s_n) + \lambda_n\gamma_n, \quad n = 0, 1, 2, \ldots, \tag{13}$$

be given where $\psi(t)$ is a continuous and strict increasing function on $[0,+\infty)$ with $\psi(0) = 0$. Then $\lim_{n\to\infty}s_n = 0$.

2. Main Results

Throughout this section, we assume the following.

(i) E is a strictly convex Banach space which has a uniformly Gâteaux differentiable norm, and C is a nonempty closed convex subset of E.

(ii) Take $S_r = (1-\lambda)J_r^A + \lambda J_r^B$, $0 < \lambda < 1$. Obviously S_r is nonexpansive mapping and $F(S_r) = A^{-1}0 \cap B^{-1}0$, if E is a strictly convex Banach space. Indeed, it is easy to see that $F(S_r) \supset A^{-1}0 \cap B^{-1}0$. Let $q \in F(S_r)$, $p \in A^{-1}0 \cap B^{-1}0$; then

$$\begin{aligned}\|q-p\| &\le (1-\lambda)\left\|J_r^A q - p\right\| + \lambda\left\|J_r^B q - p\right\| \\ &\le (1-\lambda)\left\|J_r^A q - p\right\| + \lambda\|q-p\| \le \|q-p\|.\end{aligned} \tag{14}$$

From the above formula, we obtain $(1-\lambda)\|J_r^A q-p\|+\lambda\|q-p\| = \|q-p\|$, so $\|J_r^A q-p\| = \|q-p\|$. Similarly, $\|J_r^B q-p\| = \|q-p\|$. But

$$\|q-p\| = \left\|(1-\lambda)\left(J_r^A q - p\right) + \lambda\left(J_r^B q - p\right)\right\|. \tag{15}$$

Then the strict convexity of E implies that $q-p = J_r^A q - p = J_r^B q - p$, that is, $q = J_r^A q = J_r^B q$, or $q \in A^{-1}0 \cap B^{-1}0$.

Theorem 7. *Let E be a strictly convex Banach space which has a uniformly Gâteaux differentiable norm, A, B two m-accretive maps in E such that $C = \overline{D(A)} = \overline{D(B)}$ is convex and $A^{-1}0 \cap B^{-1}0 \neq \emptyset$, and $f : C \to C$ a fixed contraction mapping with contract constant α. Suppose that $\alpha_n \subset (0,1)$, $\beta_n \subset (0,1)$, and $r_n > 0$ satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$, *as* $n \to \infty$;

(ii) $\beta_n \to 0$, *and* $r_n \to r > \varepsilon > 0$ *as* $n \to \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ be the composite viscosity process defined by

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n) S_{r_n} x_n, \\ x_{n+1} &= \beta_n f(y_n) + (1-\beta_n) y_n. \end{aligned} \tag{16}$$

Then $\{x_n\}$ converges strongly to $p \in A^{-1}0 \cap B^{-1}0$, where p is the unique solution of the following variational inequality:

$$\langle (I-f)p, J(p-q)\rangle \le 0, \quad f \in \Sigma_C,\ q \in A^{-1}0 \cap B^{-1}0. \tag{17}$$

Proof. First, by using Lemma 4, we know that there exists the unique solution p of a variational inequality

$$\langle (I-f)p, J(p-q)\rangle \le 0, \quad f \in \Sigma_C,\ q \in A^{-1}0 \cap B^{-1}0, \tag{18}$$

where $p = \lim_{t\to 0} z_t$ and z_t is defined by $z_t = tf(z_t) + (1-t)S_r(z_t)$ for each $r > 0$ and $0 < t < 1$.

Next, we will divide our discussion into the following steps.

Step 1. We will show that $\{x_n\}$ is bounded.

In fact, take $p \in A^{-1}0 \cap B^{-1}0$. Then

$$\begin{aligned}\|y_n - p\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n) S_{r_n} x_n - p\right\| \\ &\le (1-\alpha_n(1-\alpha))\|x_n - p\| + \alpha_n\|f(p) - p\|.\end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n f(y_n) + (1-\beta_n) y_n - p\| \\ &\le (1-\beta_n(1-\alpha))\|y_n - p\| + \beta_n\|f(p) - p\| \\ &\le [1-(1-\alpha)(\beta_n + \alpha_n - \alpha_n\beta_n(1-\alpha))]\|x_n - p\| \\ &\quad + [\beta_n + \alpha_n - \alpha_n\beta_n(1-\alpha)]\|f(p) - p\| \\ &\le \max\left\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_n - p\|\right\}.\end{aligned} \tag{20}$$

Using the induction method, we have

$$\|x_n - p\| \le \max\left\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_0 - p\|\right\}, \quad n \ge 0, \tag{21}$$

which implies that $\{x_n\}$, $\{f(x_n)\}$, $\{y_n\}$, and $\{f(y_n)\}$ are all bounded. Since $\|S_{r_n}x_n - p\| \le \|x_n - p\|$, then $\{S_{r_n}x_n\}$ is bounded. Following the conditions of (i) and (ii), we obtain that

$$\|x_{n+1} - y_n\| = \beta_n\|f(y_n) - y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{22}$$

$$\left\|y_n - S_{r_n}x_n\right\| = \alpha_n\left\|f(x_n) - S_{r_n}x_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{23}$$

Step 2. We show that $\|x_{n+1} - x_n\| \to 0$.

For this, we estimate $y_{n+1} - y_n$ first. From (16), we know that

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1-\alpha_n) S_{r_n} x_n, \\ y_{n-1} &= \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1}) S_{r_{n-1}} x_{n-1}. \end{aligned} \tag{24}$$

Then simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= (1-\alpha_n)\left(S_{r_n}x_n - S_{r_{n-1}}x_{n-1}\right) \\ &\quad + \alpha_n(f(x_n) - f(x_{n-1})) \\ &\quad + (\alpha_n - \alpha_{n-1})\left(f(x_{n-1}) - S_{r_{n-1}}x_{n-1}\right). \end{aligned} \tag{25}$$

It follows from (25) that

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &\quad \le (1-\alpha_n)\left\|S_{r_n}x_n - S_{r_{n-1}}x_{n-1}\right\| + \alpha_n\alpha\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1}) - S_{r_{n-1}}x_{n-1}\right\|. \end{aligned} \tag{26}$$

In view of Lemma 3, we have

$$J_{r_n}^A x_n = J_{r_{n-1}}^A\left(\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}^A x_n\right). \tag{27}$$

If $r_{n-1} \le r_n$, then

$$\begin{aligned}
&\left\|J_{r_n}^A x_n - J_{r_{n-1}}^A x_{n-1}\right\| \\
&\quad = \left\|J_{r_{n-1}}^A \left(\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n}^A x_n\right) - J_{r_{n-1}}^A x_{n-1}\right\| \\
&\quad \le \left\|\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n}^A x_n - x_{n-1}\right\| \\
&\quad \le \left\|\frac{r_{n-1}}{r_n} (x_n - x_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right) \left(J_{r_n}^A x_n - x_{n-1}\right)\right\| \\
&\quad \le \frac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{r_n} \left\|J_{r_n}^A x_n - x_{n-1}\right\| \\
&\quad \le \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{\varepsilon} \left\|J_{r_n}^A x_n - x_{n-1}\right\|.
\end{aligned} \tag{28}$$

Similarly,

$$\begin{aligned}
\left\|J_{r_n}^B x_n - J_{r_{n-1}}^B x_{n-1}\right\| \le{}& \|x_n - x_{n-1}\| \\
&+ \frac{r_n - r_{n-1}}{\varepsilon} \left\|J_{r_n}^B x_n - x_{n-1}\right\|.
\end{aligned} \tag{29}$$

Thus, let $M = \sup\{(1/\varepsilon)\|J_{r_n}^A x_n - x_{n-1}\|, (1/\varepsilon)\|J_{r_n}^B x_n - x_{n-1}\|\}$; we have

$$\begin{aligned}
&\left\|S_{r_n} x_n - S_{r_{n-1}} x_{n-1}\right\| \\
&\quad \le (1-\lambda) \left\|J_{r_n}^A x_n - J_{r_{n-1}}^A x_{n-1}\right\| + \lambda \left\|J_{r_n}^B x_n - J_{r_{n-1}}^B x_{n-1}\right\| \\
&\quad \le \|x_n - x_{n-1}\| + (r_n - r_{n-1}) M.
\end{aligned} \tag{30}$$

Substituting (30) into (26) we get

$$\begin{aligned}
\|y_n - y_{n-1}\| \le{}& (1-\alpha_n) (\|x_n - x_{n-1}\| + (r_n - r_{n-1}) M) \\
&+ \alpha_n \alpha \|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - S_{r_{n-1}} x_{n-1}\right\| \\
\le{}& (1 - \alpha_n (1-\alpha)) \|x_n - x_{n-1}\| \\
&+ (1-\alpha_n)(r_n - r_{n-1}) M \\
&+ |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1}) - S_{r_{n-1}} x_{n-1}\right\| \\
\le{}& (1 - \alpha_n (1-\alpha)) \|x_n - x_{n-1}\| \\
&+ M_1 (|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|),
\end{aligned} \tag{31}$$

where M_1 is a constant such that

$$M_1 > \max\left\{M, \left\|f(x_{n-1}) - S_{r_{n-1}} x_{n-1}\right\|\right\}. \tag{32}$$

On the other hand, we have

$$\begin{aligned}
x_{n+1} &= \beta_n f(y_n) + (1-\beta_n) y_n, \\
x_n &= \beta_{n-1} f(y_{n-1}) + (1-\beta_{n-1}) y_{n-1}.
\end{aligned} \tag{33}$$

Simple calculations show that

$$\begin{aligned}
x_{n+1} - x_n = {}& (1-\beta_n)(y_n - y_{n-1}) + \beta_n (f(y_n) - f(y_{n-1})) \\
&+ (\beta_n - \beta_{n-1})(f(y_{n-1}) - y_{n-1}).
\end{aligned} \tag{34}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| \le{}& (1-\beta_n) \|y_n - y_{n-1}\| + \beta_n \alpha \|y_n - y_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - y_{n-1}\| \\
={}& (1 - \beta_n (1-\alpha)) \|y_n - y_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - y_{n-1}\|.
\end{aligned} \tag{35}$$

Substituting (31) into (35) we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le (1 - \beta_n (1-\alpha)) \{(1 - \alpha_n (1-\alpha)) \|x_n - x_{n-1}\| \\
&\qquad + M_1 (|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}|)\} \\
&\qquad + |\beta_n - \beta_{n-1}| \|f(y_{n-1}) - y_{n-1}\| \\
&\quad \le (1 - \alpha_n (1-\alpha)) \|x_n - x_{n-1}\| \\
&\qquad + M_2 (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |r_n - r_{n-1}|),
\end{aligned} \tag{36}$$

where M_2 is a constant such that

$$M_2 > \max\{\|f(y_{n-1}) - y_{n-1}\|, M_1\}. \tag{37}$$

From conditions (i)–(iii), we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \longrightarrow 0, \text{ as } n \longrightarrow \infty, \\
&\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |r_n - r_{n-1}|) < +\infty.
\end{aligned} \tag{38}$$

Hence, noticing (36) and applying Lemma 2, we obtain $\|x_{n+1} - x_n\| \to 0$. Then by (22) we obtain

$$\|x_n - y_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{39}$$

Step 3. We prove that $\|x_n - S_r x_n\| \to 0$, $\|y_n - S_r y_n\| \to 0$.

In fact, since

$$\left\|x_n - S_{r_n} x_n\right\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \left\|y_n - S_{r_n} x_n\right\|, \tag{40}$$

from (22) and (23), we have $\|x_n - S_{r_n}x_n\| \to 0$. Then

$$\begin{aligned}
&\|S_{r_n}x_n - S_r x_n\| \\
&\quad \le (1-\lambda)\left\|J_{r_n}^A x_n - J_r^A x_n\right\| + \lambda\left\|J_{r_n}^B x_n - J_r^B x_n\right\| \\
&\quad \le (1-\lambda)\left\|J_r^A\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^A x_n\right) - J_r^A x_n\right\| \\
&\qquad + \lambda\left\|J_r^B\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^B x_n\right) - J_r^B x_n\right\| \\
&\quad \le (1-\lambda)\left\|\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^A x_n\right) - x_n\right\| \\
&\qquad + \lambda\left\|\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^B x_n\right) - x_n\right\| \\
&\quad \le \left|1-\frac{r}{r_n}\right| \max\left\{\left\|J_{r_n}^A x_n - x_n\right\|, \left\|J_{r_n}^B x_n - x_n\right\|\right\} \longrightarrow 0.
\end{aligned} \tag{41}$$

Hence, we have

$$\begin{aligned}
&\|x_n - S_r x_n\| \\
&\quad \le \left\|x_n - S_{r_n}x_n\right\| + \left\|S_{r_n}x_n - S_r x_n\right\| \longrightarrow 0, \quad n \longrightarrow \infty, \\
&\|y_n - S_r y_n\| \\
&\quad \le \|x_n - y_n\| + \|x_n - S_r x_n\| + \|S_r y_n - S_r x_n\| \\
&\quad \le 2\|x_n - y_n\| + \|x_n - S_r x_n\| \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{42}$$

Step 4. We show that $\limsup_{n\to\infty}\langle (I-f)p, J(y_n - p)\rangle \le 0$, $\limsup_{n\to\infty}\langle (I-f)p, J(x_{n+1} - p)\rangle \le 0$.

To prove this, let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle (I-f)p, J(y_n - p)\rangle \\
&\quad = \lim_{j\to\infty}\left\langle (I-f)p, J\left(y_{n_j} - p\right)\right\rangle.
\end{aligned} \tag{43}$$

By Lemma 4, $\lim_{t\to 0^+} z_t = p \in F(S_r)$, where $z_t = tf(z_t) + (1-t)S_r(z_t)$. Then

$$z_{t_n} - y_{n_j} = t_n\left(f\left(z_{t_n}\right) - y_{n_j}\right) + (1-t_n)\left(S_r\left(z_{t_n}\right) - y_{n_j}\right). \tag{44}$$

For each integer $n \ge 0$, let $t_n \in (0,1)$ such that

$$t_n \longrightarrow 0, \quad \frac{\left\|S_r y_{n_j} - y_{n_j}\right\|}{t_n} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{45}$$

Using Lemma 1, we get

$$\begin{aligned}
&\left\|z_{t_n} - y_{n_j}\right\|^2 \\
&\quad \le (1-t_n)^2\left\|S_r\left(z_{t_n}\right) - y_{n_j}\right\|^2 \\
&\qquad + 2t_n\left\langle f\left(z_{t_n}\right) - y_{n_j}, J\left(z_{t_n} - y_{n_j}\right)\right\rangle \\
&\quad \le \left(1 - 2t_n + t_n^2\right)\left(\left\|S_r\left(z_{t_n}\right) - S_r y_{n_j}\right\| + \left\|y_{n_j} - S_r y_{n_j}\right\|\right)^2 \\
&\qquad + 2t_n\left\langle f\left(z_{t_n}\right) - z_{t_n}, J\left(z_{t_n} - y_{n_j}\right)\right\rangle + 2t_n\left\|z_{t_n} - y_{n_j}\right\|^2 \\
&\quad \le \left(1 + t_n^2\right)\left\|z_{t_n} - y_{n_j}\right\|^2 + (1+t_n)^2\left\|S_r y_{n_j} - y_{n_j}\right\| \\
&\qquad \times\left(2\left\|z_{t_n} - y_{n_j}\right\| + \left\|S_r y_{n_j} - y_{n_j}\right\|\right) \\
&\qquad + 2t_n\left\langle f\left(z_{t_n}\right) - z_{t_n}, J\left(z_{t_n} - y_{n_j}\right)\right\rangle,
\end{aligned} \tag{46}$$

and hence

$$\begin{aligned}
&\left\langle z_{t_n} - f\left(z_{t_n}\right), J\left(z_{t_n} - y_{n_j}\right)\right\rangle \\
&\quad \le \frac{t_n}{2}\left\|z_{t_n} - y_{n_j}\right\|^2 + \frac{(1+t_n)^2\left\|S_r y_{n_j} - y_{n_j}\right\|}{2t_n} \\
&\qquad \times\left(2\left\|z_{t_n} - y_{n_j}\right\| + \left\|S_r y_{n_j} - y_{n_j}\right\|\right).
\end{aligned} \tag{47}$$

Since $\{x_n\}$, $\{z_{t_n}\}$, and $\{S_r y_n\}$ are bounded, then $\|S_r y_{n_j} - y_{n_j}\|/2t_n \to 0$. Therefore,

$$\limsup_{j\to\infty}\left\langle z_{t_n} - f\left(z_{t_n}\right), J\left(z_{t_n} - y_{n_j}\right)\right\rangle \le 0. \tag{48}$$

We also know that

$$\begin{aligned}
&\left\langle p - f\left(z_{t_n}\right), J\left(y_{n_j} - p\right)\right\rangle \\
&\quad = \left\langle p - f\left(z_{t_n}\right), J\left(y_{n_j} - p\right) - J\left(z_{t_n} - y_{n_j}\right)\right\rangle \\
&\qquad + \left\langle p - z_{t_n}, J\left(z_{t_n} - y_{n_j}\right)\right\rangle \\
&\qquad + \left\langle z_{t_n} - f\left(z_{t_n}\right), J\left(z_{t_n} - y_{n_j}\right)\right\rangle.
\end{aligned} \tag{49}$$

Notice that $z_{t_n} \to p$, $p \in F(S_r)$, $n \to \infty$, and J is norm to weak* uniformly continuous on bounded subset of E; then we obtain

$$\begin{aligned}
&\left\langle p - z_{t_n}, J\left(z_{t_n} - y_{n_j}\right)\right\rangle \longrightarrow 0, \quad n \longrightarrow \infty, \\
&\left\langle p - f\left(z_{t_n}\right), J\left(y_{n_j} - p\right) - J\left(z_{t_n} - y_{n_j}\right)\right\rangle \longrightarrow 0, \\
&\qquad n \longrightarrow \infty.
\end{aligned} \tag{50}$$

From (48), (49), and the two results mentioned above, we have

$$\limsup_{n\to\infty} \langle (I-f)p, J(y_n - p)\rangle \le 0. \tag{51}$$

Using (22) and the property of J, we obtain the result that

$$\begin{aligned}\limsup_{n\to\infty} &\langle (I-f)p, J(y_n - p)\rangle \\ &= \limsup_{n\to\infty} \langle (I-f)p, J(x_{n+1} - p)\rangle \le 0.\end{aligned} \tag{52}$$

Step 5. $\lim_{n\to\infty}\|x_n - p\| = 0$.

Using (16), we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &\quad = \|\alpha_n (f(x_n) - p) + (1-\alpha_n)(S_r x_n - p)\|^2 \\ &\quad = \|(1-\alpha_n)(S_r x_n - p) + \alpha_n (f(x_n) - f(p)) \\ &\qquad + \alpha_n (f(p) - p)\|^2 \\ &\quad \le (1 - \alpha_n(1-\alpha))\|x_n - p\|^2 \\ &\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle.\end{aligned} \tag{53}$$

Applying Lemma 1, we obtain

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \|\beta_n (f(y_n) - p) + (1-\beta_n)(y_n - p)\|^2 \\ &\quad = \|(1-\beta_n)(y_n - p) \\ &\qquad + \beta_n (f(y_n) - f(p)) + \beta_n (f(p) - p)\|^2 \\ &\quad \le (1 - \beta_n(1-\alpha))\|y_n - p\|^2 \\ &\qquad + 2\beta_n \langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\quad \le (1 - \alpha_n(1-\alpha))(1 - \beta_n(1-\alpha))\|x_n - p\|^2 \\ &\qquad + 2\alpha_n (1 - \beta_n(1-\alpha)) \langle f(p) - p, J(y_n - p)\rangle \\ &\qquad + 2\beta_n \langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\quad \le [1 - (\alpha_n + \beta_n)(1-\alpha)]\|x_n - p\|^2 \\ &\qquad + \alpha_n\beta_n (1-\alpha)^2 \|x_n - p\|^2 \\ &\qquad - 2\alpha_n\beta_n (1-\alpha) \langle f(p) - p, J(y_n - p)\rangle \\ &\qquad + 2(\alpha_n + \beta_n) \langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p) - J(x_{n+1} - p)\rangle \\ &\quad \le [1 - (\alpha_n + \beta_n)(1-\alpha)]\|x_n - p\|^2 \\ &\qquad + (\alpha_n + \beta_n)\frac{\alpha_n\beta_n}{\alpha_n + \beta_n}(1-\alpha)^2 L^2 \\ &\qquad + 2(\alpha_n + \beta_n)\frac{\alpha_n\beta_n}{\alpha_n + \beta_n}(1-\alpha) L^2 \\ &\qquad + 2(\alpha_n + \beta_n) \langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\qquad + (\alpha_n + \beta_n)\frac{2\alpha_n}{\alpha_n + \beta_n} \\ &\qquad \times \langle f(p) - p, J(y_n - p) - J(x_{n+1} - p)\rangle,\end{aligned} \tag{54}$$

where $L = \sup_{n\ge 0}\{\|x_n - p\|, \|y_n - p\|, \|f(p) - p\|\}$. Put

$$\begin{aligned}\gamma_n &= (\alpha_n + \beta_n)(1-\alpha), \\ \sigma_n &= \frac{\alpha_n\beta_n}{\alpha_n + \beta_n}(1-\alpha) L^2 + \frac{2\alpha_n\beta_n}{\alpha_n + \beta_n} L^2 \\ &\quad + \frac{2}{1-\alpha} \langle f(p) - p, J(x_{n+1} - p)\rangle \\ &\quad + \frac{\alpha_n}{\alpha_n + \beta_n}\frac{2}{1-\alpha} \\ &\quad \times \langle f(p) - p, J(y_n - p) - J(x_{n+1} - p)\rangle.\end{aligned} \tag{55}$$

From the conditions (i) and (ii), the result of Step 4, and the facts that $\alpha_n\beta_n/(\alpha_n + \beta_n) \to 0$ and $\langle f(p) - p, J(y_n - p) - J(x_{n+1} - p)\rangle \to 0$, we know that $\gamma_n \to 0$, $\sum_{n=1}^{\infty}\gamma_n = +\infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. In view of Lemma 2, (54) reduces to

$$\|x_{n+1} - p\| \le (1-\gamma_n)\|x_n - p\| + \gamma_n\sigma_n; \tag{56}$$

then we know that

$$\lim_{n\to\infty}\|x_n - p\| = 0. \tag{57}$$

This completes the proof. □

Remark 8. If we modify (16) as follows:

$$\begin{aligned}y_n &= \alpha_n f(x_n) + (1-\alpha_n) S_{r_n} x_n, \\ x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) y_n,\end{aligned} \tag{58}$$

$$\begin{aligned}y_n &= \alpha_n u + (1-\alpha_n) S_{r_n} x_n, \\ x_{n+1} &= \beta_n f(y_n) + (1-\beta_n) y_n,\end{aligned} \tag{59}$$

or

$$\begin{aligned}y_n &= \alpha_n u + (1-\alpha_n) S_{r_n} x_n, \\ x_{n+1} &= \beta_n f(y_n) + (1-\beta_n) S_{r_n} y_n.\end{aligned} \tag{60}$$

Then, imitating the proof of Theorem 7, we can also get the result of Theorem 7. Therefore, from the compare of iterative scheme, the conclusions of [10, 11] are special cases of Theorem 7.

Example 9. Next we study the following optimization problem:

$$\min_{x\in C} h(x), \qquad \min_{x\in C} k(x), \tag{61}$$

where C is an interior nonempty closed convex subset of a Hilbert space and $h, k : C \to R$ are two proper convex and lower semicontinuous functionals. To solve optimization problem (61), we will list the following well known results.

Proposition 10 (see [18]). *Let $\varphi : C \to R$ be a proper convex and lower semicontinuous functional. Then*

(i) *$\partial\varphi : C \to E^*$ (∂ denotes the subdifferential in the sense of convex analysis) is a maximal monotone mapping;*

(ii) *$\partial\varphi(x_0) = \min_{x\in C}\varphi(x)$ if and only if $0 \in \partial\varphi(x_0)$.*

In Hilbert space $\partial\varphi$ is a m-accretive mapping. Thus $A = \partial h$, $B = \partial k$ are two m-accretive mappings. Solving optimization problem (61) is equivalent to finding a common zero of A and B.

Let

$$\begin{gathered} J_r^A = (I + rA)^{-1} = (I + r\partial h)^{-1}, \\ J_r^B = (I + rB)^{-1} = (I + r\partial k)^{-1}, \\ S_r = (1-\lambda) J_r^A + \lambda J_r^B, \qquad \alpha_n = \beta_n = \frac{1}{n}, \\ r_n = \frac{rn}{n+1}, \quad (r > \varepsilon). \end{gathered} \tag{62}$$

Then the conditions (i), (ii), and (iii) of Theorem 7 are satisfied. For arbitrary $f \in \Sigma_C$ the sequence $\{x_n\}$ generated by (16) converges strongly to a common zero of A and B, which is also the solution of the optimization problem (60).

Theorem 11. *Let A, B be two accretive maps in E with $A^{-1}0 \cap B^{-1}0 \neq \emptyset$ and satisfying the following range conditions: $\overline{D(A)} \subseteq C \subset R(I + rA) \cap R(I + rB)$, $\overline{D(B)} \subseteq C \subset R(I + rA) \cap R(I + rB)$ which are convex. Let f, $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{x_n\}$, $\{y_n\}$ be the same as those in Theorem 7. Let $\{x_n\}$ be a sequence generated by (16). Then $\{x_n\}$ converges strongly to $p \in A^{-1}0 \cap B^{-1}0$, where p is the unique solution of the following variational inequality:*

$$\langle (I - f) p, J(p - q)\rangle \le 0, \quad f \in \Sigma_C,\ q \in A^{-1}0 \cap B^{-1}0. \tag{63}$$

Theorem 12. *Let A, B be two accretive maps in E with $F = A^{-1}0 \cap B^{-1}0 \neq \emptyset$ and satisfying the following range conditions: $\overline{D(A)} \subseteq C \subset R(I+rA) \cap R(I+rB)$, $\overline{D(B)} \subseteq C \subset R(I+rA) \cap R(I+rB)$ which are convex. Let f, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ be the same as those in Theorem 7. Let $g : C \to C$ be a weakly contractive mapping with the function ψ. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} y_n = \alpha_n g(x_n) + (1 - \alpha_n) S_{r_n} x_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n. \end{gathered} \tag{64}$$

Then $\{x_n\}$ converges strongly to $p = Q(g(p)) \in A^{-1}0 \cap B^{-1}0$, where Q is a sunny nonexpansive retraction from C onto F.

Proof. Since E is a uniformly smooth Banach space, then there is a sunny nonexpansive retraction Q from C onto F. Then $Q \circ g$ is a weakly contractive mapping of C into itself. Indeed, for all $x, y \in C$,

$$\begin{aligned} \|Q(g(x)) - Q(g(y))\| &\le \|g(x) - g(y)\| \\ &\le \|x - y\| - \psi(\|x - y\|). \end{aligned} \tag{65}$$

Lemma 5 assures that there exists a unique element $p \in C$ such that $p = Q(g(p))$. Such $p \in C$ is an element of $A^{-1}0 \cap B^{-1}0$. Now we define a iterative scheme as follows:

$$\begin{gathered} z_n = \alpha_n g(p) + (1 - \alpha_n) S_{r_n} w_n, \\ w_{n+1} = \beta_n f(z_n) + (1 - \beta_n) z_n. \end{gathered} \tag{66}$$

Let w_n be the sequence generated by (66). Then Remark 8 (59) assures that w_n converges strongly to $p = Q(g(p))$ as $n \to \infty$. For any n, we have

$$\begin{aligned} &\|x_{n+1} - w_{n+1}\| \\ &\quad \le \beta_n \|f(y_n) - f(z_n)\| + (1 - \beta_n)\|y_n - z_n\| \\ &\quad \le \|y_n - z_n\| \\ &\quad \le \alpha_n \|g(x_n) - g(p)\| + (1 - \alpha_n)\|S_{r_n} x_n - S_{r_n} w_n\| \\ &\quad \le \alpha_n \|g(x_n) - g(w_n)\| + \alpha_n \|g(w_n) - g(p)\| \\ &\qquad + (1 - \alpha_n)\|x_n - w_n\| \\ &\quad \le \|x_n - w_n\| - \alpha_n \psi(\|x_n - w_n\|) \\ &\qquad + \alpha_n (\|w_n - p\| - \psi(\|w_n - p\|)) \\ &\quad \le \|x_n - w_n\| - \alpha_n \psi(\|x_n - w_n\|) + \alpha_n \|w_n - p\|. \end{aligned} \tag{67}$$

Thus, we obtain for $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} = s_n - \alpha_n \psi(s_n) + \alpha_n \|w_n - p\|. \tag{68}$$

Since $\|w_n - p\| \to 0$, it follows from Lemma 6 that $\lim_{n\to\infty}\|x_n - w_n\| = 0$. Hence

$$\lim_{n\to\infty} \|x_n - p\| \le \lim_{n\to\infty} (\|x_n - w_n\| + \|w_n - p\|) = 0. \tag{69}$$

This completes the proof. □

In virtue of the weakly contractive mapping g being a contraction, using Theorem 12 we may obtain the following.

Corollary 13. *Let A, B be two accretive maps in E with $F = A^{-1}0 \cap B^{-1}0 \neq \emptyset$ and satisfying the following range conditions: $\overline{D(A)} \subseteq C \subset R(I+rA) \cap R(I+rB)$, $\overline{D(B)} \subseteq C \subset R(I+rA) \cap R(I+rB)$ which are convex. Let $f, g \in \Sigma_C$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ be the same as those in Theorem 7. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} y_n = \alpha_n g(x_n) + (1 - \alpha_n) S_{r_n} x_n, \\ x_{n+1} = \beta_n f(y_n) + (1 - \beta_n) y_n. \end{gathered} \tag{70}$$

Then $\{x_n\}$ converges strongly to $p = Q(g(p)) \in A^{-1}0 \cap B^{-1}0$, where Q is a sunny nonexpansive retraction from C onto F.

Example 14. Next we give an essential example.

Let Ω be an bounded domain in a Euclidean space R^N with Lipschitz boundary Γ. Let $\phi : \Gamma \times R \to R$ be a given function shch that for each $x \in \Gamma$

(i) $\phi_x = \phi(x, \cdot) : R \to R$ is a proper, convex, lower semicontinuous function with $\phi_x(0) = 0$.

(ii) $\beta_x = \partial\phi_x$ (subdifferential of ϕ_x) is the maximal monotone mapping on R with $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Let $\alpha : R^N \to R^N$ be a continuous, monotone function such that there exist constants k_1, k_2 satisfying (i) $|\alpha(\xi)| \leq k_1|\xi|$ and (ii) $\langle \alpha(\xi), \xi\rangle \geq k_2|\xi|^2$ for each $\xi \in R^N$.

Definition 15 (see [19]). One first defines a mapping $A^\alpha : H^1(\Omega) \to (H^1(\Omega))^*$ ($H^1(\Omega)$ is a sobolev space) by

$$\left(A^\alpha u, v\right) = \int_\Omega \left\langle \alpha\left(\operatorname{grad} u\right), \operatorname{grad} v\right\rangle dx \tag{71}$$

for $u, v \in H^1(\Omega)$. Clearly A^α is an everywhere defined, monotone, demicontinuous operator from $H^1(\Omega)$ into $(H^1(\Omega))^*$. Second one defines an operator $A_p^\alpha : L^p(\Omega) \to 2^{L^p(\Omega)}$ for $1 < p < +\infty$ as follows.

(i) For $p \geq 2$ one defines the domain of A_p^α by

$$\begin{aligned} D\left(A_p^\alpha\right) = \{&u \in L^p(\Omega) : \text{there exists an } f \in L^p(\Omega) \\ &\text{such that } A^\alpha u + \partial\Phi(u) \ni f\}. \end{aligned} \tag{72}$$

Here $\Phi(u) = \int_\Gamma \phi_x(u|_\Gamma(x))d\Gamma(x)$ is the proper, convex, l.s.c. function (see [19, Lemma 3.1]). For $u \in D(A_p^\alpha)$ we set

$$A_p^\alpha(u) = \left\{f \in L^p(\Omega) \mid A^\alpha u + \partial\Phi(u) \ni f\right\}. \tag{73}$$

(ii) For $1 < p < 2$, one defines A_p^α as the L^p-closure of A_2^α defined in (i) above.

For the operator A_p^α one has following results.

Lemma 16 (see [19, Lemma 3.4]). *$A_p^\alpha : L^p(\Omega) \to 2^{L^p(\Omega)}$ is m-accretive operator ($1 < p < +\infty$).*

Lemma 17 (see [19, Proposition 3.2]). *Let $f \in L^p(\Omega), u \in L^p(\Omega)$ such that $f \in A_p^\alpha u$. Then*

(i) $\operatorname{div}(\alpha(\operatorname{grad} u)) = f$, *a.e.* on Ω *and*

(ii) $\langle n, \alpha(\operatorname{grad} u)\rangle \in \beta_x(u(x))$ *for a.e.* $x \in \Gamma$.

Lemma 18 (see [19, Proposition 3.3]). *Let $\beta_x \equiv 0$ for $x \in \Gamma$. Then $(A_p^\alpha)^{-1}0 = \{u \in L^p(\Omega) \mid u = a\ constant\ function\}$.*

Clearly for different α_1, α_2, $A_p^{\alpha_1}, A_p^{\alpha_2}$ are two m-accretive operators. The above results show that

$$\begin{aligned} \emptyset \neq &\left(A_p^{\alpha_1}\right)^{-1}0 \cap \left(A_p^{\alpha_2}\right)^{-1}0 \\ &= \left\{u \in L^p(\Omega) \mid u = a\ constant\ function\right\}. \end{aligned} \tag{74}$$

For the sake of finding a common zero, Theorems 7, 11, and 12 provided three different iterative algorithms. Therefore the study of a common zero of two accretive operators makes sense.

Remark 19. The results presented in this paper substantially improve and extend the results of Ceng et al. [10] from the following aspects.

(1) Theorems 7 and 12 extend the result on the iterative construction of the zero for a single accretive operator to the case of that for common zeros of two accretive operators. If we modify two accretive operators as finite accretive operators, then, imitating the proof of Theorem 7, we can also get the result of Theorem 7.

(2) Our results include one or two different viscosity items. Remark 8 shows that the conclusions of Ceng et al. are special cases of this paper.

(3) The viscosity item is changed from a contractive mapping f to weakly contractive mapping g in Theorem 12.

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