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Research Article

Frequently Hypercyclic and Chaotic Behavior of Some First-Order Partial Differential Equation

Cheng-Hung Hung[1] and Yu-Hsien Chang[2]

[1] *Cheng Shih University, No. 840 Cheng Cing Road, Kaohsiung 833, Taiwan*
[2] *Department of Mathematics, National Taiwan Normal University, Section 4, 88 Ting Chou Road, Taipei 116, Taiwan*

Correspondence should be addressed to Cheng-Hung Hung; hong838@yahoo.com.tw

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We study a particular first-order partial differential equation which arisen from a biologic model. We found that the solution semigroup of this partial differential equation is a frequently hypercyclic semigroup. Furthermore, we show that it satisfies the frequently hypercyclic criterion, and hence the solution semigroup is also a chaotic semigroup.

1. Introduction

The first-order partial differential equations appear in different branches of science and succeed to demonstrate events of nature. In this paper we focus on the particular form of partial differential equation

$$\frac{\partial}{\partial t}u + c(x)\frac{\partial}{\partial x}u = f(t,u), \quad t \ge 0,\ 0 \le x \le 1, \tag{1}$$

with an initial condition

$$u(0,x) = v(x), \quad 0 \le x \le 1, \tag{2}$$

where v and c are given continuous function defined on $[0,1]$ with

$$c(0) = 0, \quad c(x) > 0 \quad \text{for } x \in (0,1], \quad \int_0^1 \frac{dx}{c(x)} = \infty. \tag{3}$$

The function $f : [0,\infty) \times [0,\infty) \to R$ in (1) is a given continuously differentiable function satisfying

$$f(t,y) \le k_1 y + k_2 \quad \text{for } t \in [0,\infty),\ y \ge 0, \tag{4}$$

$$f(t,0) = 0, \quad f_y'(t,0) > 0 \quad \text{for } t \in [0,\infty), \tag{5}$$

where k_1, k_2 are nonnegative constants.

This equation is usually used to represent the models of age-structured populations. Populations of replicating and maturing cells are age structured in that the replenishment of new individuals into the population depends on the density of a cohort of older individuals. Many biological populations have similar models; the Lasota equation is the famous example which is an application of (1). It can be written as

$$\frac{\partial}{\partial t}u + x\frac{\partial}{\partial x}u = \lambda u, \quad t \ge 0,\ 0 \le x \le 1,\ \lambda > 1 \tag{6}$$

with the initial condition

$$u(0,x) = \varphi(x), \quad 0 \le x \le 1. \tag{7}$$

Equation (6) has been developed as a model for the dynamics of a self-reproducing cell population, such as the population of developing red blood cell (erythrocyte precursors). It has also been applied to a conceptualization of abnormal blood cell production such as leukemia. Although this equation is linear, the solution has chaotic behavior. Recently, there has been many authors studying this problem (e.g., [1–20]).

A lot of researchers are interested in the chaotic behavior of differential equation and chaotic C_0-semigroup. In this paper, we would like to study a type of semigroup, so-called frequently hypercyclic semigroup. The frequently hypercyclic semigroup has some restricted property to the chaotic C_0-semigroup.

Motivated by Birkhoff's ergodic Theorem, Bayart and Grivaux introduced the notion of frequently hypercyclic operators in [5]. In that paper, they quantified the frequency with which an orbit meets the open sets, and several examples of frequently hypercyclic operator are given. Moreover, the authors also give an operator which is hypercyclic but not frequently hypercyclic. Mangino and Peris extended the concept of single operator to the continuous case and defined the frequently hypercyclic semigroup in [17]. When a semigroup $\{S(t)\}_{t\geq 0}$ is a frequently hypercyclic semigroup, then for every $t_0 > 0$ the operator $S(t_0)$ is frequently hypercyclic, but the chaotic semigroup does not necessary satisfy this condition. By recent results of Bayart and Bermúdez [6], there are chaotic C_0-semigroup $\{S(t)\}_{t\geq 0}$ such that no single operator $S(t)$ is chaotic and a C_0-semigroup $\{S(t)\}_{t\geq 0}$ containing a nonchaotic operator $S(t_0)$, $t_0 > 0$ and a chaotic operator $S(t_1)$ for some $t_1 > 0$.

However, if a frequently hypercyclic semigroup $\{S(t)\}_{t\geq 0}$ satisfies frequently hypercyclic criterion, then $S(t)$ is also chaotic for every $t > 0$ [17, Proposition 2.7]. That is one of the reasons for us to study frequently hypercyclic semigroup.

The arrangement of this paper as follows. we will find the solution semigroup $\{S(t)\}_{t\geq 0}$ of (1) and some prosperities of it in Section 2. We prove that the solution semigroup is a frequently hypercyclic semigroup and some useful propositions in Section 3. In Section 4, we find the set of period points of the solution semigroup of (1) and prove this solution semigroup is chaotic directly.

2. The Solution Semigroup $\{S(t)\}_{t\geq 0}$ of (1)

Using the method of characteristics to find the unique solution of problem (1)~(3) is equivalent to solving the following two initial value problems:

$$\frac{\partial}{\partial t}x = c(x); \qquad x(p) = s, \tag{8}$$

$$\frac{\partial}{\partial t}y = f(t, y); \qquad y(0) = r. \tag{9}$$

Under condition (3), there exists a unique solution of the initial value problem (8). Denote it by $\varphi(t; p, s)$ for $p > 0$ and $s \in [0, 1]$. For simplicity, we denote $\varphi_s(t) = \varphi(t; 0, s)$ for $s \in [0, 1]$ and $t \in [0, \tau(s)]$, where $\tau(s)$ is the first point such that $\varphi_s(\tau(s)) = 1$. We admit that $\tau(0) = \infty$.

For describing $\varphi(t; p, s)$, we set

$$G(x) = \int_x^1 \frac{d\xi}{c(\xi)} \quad \text{for } x \in (0, 1]. \tag{10}$$

From (3), it follows that G is strictly decreasing and hence G^{-1}exist. Moreover,

$$\varphi(t; p, s) = G^{-1}(p - t + G(s)) \tag{11}$$

for $s \in [0, 1]$, $t \in [0, p + G(s)]$. In particular,

$$\varphi_s(t) = G^{-1}(G(s) - t) \quad \text{for } s \in [0, 1],\ t \in [0, G(s)]. \tag{12}$$

From (11), $\varphi(t; p, s)$ is nonnegative and nondecreasing in t. Furthermore, it is increasing and positive for $s > 0$.

Secondly, we are going to solve the initial value problem (9). According to conditions (4) and (5), the unique solution $\Psi(t; s, r)$ of (9) exists for all $t \in [0, \tau(s)]$ and $(s, r) \in [0, 1] \times [0, \infty)$.

The characteristic of (1) is given by $(\varphi_s(t), \Psi(t; s, r))$. Thus, for each pair of (1) and (3), we have that

$$u(t, \varphi_s(t)) = \Psi(t; s, r) \quad \text{for } t \in [0, \tau(s)], \tag{13}$$

where $r = u(0, s) = v(s)$. Setting $s = \varphi(0; t, x)$, we obtain $\varphi_s(t) = x$, and hence

$$u(t, x) = \Psi(t; \varphi(0; t, x), v(\varphi(0; t, x))) \quad \text{for } (t, x) \in [0, \infty) \times [0, 1]. \tag{14}$$

Formula (14) shows the existence and uniqueness of solution of (1) and (3).

Let $C_+([0, 1])$ be the space of nonnegative and continuous functions defined on $[0, 1]$ and $C_+^1([0, 1])$ the subspace of $C_+([0, 1])$ which contains all continuously differentiable functions. Because the functions c and f do not depend explicitly on t, formula (14) can define a semigroup on $C_+^1([0, 1])$ of the form

$$S(t)v(x) = \Psi(t; \varphi(0; t, x), v(\varphi(0; t, x))) \quad \text{for } (t, x) \in [0, \infty) \times [0, 1]. \tag{15}$$

The space $C_+([0, 1])$ with the supremum norm is a Banach space. Since the transformations $\{S(t)\}_{t\geq 0}$ are continuous operators and $C_+^1([0, 1])$ is a dense subspace of $C_+([0, 1])$, hence, $\{S(t)\}_{t\geq 0}$ is also a semigroup on $C_+([0, 1])$.

Let

$$V = \{v \in C_+([0, 1]) : v(0) = 0\} \tag{16}$$

with the metric

$$\rho(u, v) = \|u - v\|, \quad \forall u, v \in V, \tag{17}$$

where $\|\cdot\|$ is the supremum norm on $C_+([0, 1])$. Then, (V, ρ) is a complete metric space. For convenience, we will denote it as V.

Lemma 1. *The space V is invariant with respect to the semigroup $\{S(t)\}_{t\geq 0}$ given by* (15).

Proof. We need only to show that $S(t)v(0) = 0$ for $v \in V$ and $t \geq 0$. By (3), (11), and (15), we have $\varphi(0; t, 0) = 0$ and $S(t)\ v(0) = \Psi(t; \varphi(0; t, 0), v(\varphi(0; t, 0))) = \Psi(t; 0, 0)$. From (5), $y = 0$ is the unique solution of (9) for initial value $y(0) = r = 0$. This implies that $S(t)v(0) = \Psi(t; 0, 0) = 0$ and the proof of this lemma is completed. □

Since V is complete separable metric space and $S(t)(V) \subset V$ for $t \geq 0$ (Lemma 1), the semigroup $\{S(t)\}_{t\geq 0}$ can be considered on the space V.

For proving Lemma 2, given any $t_0 > 0$ with $t_0 \geqslant G(\alpha)$, we will define a transformation T_0 on V. From (4) and (5), it follows that $\Psi(t;s,\cdot)$ is a bijection. Thus, there exists a unique function $\sigma(t;s,\cdot): R_+ \rightarrow R_+$ satisfying

$$\Psi(t;s,\sigma(t;s,r)) = r \quad \text{for } s \in [0,1], \ t \in [0,\tau(s)]. \tag{18}$$

For every $t_0 \geq G(\alpha)$ and $d_0 \equiv G^{-1}(t_0) = \varphi(0;t_0,1)$, we can define a transformation $T_0 : V \rightarrow V$ by the formula

$$T_0 v(x) : \begin{cases} \sigma(t_0; x, v(\varphi(t_0;0,x))) & \text{for } 0 \leq x \leq d_0, \\ \sigma(t_0; d_0, v(1)) & \text{for } d_0 \leq x \leq 1, \end{cases} \tag{19}$$

for every $v \in V$.

Lemma 2. *There exists a closed subset $U \subseteq V$ which is invariant with respect to T_0.*

Proof. Let $U = \{v \in V : 0 \leq v(x) \leq M_0, \text{for } 0 \leq x \leq 1\}$, where M_0 will be determined later. By the differentiation of (18) with respect to r and the fact that $\Psi(t;s,\cdot)$ is a bijection, we obtain

$$\sigma_r'(t;s,r) = \left(\Psi_r'(t;s,\sigma(t;s,r))\right)^{-1}. \tag{20}$$

Using (9), we have

$$\begin{aligned} &\Psi_r'(t;s,\sigma(t;s,r)) \\ &\quad = \exp\left(\int_0^t f_y'(\varphi_s(p), \Psi(p;s,\sigma(t;s,r)))\,dp\right) \end{aligned} \tag{21}$$

and consequently

$$\sigma_r'(t;x,\theta) = \exp\left(-\int_0^t f_y'(\varphi_s(p), \Psi(p;s,\sigma(t;s,r)))\,dp\right) \tag{22}$$

for $s \in [0,1]$, $t \in [0,\tau(s)]$, and $r \geq 0$. Since $\Psi(t;s,0) = 0$ for $t \in [0,\tau(s)]$ and from (18), we get $\sigma(t;s,0) = 0$ for $t \in [0,\tau(s)]$. For a given $x \in [0,d_0]$, by mean-value theorem, we have that

$$\begin{aligned} T_0 v(x) &= \sigma(t_0;x,v(\varphi(t_0;0,x))) - \sigma(t_0;x,0) \\ &= \sigma_r'(t_0;x,\theta)\, v(\varphi(t_0;0,x)) \\ &\qquad \text{for some } \theta \in [0, v(\varphi(t_0;0,x))]. \end{aligned} \tag{23}$$

For estimate T_0, according to (22) and (23), we need to estimate σ_r' and f_y'.

From (5) there exist numbers $\alpha \in (0,1]$ and $\beta > 0$ such that

$$f_y'(t,y) \geq \beta \quad \text{for } t \in [0,\infty), \ y \in [0,\alpha]. \tag{24}$$

By the fact, $\Psi(p;x,0) = 0$, and $\sigma(t_0;x,0) = 0$, we may choose $M_0 > 0$ such that

$$\Psi(p;x,\sigma(t_0;x,\theta)) \leq \alpha \tag{25}$$

for $p \in [0,t_0]$, $x \in [0,d_0]$, $\theta \in [0,M_0]$ and α is chosen in (24). From (22) and (24), we have

$$\begin{aligned} &\sigma_r'(t_0;x,\theta) \leq L, \\ &\text{for } x \in [0,d_0], \ \theta \in [0,M_0], \text{ where } L = e^{-\beta t_0}. \end{aligned} \tag{26}$$

Using (19), (23), and (26), it is easy to see $T_0(U) \subset U$, and the proof of this lemma is complete. □

From the properties of $S(t)$ and T_0, we have the following lemma.

Lemma 3. *For every $t_o \geq G(\alpha)$,*

$$S(t_0)\, T_0 v = v \quad \text{for every } v \in V. \tag{27}$$

Proof. Using formula (15), (19), and the definition of φ, we have

$$\begin{aligned} &S(t)\, T_0 v(x) \\ &\quad = \Psi(t;\varphi(0;t,x), \sigma(t_0;\varphi(0;t,x), v(\varphi(t_0;t,x)))) \end{aligned} \tag{28}$$

for $0 \leq t \leq t_0$, $0 \leq x \leq \varphi(t;0,d_0)$, and $v \in V$.

Pluging $t = t_0$ into (28), we have

$$\begin{aligned} &S(t_0)\, T_0 v(x) \\ &\quad = \Psi(t_0;\varphi(0;t_0,x), \sigma(t_0;\varphi(0;t_0,x), v(\varphi(t_0;t_0,x)))). \end{aligned} \tag{29}$$

Furthermore, since $\varphi(t_0;t_0,x) = x$ and $\varphi(t_0;0,d_0) = 1$,

$$\begin{aligned} &S(t_0)\, T_0 v(x) \\ &\quad = \Psi(t_0;\varphi(0;t_0,x), \sigma(t_0;\varphi(0;t_0,x), v(\varphi(t_0;t_0,x)))) \\ &\quad = \Psi(t_0;\varphi(0;t_0,x), \sigma(t_0;\varphi(0;t_0,x), v(x))) \end{aligned} \tag{30}$$

for $0 \leq x \leq \varphi(t;0,d_0) = 1$. By (18) we have $S(t_0)T_0 v = v$ for every $v \in V$, and the assertion of this lemma is established now. □

3. The Frequently Hypercycle Property of $\{S(t)\}_{t \geq 0}$

At beginning of this section, we introduce some terminologies and propositions which will be used later. According to Devaney's definition, a semigroup $\{K(t)\}_{t\geq 0}$ defined in a metric space (X,d) is chaotic if it has following properties:

(1) $\{K(t)\}_{t\geq 0}$ has a property of sensitive dependence on initial conditions; that is, there is a positive real number M such that for every point $v \in X$ and every $\varepsilon > 0$ there is $w \in B(v,\varepsilon)$ and $t > 0$, such that $d(K(t)v, K(t)w) \geq M$;

(2) $\{K(t)\}_{t\geq 0}$ is transitive; that is, for all nonempty open subsets U_1, U_2 there is $t > 0$ such that $(K(t)U_1) \cap U_2 \neq \phi$;

(3) the set of periodic points of $\{K(t)\}_{t\geq 0}$ is dense in X.

We recall that the lower density of a measurable set $M \subset R^+$ is defined by

$$\underline{\text{Dens}}(M) = \frac{\liminf_{N\to\infty} \mu(M \cap [0,N])}{N}, \tag{31}$$

where μ is the Lebesgue measure on R^+. A C_0-semigroup $\{S(t)\}_{t\geq 0}$ is called frequently hypercyclic if there exists $x \in X$ such that

$$\underline{\mathrm{Dens}}\left(\{t \in R^+ : S(t)\,x \in U\}\right) > 0 \text{ for any nonempty open set } U \subset X. \tag{32}$$

The lower density of a set $A \subset \mathbf{N}$ is defined by

$$\underline{\mathrm{Dens}}\,(A) = \lim \inf_{N\to\infty} \sharp \frac{\{n \leq N : n \in A\}}{N}. \tag{33}$$

An operator $S \subset L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ (called frequently hypercyclic vector) such that, for any non-empty open set $U \subset X$, the set $\{n \in \mathbf{N} : S^n x \in U\}$ has positive lower density. In [6], Bayart and Bermúdez proved that if $x \in X$ is a frequently hypercyclic vector for $\{S(t)\}_{t\geq 0}$, then for $t > 0$ the x is a frequently hypercyclic vector for the operator $S(t)$.

Proposition 4 (see [17, Proposition 2.1]). *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Then, the following conditions are equivalent:*

(1) *$\{S(t)\}_{t\geq 0}$ is frequently hypercyclic;*
(2) *for every $t > 0$ the operator $S(t)$ is frequently hypercyclic;*
(3) *there exist $t_0 > 0$ such that the operator $S(t_0)$ is frequently hypercyclic.*

According to this proposition one wants to show that a semigroup $\{S(t)\}_{t\geq 0}$ is frequently hypercyclic just needed to check the operator $S(t_0)$ for some fixed t_0 is frequently hypercyclic. The following proposition described the sufficient condition for frequently hypercyclic operator. It is also called frequently hypercyclic criterion. Frequently hypercyclic criterion builds the relation between frequently hypercyclic semigroup and chaotic semigroup.

Proposition 5. *Let S be a continuous operator on a separable Banach space X. If there exist a dense subset $X_0 \subseteq X$ and a map $T : X_0 \to X_0$ satisfying*

(1) *$STx = x$, for all $x \in X_0$;*
(2) *$\sum_{n=1}^{\infty} S^n x$ is unconditionally convergent for all $x \in X_0$;*
(3) *$\sum_{n=1}^{\infty} T^n x$ is unconditionally convergent for all $x \in X_0$;*

then S is frequently hypercyclic.

The proof of this proposition can be found in [3].

Theorem 6. *Suppose that U is the closed set in Lemma 2; then the solution semigroup $\{S(t)\}_{t\geq 0}$ in Section 2 is frequently hypercyclic on U.*

Proof. To show the conclusion of this theorem to be true, we are planning to apply Proposition 5.

According to Proposition 4, to show that $S(t)$ is frequently hypercyclic, we need only to prove that $S(t_0)$ is frequently hypercyclic operator for some fixed t_0.

For this purpose, we defined an operator S_0 on V by

$$\begin{aligned} S_0 v(x) &= S(t_0)\, v(x) \\ &= \Psi\left(t_0; \varphi\left(0; t_0, x\right), v\left(\varphi\left(0; t_0, x\right)\right)\right) \\ &\qquad \text{for } 0 \leq x \leq 1, v \in V. \end{aligned} \tag{34}$$

It is obvious that the operator T_0 defined by (19) is a good candidate for checking condition (1) of Proposition 5. In fact, by Lemma 3 we have $S_0 T_0 = I$ and condition (1) of Proposition 5 is satisfied.

For checking condition (2) of Proposition 5, we are going to find a dense subset of $C_+([0,1])$. The characteristic functions $\chi_{[a,b]}$, a, $b \in [0,1]$, are candidates. However, $\chi_{[a,b]}$ does not belong to $C_+([0,1])$. So we need to modify $\chi_{[a,b]}$. For a suitable small positive constant ε and a, $b \in [0,1]$, we define $\chi_{[a,b],\varepsilon} = 1$ for $x \in [a+\varepsilon, b-\varepsilon]$, $\chi_{[a,b],\varepsilon} = 0$ for $x \in [0,1] - [a,b]$, and smooth connecting the graph of $\chi_{[a,b],\varepsilon}$ for $x \in [a, a+\varepsilon] \cup [b-\varepsilon, b]$ such that $\chi_{[a,b],\varepsilon} \in C_+([0,1])$. We choose some sequences $\{a_i\}$ and $\{b_i\}$, $i \in \mathbf{N}$, $a_i, b_i \in (0,1)$ such that $a_i \to 0$ and $b_i \to 1$ as $i \to \infty$. Let

$$I_i = \chi_{[a_i, b_i],\varepsilon}; \tag{35}$$

we have $\chi_{[0,1]} = \lim_{i\to\infty} I_i$ and $v = \lim_{i\to\infty} v \cdot I_i$ for $v \in C_+([0,1])$.

Let $W = \{v_i : v_i = v \cdot I_i,\ v \in C_+([0,1])$, where I_i were defined as in (35)$\}$. It is clear that W is dense in V, and hence W is dense in U also.

According to the definitions of G and φ, for $k \in \mathbf{N}$, we have that

$$S_0^k v(x) = \Psi\left(kt_0; \varphi\left(0; kt_0, x\right), v\left(\varphi\left(0; kt_0, x\right)\right)\right) \tag{36}$$

for $0 \leq x \leq 1$ and $v \in V$, provided $S(t)$ is the solution semigroup. From (11) and the fact that G is strictly decreasing, for any fixed I_i, there exists $k_0 \in \mathbf{N}$ such that for all $k > k_0$, $k \in \mathbf{N}$, we have that $\varphi(0; kt_0, 1) < a_i$ and $I_i(\varphi(0; kt_0, x)) = 0$ for $0 \leq x \leq 1$. This implies that

$$\begin{aligned} S_0^k I_i(x) &= \Psi\left(kt_0; \varphi\left(0; kt_0, x\right), I_i\left(\varphi\left(0; kt_0, x\right)\right)\right) = 0, \\ &\quad \text{for } 0 \leq x \leq 1, \forall k > k_0 k \in \mathbf{N}, \text{ and consequently,} \end{aligned} \tag{37}$$

$$S_0^k v_i(x) = \Psi\left(kt_0; \varphi\left(0; kt_0, x\right), v_i\left(\varphi\left(0; kt_0, x\right)\right)\right) = 0. \tag{38}$$

From the previously mentioned, there have been only finite many $k \in \mathbf{N}$ such that $S_0^k v_i(x) \neq 0$, so $\sum_{k=1}^{\infty} S_0^k v_i$ is unconditionally convergent. This proves that condition (2) of Proposition 5 is satisfied.

From (19), (27), and the definitions of G and φ, for $k \in \mathbf{N}$, we have

$$T_0^k v(x) : \begin{cases} \sigma\left(kt_0; x, v\left(\varphi\left(kt_0; 0, x\right)\right)\right) & \text{for } 0 \leq x \leq d_{k-1}, \\ \sigma\left(kt_0; d_{k-1}, v(1)\right) & \text{for } d_0 \leq x \leq 1, \end{cases} \tag{39}$$

where $d_{k-1} = G^{-1}(kt_0) = \varphi(0; kt_0, 1)$ for every $k \in \mathbf{N}$. In fact, $T_0^k v(x)$ for $0 \leq x \leq d_0$ is equal to $\sigma(kt_0; x, v(\varphi(kt_0; 0, x)))$ for $0 \leq x \leq d_{k-1}$.

Using similar estimation of (23), we have that

$$T_0^k v(x) = \sigma_r'(kt_0; x, \theta)\, v(\varphi(kt_0; 0, x)) \quad \text{for } x \in [0, d_0],$$
$$\sigma_r'(kt_0; x, \theta) \le L^k, \quad \text{for } x \in [0, d_{k-1}],\ \theta \in [0, M_0]. \tag{40}$$

This implies that $\|T_0^k\| \le L^k$, for every $k \in \mathbf{N}$, and hence $\sum_{k=1}^{\infty} \|T_0^k\| \le L/(1-L)$. This shows that $\sum_{k=1}^{\infty} T_0^k v$ is unconditionally convergent. Thus, condition (3) of Proposition 5 is also satisfied. We complete the proof of this theorem now. □

Although from frequently hypercyclic criterion we can get $S(t)$ is chaotic for every $t > 0$, we can directly prove the conclusion without using frequently hypercyclic criterion and we state in next section.

4. The Chaotic Property of $\{S(t)\}_{t\ge 0}$

From the definition of chaotic semigroup, we need to find the set of period points of $S(t_0)$ for some $t_0 > 0$ which is dense in V. For this purpose, we first find a special function such that

$$\begin{aligned} T_{t_0} v_0(x) &= \sigma(t_0; x, v_0(\varphi(t_0; 0, x))), \\ &= v_0(x) \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{41}$$

We restrict T_{t_0} on $U = \{v \in V : 0 \le v(x) \le M_0, 0 \le x \le 1\}$; then by (22), (23), and (24), we have

$$\begin{aligned} T_{t_0} v(x) &= \sigma_r'(t_0; x, \theta)\, v(\varphi(t_0; 0, x)) \\ &\le e^{-\beta t_0} v(\varphi(t_0; 0, x)), \quad \text{for } v \in U. \end{aligned} \tag{42}$$

Since $e^{-\beta t_0} < 1$, this implies that T_{t_0} is contraction on U, and hence there exists a $v_0 \in U$ such that $T_{t_0} v_0(x) = v_0(x)$.

For any $w \in V$, we define $\widetilde{w_\delta}$ as

$$\widetilde{w_\delta}(x) : \begin{cases} v_0(x) & \text{for } 0 \le x \le d_0, \\ \dfrac{1}{\delta}(w(d_0+\delta) - v_0(d_0)) & \\ (x - d_0) + v_0(d_0) & \text{for } d_0 \le x \le d_0 + \delta, \\ w(x) & \text{for } d_0 + \delta \le x \le 1. \end{cases} \tag{43}$$

From (43), $\widetilde{w_\delta}$ is also belonging to V. Combining (41) and (43), we obtain

$$T_{t_0} \widetilde{w_\delta}(x) = v_0(x) = \widetilde{w_\delta}(x), \quad \text{for } 0 \le x \le d_0. \tag{44}$$

Using (27), (44), and the fact that $0 \le \varphi(0; t_0, x) \le d_0$ for $0 \le x \le 1$, we have

$$\begin{aligned} S(t_0)\widetilde{w_\delta}(x) &= \Psi(t_0; \varphi(0; t_0, x), \widetilde{w_\delta}(\varphi(0; t_0, x))) \\ &= \Psi\left(t_0; \varphi(0; t_0, x), T_{t_0}\widetilde{w_\delta}(\varphi(0; t_0, x))\right) \\ &= S(t_0) T_{t_0} \widetilde{w_\delta}(x) = \widetilde{w_\delta}(x) \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{45}$$

In other words, $\widetilde{w_\delta}$ is a periodic point of $S(t_0)$.

Remark 7. It is not hard to prove that the set of periodic points of (45) is dense in V and the solution semigroup defined by (15) is transitive in V. As proved by Bayart and Matheron [4], the sensitive dependence of the C_0-semigroup on initial conditions in the sense of Guckenheimer appears immediately from its transitivity and density of the set of its periodic points. This is expressed by the following corollary.

Corollary 8. *The solution C_0-semigroup $\{S(t)\}_{t\ge 0}$ defined by (15) is chaotic in V.*

Finally, we demonstrate two simple examples. The first one is

$$\frac{\partial}{\partial t}u + rx\frac{\partial}{\partial x}u = \alpha u(1-u), \quad t \ge 0,\ 0 \le x \le 1, \tag{46}$$

where $r, \alpha > 0$ are constants, and with the initial condition

$$u(0, x) = \varphi(x), \quad 0 \le x \le 1. \tag{47}$$

It is easy to see that condition (3) is satisfied.

In fact, the solution semigroup $\{S(t)\}_{t\ge 0}$ of (46) is given by

$$\begin{aligned} S(t)\varphi(x) &= \frac{\varphi(xe^{-rt})\, e^{\alpha t}}{1 - \varphi(xe^{-rt})[1 - e^{\alpha t}]} \\ &= \left[\left(\varphi\left(xe^{-rt}\right)\right)^{-1} \cdot e^{-\alpha t} - e^{-\alpha t} + 1\right]^{-1}. \end{aligned} \tag{48}$$

From the previous results, we know that $\{S(t)\}_{t\ge 0}$ is not only a frequently hypercyclic semigroup but also chaotic.

Another example is the Lasota equation (6) in Section 1. It is easy to see that condition (3) is satisfied. The solution semigroup of (6) is frequently hypercyclic and chaotic as well.

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References

[1] A. A. Albanese, X. Barrachina, E. M. Mangino, and A. Peris, "Distributional chaos for strongly continuous semigroups of operators," *Communications on Pure and Applied Analysis*, vol. 12, no. 5, pp. 2069–2082, 2013.

[2] A. Bonilla and K.-G. Grosse-Erdmann, "Frequently hypercyclic operators and vectors," *Ergodic Theory and Dynamical Systems*, vol. 27, no. 2, pp. 383–404, 2007, Erratum in *Ergodic Theory and Dynamical Systems*, vol. 29, 1993–1994, 2009.

[3] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, vol. 179 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 2009.

[4] F. Bayart and E. Matheron, "Mixing operators and small subsets of the circle," http://arxiv.org/abs/1112.1289.

[5] F. Bayart and S. Grivaux, "Frequently hypercyclic operators," *Transactions of the American Mathematical Society*, vol. 358, pp. 5083–5117, 2006.

[6] F. Bayart and T. Bermúdez, "Semigroups of chaotic operators," *Bulletin of the London Mathematical Society*, vol. 41, no. 5, pp. 823–830, 2009.

[7] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, "On Devaney's definition of chaos," *The American Mathematical Monthly*, vol. 99, no. 4, pp. 332–334, 1992.

[8] J. Banasiak and M. Moszyński, "Dynamics of birth-and-death processes with proliferation—stability and chaos," *Discrete and Continuous Dynamical Systems A*, vol. 29, no. 1, pp. 67–79, 2011.

[9] Y.-H. Chang and C.-H. Hong, "The chaos of the solution semigroup for the quasi-linear lasota equation," *Taiwanese Journal of Mathematics*, vol. 16, no. 5, pp. 1707–1717, 2012.

[10] J. A. Conejero and E. M. Mangino, "Hypercyclic semigroups generated by ornstein-uhlenbeck operators," *Mediterranean Journal of Mathematics*, vol. 7, no. 1, pp. 101–109, 2010.

[11] J. A. Conejero, V. Müller, and A. Peris, "Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup," *Journal of Functional Analysis*, vol. 244, no. 1, pp. 342–348, 2007.

[12] R. I. Devancy, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, New York, NY, USA, 2nd edition, 1989.

[13] A. L. Dawidowicz, N. Haribash, and A. Poskrobko, "On the invariant measure for the quasi-linear lasota equation," *Mathematical Methods in the Applied Sciences*, vol. 30, no. 7, pp. 779–787, 2007.

[14] H. Emamirad, G. R. Goldstein, and J. A. Goldstein, "Chaotic solution for the black-scholes equation," *Proceedings of the American Mathematical Society*, vol. 140, no. 6, pp. 2043–2052, 2012.

[15] L. Ji and A. Weber, "Dynamics of the heat semigroup on symmetric spaces," *Ergodic Theory and Dynamical Systems*, vol. 30, no. 2, pp. 457–468, 2010.

[16] A. Lasota and T. Szarek, "Dimension of measures invariant with respect to the ważewska partial differential equation," *Journal of Differential Equations*, vol. 196, no. 2, pp. 448–465, 2004.

[17] E. M. Mangino and A. Peris, "Frequently hypercyclic semigroups," *Studia Mathematica*, vol. 202, no. 3, pp. 227–242, 2011.

[18] M. Murillo-Arcila and A. Peris, "Strong mixing measures for linear operators and frequent hypercyclicity," *Journal of Mathematical Analysis and Applications*, vol. 398, no. 2, pp. 462–465, 2013.

[19] R. Rudnicki, "Chaos for some infinite-dimensional dynamical systems," *Mathematical Methods in the Applied Sciences*, vol. 27, no. 6, pp. 723–738, 2004.

[20] R. Rudnicki, "Chaoticity and invariant measures for a cell population model," *Journal of Mathematical Analysis and Applications*, vol. 393, no. 1, pp. 151–165, 2012.