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Research Article

Best Proximity Points for Some Classes of Proximal Contractions

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Given a self-mapping $g : A \to A$ and a non-self-mapping $T : A \to B$, the aim of this work is to provide sufficient conditions for the existence of a unique point $x \in A$, called g-best proximity point, which satisfies $d(gx, Tx) = d(A, B)$. In so doing, we provide a useful answer for the resolution of the nonlinear programming problem of globally minimizing the real valued function $x \to d(gx, Tx)$, thereby getting an optimal approximate solution to the equation $Tx = gx$. An iterative algorithm is also presented to compute a solution of such problems. Our results generalize a result due to Rhoades (2001) and hence such results provide an extension of Banach's contraction principle to the case of non-self-mappings.

1. Introduction

A fundamental result in the fixed point theory is the Banach contraction principle, which has various nontrivial implications in many branches of pure and applied sciences.

Let A and B be nonempty subsets of a metric space (X, d). We say that a non-self-mapping $T : A \to B$ is a contraction if there exists $k \in [0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \le kd(x, y). \tag{1}$$

The Banach contraction principle asserts that if a self-mapping $T : X \to X$ is a contraction and (X, d) is complete, then T has a unique fixed point $x \in X$. This result was extended to other important classes of mappings and has numerous applications. For some important and interesting generalizations of Banach contraction principle, one can refer to [1, 2]. The following notion of weakly contractive self-mapping was introduced by Alber and Guerre-Delabriere in [3].

Definition 1 (see [3]). Let (X, d) be a metric space and let A be a nonempty subset of X. A self-mapping $T : A \to A$ is said to be weakly contractive if

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)), \tag{2}$$

for all $x, y \in A$, where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to+\infty}\psi(t) = +\infty$. If A is bounded, then the infinity condition can be omitted.

Since all contractions are weakly contractive with the function $\psi(t) = (1 - k)t$, the above theorem extends Banach contraction principle. In fact, the class of weakly contractive mappings lies between the classes of mappings called contraction ones and contractive ones ($d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ with $x \neq y$).

Generally, the solution of the equation $Tx = x$, where $T : A \to X$ is a non-self-mapping, is called a fixed point of T. Hence, the condition $T(A) \cap A \neq \emptyset$ is necessary for the existence of a fixed point of T. Clearly, when $T(A) \cap A = \emptyset$, we have $d(x, Tx) > 0$, for all $x \in A$. In such a situation it is natural to search for a point $x \in A$ such that x the is closest to Tx in some sense. The following well-known best approximation theorem, due to Fan [4], explores the existence of an approximate solution to the equation $Tx = x$.

Theorem 2 (see [4]). *Let A be a nonempty compact convex subset of a normed linear space X and let $T : A \to X$ be a continuous mapping. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A)$.*

The point $x \in A$ in Theorem 2 is called a best approximant of T in A. Again, let A, B be nonempty subsets of a metric space (X, d) and let $T : A \to B$ be a non-self-mapping. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0, Tx_0) = d(A, B)$. Some interesting results in approximation theory can be found in [4–23].

The aim of this paper is to prove some best proximity point theorems for proximal contractions which are extensions of Banach contraction principle to the case of non-self-mappings. Precisely, given a self-mapping $g : A \to A$ and a non-self-mapping $T : A \to B$, this work focuses on g-best proximity point theorems for some classes of proximal contractions and a new family of mappings known as g-weak contractions. In fact, we provide sufficient conditions for the existence of a unique point $x \in A$, called g-best proximity point, which satisfies the condition $d(gx, Tx) = d(A, B)$. Further, an iterative algorithm is furnished to determine an optimal approximate solution in the guise of a g-best proximity point. As a consequence, one can compute an optimal approximate solution to some coincidence point equations.

2. Preliminaries

Let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{N}$ denote the set of all positive integers. Let A, B be two nonempty subsets of a metric space (X, d). Let us fix the following notation which will be needed throughout this paper:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}, \tag{3}$$

where $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$. In [11], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Also, in [20], the authors proved that A_0 is contained in the boundary of A.

We denote by Ψ the set of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following condition:

(ψ1) $\lim_{n \to +\infty} \psi^n(t) = 0$, for all $t > 0$, where ψ^n is the nth iterate of ψ.

Note that if $\psi \in \Psi$, then the following conditions hold:

(ψ2) $\psi(t) < t$, for all $t > 0$; $\psi(0) = 0$; ψ is continuous at $t = 0$.

We denote by Φ the set of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) = 0$ if and only if $t = 0$ and with $\Phi_c = \{\psi \in \Phi : \psi \text{ is continuous at } t = 0\}$.

Definition 3 (see [21]). Let A and B be two nonempty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a proximal ψ-contraction of the first kind if

$$d(u, Tx) = d(A, B) = d(v, Ty) \Longrightarrow d(u, v) \leq \psi(d(x, y)), \tag{4}$$

for all $u, v, x, y \in A$, where $\psi \in \Psi$. If $\psi(t) = \alpha t$ for some $\alpha \in [0, 1)$, then T is said to be a proximal contraction of the first kind.

Definition 4 (see [21]). Let A and B be two nonempty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a proximal ψ-contraction of the second kind if

$$d(u, Tx) = d(A, B) = d(v, Ty) \Longrightarrow d(Tu, Tv) \leq \psi(d(Tx, Ty)), \tag{5}$$

for all $u, v, x, y \in A$, where $\psi \in \Psi$. If $\psi(t) = \alpha t$ for some $\alpha \in [0, 1)$, then T is said to be a proximal contraction of the second kind.

Definition 5 (see [14]). Let A and B be two nonempty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a weak proximal ψ-contraction of the first kind if

$$d(u, Tx) = d(A, B) = d(v, Ty) \Longrightarrow d(u, v) \leq d(x, y) - \psi(d(x, y)), \tag{6}$$

for all $u, v, x, y \in A$, where $\psi \in \Phi$.

Definition 6 (see [14]). Let A and B be two nonempty subsets of a metric space (X, d). A non-self-mapping $T : A \to B$ is said to be a weak proximal ψ-contraction of the second kind if

$$d(u, Tx) = d(A, B) = d(v, Ty) \Longrightarrow d(Tu, Tv) \leq d(Tx, Ty) - \psi(d(Tx, Ty)), \tag{7}$$

for all $u, v, x, y \in A$, where $\psi \in \Phi$.

An example of a non-self-mapping T that is weak proximal ψ-contraction of the first and second kinds can be found in [14].

The following result is a best proximity point theorem for weak proximal ψ-contraction of the first and second kinds.

Theorem 7 (see [14, Theorem 3.1]). *Let A and B be closed subsets of a complete metric space (X, d) such that A_0 and B_0 are nonvoid. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) *T is a weak proximal ψ-contraction of the first and second kinds;*

(b) *g is an isometry;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$;*

(e) *T preserves the isometric distance with respect to g.*

Then, there exists a unique element x^* in A such that $d(gx^*, Tx^*) = d(A, B)$. Further, for any fixed element x_0 in A_0, the iterative sequence $\{x_n\}$, defined by $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$, converges to the element x^*.

Note that in Theorem 7, Sadiq Basha assumes that the function $\psi \in \Phi$ is continuous such that $\lim_{t \to +\infty} \psi(t) = +\infty$.

Let us define the notion of non-self-g-weakly contractive mappings as follows.

Definition 8. Let (X, d) be a metric space, let A, B be two nonempty subsets of X, and let $g : A \to A$. A non-self-mapping $T : A \to B$ is said to be a g-weakly contractive mapping if there exists $\psi \in \Phi_c$ such that

$$d(Tx, Ty) \leq d(gx, gy) - \psi(d(gx, gy)), \qquad (8)$$

for all $x, y \in A$.

Note that

$$d(Tx, Ty) \leq d(gx, gy) - \psi(d(gx, gy)) < d(gx, gy) \quad (9)$$

if $x, y \in A$ with $gx \neq gy$; that is, T is a g-contractive mapping.

Sankar Raj, in [22], introduced the notion called P-property, which was used to prove an extended version of Banach contraction principle.

Definition 9. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$.

(i) The pair (A, B) is said to have the P-property if and only if $d(x_1, y_1) = d(A, B) = d(x_2, y_2)$ implies $d(x_1, x_2) = d(y_1, y_2)$, where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ (see [22]).

(ii) The pair (A, B) is said to have the weak P-property if and only if $d(x_1, y_1) = d(A, B) = d(x_2, y_2)$ implies $d(x_1, x_2) \leq d(y_1, y_2)$, where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ (see [24]).

It is easy to see that, for any nonempty subset A of X, the pair (A, A) has the P-property.

Definition 10. Let A and B be two nonempty subsets of a metric space (X, d). Let $g : A \to A$ be a self-mapping and $T : A \to B$ a non-self-mapping. Then

(i) $g \in \mathcal{G}_A$ if g is continuous and $d(x, y) \leq d(gx, gy)$, for all $x, y \in A$;

(ii) $T \in \mathcal{T}_g$ if $d(Tx, Ty) \leq d(Tgx, Tgy)$ for all $x, y \in A$;

(iii) T is said to preserve (isometric) distance with respect to g if $d(Tgx, Tgy) = d(Tx, Ty)$, for every $x, y \in A$ (see [9]).

3. Best Proximity Point Theorems for Proximal Contractions

In this section, we establish some results of best proximity point for proximal ψ-contractions and weak proximal ψ-contractions.

Theorem 11. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that A_0 is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a proximal ψ-contraction of the first kind;*

(b) *$g \in \mathcal{G}_{A_0}$;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A_0$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B). \qquad (10)$$

Again, for $x_1 \in A_0$, there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \qquad (11)$$

By repeating this process, for $x_n \in A_0$, we can find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N}. \qquad (12)$$

Since T is a proximal ψ-contraction of the first kind and $g \in \mathcal{G}_{A_0}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(gx_{n+1}, gx_n) \\ &\leq \psi(d(x_n, x_{n-1})) \end{aligned} \qquad (13)$$

for every $n \in \mathbb{N} \cup \{0\}$. Since ψ is nondecreasing, we get by induction that

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)). \qquad (14)$$

By the definition of ψ, letting $n \to +\infty$, we obtain that

$$\lim_{n \to +\infty} d(x_{n+1}, x_n) = 0. \qquad (15)$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Given that $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \varepsilon - \psi(\varepsilon), \quad \forall n \geq n(\varepsilon). \qquad (16)$$

Now, fix $m \geq n(\varepsilon)$ and we prove that

$$d(x_m, x_{n+1}) < \varepsilon, \quad \forall n \geq m. \qquad (17)$$

Note that (17) holds if $n = m$, by (16). Assume that (17) holds for some $n \geq m$. Since T is a proximal ψ-contraction of the first kind,

$$\begin{aligned} d(x_m, x_{n+2}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+2}) \\ &\leq d(x_m, x_{m+1}) + d(gx_{m+1}, gx_{n+2}) \\ &\leq d(x_m, x_{m+1}) + \psi(d(x_m, x_{n+1})) \\ &< \varepsilon - \psi(\varepsilon) + \psi(\varepsilon) = \varepsilon. \end{aligned} \tag{18}$$

This implies that (17) holds, for all $n \geq m$, and hence

$$\lim_{m \to +\infty} d(x_m, x_{n+1}) = 0. \tag{19}$$

That is, $\{x_n\}$ is a Cauchy sequence. By the completeness of X and since A_0 is closed, we have $x_n \to x \in A_0$. Moreover, by the continuity of g, we have $gx_n \to gx$ and thus $gx \in A_0$, since $gx_n \in A_0$, for all $n \in \mathbb{N}$. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A$ such that

$$d(z, Tx) = d(A, B). \tag{20}$$

Clearly $z \in A_0$. Again, since T is a proximal ψ-contraction of the first kind, we get

$$d(z, gx_{n+1}) \leq \psi(d(x, x_n)) \leq d(x, x_n), \tag{21}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$, we obtain that $d(z, gx_{n+1}) \to 0$ and then $z = gx$. This implies that

$$d(gx, Tx) = d(A, B). \tag{22}$$

To prove the uniqueness, let x^* be another point in A_0 such that

$$d(gx^*, Tx^*) = d(A, B). \tag{23}$$

If $x \neq x^*$, since $g \in \mathcal{G}_{A_0}$ and T is a proximal ψ-contraction of the first kind, we get

$$\begin{aligned} d(x, x^*) &\leq d(gx, gx^*) \leq \psi(d(x, x^*)) \\ &< d(x, x^*), \end{aligned} \tag{24}$$

which is a contradiction; thus we have $x = x^*$. □

Remark 12. If in Theorem 11 we assume $g \in \mathcal{G}_A$, then we get that there exists a unique $x \in A$ such that $d(gx, Tx) = d(A, B)$.

From Theorem 11 and the above remark, we obtain the following corollary.

Corollary 13 (see [9, Theorem 3.1]). *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that A_0 is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a proximal contraction of the first kind;*

(b) *g is an isometry;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.

If in Theorem 11 the mapping g is the identity on A, then we get the following corollary.

Corollary 14. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that A_0 is nonempty and closed. Let $T : A \to B$ satisfy the following conditions:*

(a) *T is a proximal ψ-contraction of the first kind;*

(b) *$T(A_0) \subseteq B_0$.*

Then there exists a unique point $x \in A$ such that $d(x, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.

The following theorem is our main result for proximal ψ-contractions of the second kind.

Theorem 15. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that $T(A_0)$ is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a proximal ψ-contraction of the second kind;*

(b) *$T \in \mathcal{T}_g$;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, if T is injective, then the point x such that $d(gx, Tx) = d(A, B)$ is unique.

Proof. Similar to the proof of Theorem 11, we can find a sequence $\{x_n\} \subseteq A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{25}$$

Since T is a proximal ψ-contraction of the second kind, we have

$$d(Tgx_{n+1}, Tgx_n) \leq \psi(d(Tx_n, Tx_{n-1})) \tag{26}$$

for every $n \in \mathbb{N}$. Since $T \in \mathcal{T}_g$, we get

$$d(Tx_{n+1}, Tx_n) \leq \psi(d(Tx_n, Tx_{n-1})) \tag{27}$$

for every $n \in \mathbb{N}$. Since ψ is nondecreasing, we get by induction that

$$d(Tx_{n+1}, Tx_n) \leq \psi^n(d(Tx_1, Tx_0)). \tag{28}$$

By definition of ψ, letting $n \to +\infty$, we obtain that

$$\lim_{n \to +\infty} d(Tx_{n+1}, Tx_n) = 0. \tag{29}$$

Similar to the proof of Theorem 11, we prove that $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X and since $T(A_0)$ is closed, we have $Tx_n \to Tu \in B_0$. Moreover, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A, B). \tag{30}$$

Since $A_0 \subseteq g(A_0)$, we obtain that $z = gx$ for some $x \in A_0$, and then

$$d(gx, Tu) = d(A, B). \tag{31}$$

Again, since T is a proximal ψ-contraction of the second kind, we get

$$\begin{aligned} d(Tx, Tx_{n+1}) &\leq d(Tgx, Tgx_{n+1}) \\ &\leq \psi(d(Tu, Tx_n)) \\ &\leq d(Tu, Tx_n). \end{aligned} \tag{32}$$

Letting $n \to +\infty$, we obtain that $d(Tx, Tx_{n+1}) \to 0$ and hence $Tx = Tu$. This implies that

$$d(gx, Tx) = d(A, B). \tag{33}$$

To prove the uniqueness, let x^* be another point in A such that

$$d(gx^*, Tx^*) = d(A, B). \tag{34}$$

If $x \neq x^*$, since $T \in \mathcal{T}_g$ is injective, we deduce

$$\begin{aligned} d(Tx, Tx^*) &\leq d(Tgx, Tgx^*) \\ &\leq \psi(d(Tx, Tx^*)) \\ &< d(Tx, Tx^*), \end{aligned} \tag{35}$$

which is a contradiction; thus we have $Tx = Tx^*$ and hence $x = x^*$. □

From Theorem 15, we deduce the following corollary.

Corollary 16 (see [15, Theorem 3.2]). *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that $T(A_0)$ is nonempty and closed. Assume also the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a proximal contraction of the second kind;*

(b) *g is an isometry;*

(c) *T preserves isometric distance with respect to g;*

(d) *$T(A_0) \subseteq B_0$;*

(e) *$A_0 \subseteq g(A_0)$.*

Then there exists a point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, if $z \in A$ is another point for which $d(gz, Tz) = d(A, B)$, then $Tx = Tz$.

If in Theorem 15 the mapping g is the identity on A, then we get the following corollary.

Corollary 17. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that $T(A_0)$ is nonempty and closed. Let $T : A \to B$ satisfy the following conditions:*

(a) *T is a proximal ψ-contraction of the second kind;*

(b) *$T(A_0) \subseteq B_0$.*

Then there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$. Moreover, if T is injective on A, then the point x such that $d(x, Tx) = d(A, B)$ is unique.

The following is a theorem for weak proximal ψ-contractions of the first kind.

Theorem 18. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that A_0 is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a weak proximal ψ-contraction of the first kind;*

(b) *$g \in \mathcal{G}_{A_0}$;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A_0$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B). \tag{36}$$

Again, for $x_1 \in A_0$, there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \tag{37}$$

By repeating this process, for $x_n \in A_0$, we can find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{38}$$

Since T is a weak proximal ψ-contraction of the first kind and $g \in \mathcal{G}_{A_0}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(gx_{n+1}, gx_n) \\ &\leq d(x_n, x_{n-1}) - \psi(d(x_n, x_{n-1})) \\ &\leq d(x_n, x_{n-1}), \end{aligned} \tag{39}$$

for every $n \in \mathbb{N}$. Let $t_n = d(x_n, x_{n+1})$; then $\{t_n\}$ is a bounded nonincreasing sequence of nonnegative real numbers. Therefore, $\{t_n\}$ converges to t, where $t \geq 0$. Now let us claim that $t = 0$. Suppose that $t > 0$. Since $\psi \in \Phi$, we get $0 < \psi(t) \leq \psi(t_n)$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} t_n = d(x_n, x_{n+1}) &\leq d(gx_n, gx_{n+1}) \\ &\leq d(x_{n-1}, x_n) - \psi(d(x_{n-1}, x_n)) \\ &= t_{n-1} - \psi(t_{n-1}) \\ &\leq t_{n-1} - \psi(t). \end{aligned} \tag{40}$$

Inductively we obtain $t_{n+p} \leq t_n - p\psi(t)$, which is a contradiction for p large enough. Therefore $t = 0$ and hence $d(x_n, x_{n+1}) \to 0$.

Now let us claim that $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$r_k = d\left(x_{m_k}, x_{n_k}\right) \geq \varepsilon, \qquad d\left(x_{m_k}, x_{n_k-1}\right) < \varepsilon, \tag{41}$$

and $n_k > m_k \geq k$, for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} \varepsilon \leq r_k &\leq d\left(x_{m_k}, x_{n_k-1}\right) + d\left(x_{n_k-1}, x_{n_k}\right) \\ &< \varepsilon + t_{n_k-1}. \end{aligned} \tag{42}$$

By letting $k \to +\infty$, we have

$$\lim_{k \to +\infty} r_k = \varepsilon. \tag{43}$$

Since

$$\begin{aligned} d\left(gx_{m_k+1}, Tx_{m_k}\right) &= d(A, B), \\ d\left(gx_{n_k+1}, Tx_{n_k}\right) &= d(A, B), \end{aligned} \tag{44}$$

and T is a weak proximal ψ-contraction of the first kind, we obtain that

$$\begin{aligned} d\left(x_{m_k+1}, x_{n_k+1}\right) &\leq d\left(gx_{m_k+1}, gx_{n_k+1}\right) \\ &\leq d\left(x_{m_k}, x_{n_k}\right) - \psi\left(d\left(x_{m_k}, x_{n_k}\right)\right). \end{aligned} \tag{45}$$

Thus,

$$\begin{aligned} \varepsilon \leq r_k &\leq d\left(x_{m_k}, x_{m_k+1}\right) + d\left(x_{m_k+1}, x_{n_k+1}\right) + d\left(x_{n_k+1}, x_{n_k}\right) \\ &= t_{m_k} + t_{n_k} + d\left(x_{m_k+1}, x_{n_k+1}\right) \\ &\leq t_{m_k} + t_{n_k} + d\left(x_{m_k}, x_{n_k}\right) - \psi\left(d\left(x_{m_k}, x_{n_k}\right)\right) \\ &\leq t_{m_k} + t_{n_k} + d\left(x_{m_k}, x_{n_k}\right) - \psi(\varepsilon). \end{aligned} \tag{46}$$

Letting $k \to +\infty$, we have $\varepsilon \leq \varepsilon - \psi(\varepsilon)$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X and since A_0 is closed, we have $x_n \to x \in A_0$. Moreover, by the continuity of g, we have $gx_n \to gx$ and thus $gx \in A_0$, since $gx_n \in A_0$, for all $n \in \mathbb{N}$.

On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A_0$ such that

$$d(z, Tx) = d(A, B). \tag{47}$$

Again, since T is a weak proximal ψ-contraction of the first kind, we get

$$d\left(z, gx_{n+1}\right) \leq d\left(x, x_n\right) - \psi\left(d\left(x, x_n\right)\right) \leq d\left(x, x_n\right). \tag{48}$$

Letting $n \to +\infty$, we obtain that $d(z, gx_{n+1}) \to 0$ and then $z = gx$. This implies that

$$d(gx, Tx) = d(A, B). \tag{49}$$

To prove the uniqueness, let x^* be another point in A_0 such that

$$d\left(gx^*, Tx^*\right) = d(A, B). \tag{50}$$

If $x \neq x^*$, since $g \in \mathscr{G}_{A_0}$ and T is a weak proximal ψ-contraction of the first kind, we get

$$\begin{aligned} d\left(x, x^*\right) &\leq d\left(gx, gx^*\right) \\ &\leq d\left(x, x^*\right) - \psi\left(d\left(x, x^*\right)\right) \\ &< d\left(x, x^*\right), \end{aligned} \tag{51}$$

which is a contradiction; thus we have $x = x^*$. □

Remark 19. If in Theorem 18 we assume $g \in \mathscr{G}_A$, then we get that there exists a unique $x \in A$ such that $d(gx, Tx) = d(A, B)$.

If we take g as the identity mapping on A in Theorem 18, then we get the following corollary, which extends a result of Rhoades [25] to non-self-mappings.

Corollary 20. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that A_0 is nonempty and closed. Let $T : A \to B$ satisfy the following conditions:*

(a) *T is a weak proximal ψ-contraction of the first kind;*

(b) *$T(A_0) \subseteq B_0$.*

Then there exists a unique point $x \in A_0$ such that $d(x, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.

The following theorem is our main result for weak proximal ψ-contractions of the second kind.

Theorem 21. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that $T(A_0)$ is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) *T is a weak proximal ψ-contraction of the second kind;*

(b) *$T \in \mathscr{T}_g$;*

(c) *$T(A_0) \subseteq B_0$;*

(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, if T is injective on A, then the point x such that $d(gx, Tx) = d(A, B)$ is unique.

Proof. Similar to the proof of Theorem 18, we can find a sequence $\{x_n\} \subseteq A_0$ such that

$$d\left(gx_{n+1}, Tx_n\right) = d(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{52}$$

Since T is a weak proximal ψ-contraction of the second kind, we have

$$\begin{aligned} d\left(Tgx_{n+1}, Tgx_n\right) &\leq d\left(Tx_n, Tx_{n-1}\right) - \psi\left(d\left(Tx_n, Tx_{n-1}\right)\right) \\ &\leq d\left(Tx_n, Tx_{n-1}\right) \end{aligned} \tag{53}$$

for every $n \in \mathbb{N}$. Since $T \in \mathcal{T}_g$, we get

$$d(Tx_{n+1}, Tx_n) \leq d(Tgx_{n+1}, Tgx_n) \leq d(Tx_n, Tx_{n-1}) \quad (54)$$

for every $n \in \mathbb{N}$. Let $t_n = d(Tx_n, Tx_{n+1})$; then $\{t_n\}$ is a bounded nonincreasing sequence of nonnegative real numbers. Therefore, $\{t_n\}$ converges to t, where $t \geq 0$. Now let us claim that $t = 0$. Suppose that $t > 0$. Since $\psi \in \Phi$, we get $0 < \psi(t) \leq \psi(t_n)$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} t_n &= d(Tx_n, Tx_{n+1}) \leq d(Tgx_n, Tgx_{n+1}) \\ &\leq d(Tx_{n-1}, Tx_n) - \psi(d(Tx_{n-1}, Tx_n)) \\ &= t_{n-1} - \psi(t_{n-1}) \\ &\leq t_{n-1} - \psi(t). \end{aligned} \quad (55)$$

Inductively we obtain $t_{n+p} \leq t_n - p\,\psi(t)$, which is a contradiction for p large enough. Therefore $t = 0$ and hence $d(Tx_n, Tx_{n+1}) \to 0$.

Now let us claim that $\{Tx_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist $\varepsilon > 0$ and subsequences $\{Tx_{m_k}\}, \{Tx_{n_k}\}$ of $\{Tx_n\}$ such that

$$r_k = d(Tx_{m_k}, Tx_{n_k}) \geq \varepsilon, \qquad d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon, \quad (56)$$

and $n_k > m_k \geq k$, for all $k \in \mathbb{N}$. Therefore, we get

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}) \\ &< \varepsilon + t_{n_k-1}. \end{aligned} \quad (57)$$

By letting $k \to +\infty$, we have

$$\lim_{k\to+\infty} r_k = \varepsilon. \quad (58)$$

Since

$$\begin{aligned} d(gx_{m_k+1}, Tx_{m_k}) &= d(A, B), \\ d(gx_{n_k+1}, Tx_{n_k}) &= d(A, B), \end{aligned} \quad (59)$$

and T is a weak proximal ψ-contraction of the second kind, we obtain that

$$\begin{aligned} d(Tx_{m_k+1}, Tx_{n_k+1}) &\leq d(Tgx_{m_k+1}, Tgx_{n_k+1}) \\ &\leq d(Tx_{m_k}, Tx_{n_k}) - \psi(d(Tx_{m_k}, Tx_{n_k})). \end{aligned} \quad (60)$$

Thus,

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Tx_{m_k}, Tx_{m_k+1}) + d(Tx_{m_k+1}, Tx_{n_k+1}) \\ &\quad + d(Tx_{n_k+1}, Tx_{n_k}) \\ &= t_{m_k} + t_{n_k} + d(Tx_{m_k+1}, Tx_{n_k+1}) \\ &\leq t_{m_k} + t_{n_k} + d(Tx_{m_k}, Tx_{n_k}) - \psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq t_{m_k} + t_{n_k} + d(Tx_{m_k}, Tx_{n_k}) - \psi(\varepsilon). \end{aligned} \quad (61)$$

Letting $k \to +\infty$, we have $\varepsilon \leq \varepsilon - \psi(\varepsilon)$, which is a contradiction. Therefore, $\{Tx_n\}$ is a Cauchy sequence. By the completeness of X and since $T(A_0)$ is closed, we have $Tx_n \to Tu \in B_0$. Moreover, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A, B). \quad (62)$$

Since $A_0 \subseteq g(A_0)$, we obtain that $z = gx$ for some $x \in A_0$, and then

$$d(gx, Tu) = d(A, B). \quad (63)$$

Again, since T is a weak proximal ψ-contraction of the second kind, we get

$$\begin{aligned} d(Tx, Tx_{n+1}) &\leq d(Tgx, Tgx_{n+1}) \\ &\leq d(Tu, Tx_n) - \psi(d(Tu, Tx_n)) \\ &\leq d(Tu, Tx_n). \end{aligned} \quad (64)$$

Letting $n \to +\infty$, we obtain that $d(Tx, Tx_{n+1}) \to 0$ and hence $Tx = Tu$. This implies that

$$d(gx, Tx) = d(A, B). \quad (65)$$

To prove the uniqueness, let x^* be another point in A such that

$$d(gx^*, Tx^*) = d(A, B). \quad (66)$$

If $x \neq x^*$, since $T \in \mathcal{T}_g$ is injective on A, we have

$$\begin{aligned} d(Tx, Tx^*) &\leq d(Tgx, Tgx^*) \\ &\leq d(Tx, Tx^*) - \psi(d(Tx, Tx^*)) \\ &< d(Tx, Tx^*) \end{aligned} \quad (67)$$

which is a contradiction; thus we have $Tx = Tx^*$ and hence $x = x^*$. □

If in Theorem 21 the mapping g is the identity on A, we get the following corollary.

Corollary 22. *Let A and B be two nonempty subsets of a complete metric space (X, d). Suppose that $T(A_0)$ is nonempty and closed. Let $T : A \to B$ satisfy the following conditions:*

(a) *T is a weak proximal ψ-contraction of the second kind;*

(b) *$T(A_0) \subseteq B_0$.*

Then there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$. Moreover, if T is injective on A, then the point x such that $d(x, Tx) = d(A, B)$ is unique.

4. Best Proximity Point Theorem for g-Weak Contractions

The following result is a best proximity point theorem for g-weak contractions. Recall that a non-self-mapping $T : A \to B$ is g-weakly contractive if there exists $\psi \in \Phi_c$ such that $d(Tx, Ty) \leq d(gx, gy) - \psi(d(gx, gy))$, for all $x, y \in A$, where $g : A \to A$.

Theorem 23. *Let A and B be closed subsets of a complete metric space (X,d) such that $A_0, B_0 \neq \emptyset$ and the pair (A,B) has the weak P-property. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) *T is a g-weak contraction;*

(b) *$T(A_0) \subset B_0$;*

(c) *$A_0 \subset g(A_0)$.*

Then, there exists an element $x^* \in A_0$ such that $d(gx^*, Tx^*) = d(A,B)$. Further, if g is one to one then we have a unique element $x^* \in A$ such that $d(gx^*, Tx^*) = d(A,B)$.

Proof. Let x_0 be an element of A_0. In light of the fact that $T(A_0) \subset B_0$ and $A_0 \subset g(A_0)$, it is ensured that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A,B). \tag{68}$$

Again, in view of the fact that $T(A_0) \subset B_0$ and $A_0 \subset g(A_0)$, it is guaranteed that there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A,B). \tag{69}$$

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_n, Tx_{n-1}) = d(A,B), \quad \forall n \in \mathbb{N}. \tag{70}$$

Since (A,B) has the weak P-property, we conclude that

$$d(gx_n, gx_{n+1}) \leq d(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \tag{71}$$

Now, as T is a g-weak contraction, we get

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq d(gx_{n-1}, gx_n) - \psi(d(gx_{n-1}, gx_n)), \end{aligned} \tag{72}$$

where $\psi \in \Phi_c$ (see Definition 8). If we set $t_n = d(gx_n, gx_{n+1})$, then $\{t_n\}$ is a nonincreasing sequence of nonnegative real numbers and hence converges. Let $t \geq 0$ be the limit of the sequence $\{t_n\}$. Now let us claim that $t = 0$. Suppose that $t > 0$. Since ψ is a nondecreasing function, we deduce that $\psi(t_n) \geq \psi(t) > 0$, for all $n \in \mathbb{N}$. Then for any positive integer n, by (72), we get that

$$t_{n+1} \leq t_n - \psi(t). \tag{73}$$

Now, for all $n > t_1/\psi(t)$, by (73), we obtain that

$$t_{n+1} \leq t_1 - n\psi(t) < 0, \tag{74}$$

a contradiction. Therefore $t = 0$ and hence the sequence $\{d(gx_n, gx_{n+1})\}$ converges to 0. As

$$d(gx_n, gx_{n+1}) \leq d(Tx_{n-1}, Tx_n) \leq d(gx_{n-1}, gx_n), \tag{75}$$

we deduce that the sequence $\{d(Tx_{n-1}, Tx_n)\}$ converges to 0. Now, let us prove that $\{Tx_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and we choose a positive integer $n(\varepsilon)$ such that

$$d(Tx_n, Tx_{n+1}) \leq \min\left\{\frac{\varepsilon}{2}, \psi\left(\frac{\varepsilon}{2}\right)\right\}, \tag{76}$$

for all $n \geq n(\varepsilon)$. Fix $n \geq n(\varepsilon)$ and let

$$A(n,\varepsilon) := \{x \in A : d(Tx_n, Tx) \leq \varepsilon\}. \tag{77}$$

Now, it is asserted that if $x \in A(n,\varepsilon)$ and $u \in A$ is such that $d(gu, Tx) = d(A,B)$, then $u \in A(n,\varepsilon)$. First, we note that as $d(gx_{n+1}, Tx_n) = d(A,B)$, then by the weak P-property $d(gx_{n+1}, gu) \leq d(Tx_n, Tx)$. Two cases will be considered to establish this fact. Precisely, if $d(gx_{n+1}, gu) \leq \varepsilon/2$, then it follows that

$$\begin{aligned} d(Tx_n, Tu) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tu) \\ &\leq \frac{\varepsilon}{2} + d(gx_{n+1}, gu) - \psi(d(gx_{n+1}, gu)) \\ &\leq \frac{\varepsilon}{2} + d(gx_{n+1}, gu) \leq \varepsilon. \end{aligned} \tag{78}$$

On the other hand if $\varepsilon/2 < d(gx_{n+1}, gu) \leq \varepsilon$, then it follows that

$$\begin{aligned} d(Tx_n, Tu) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tu) \\ &\leq \psi\left(\frac{\varepsilon}{2}\right) + d(gx_{n+1}, gu) - \psi(d(gx_{n+1}, gu)) \\ &\leq \psi\left(\frac{\varepsilon}{2}\right) + d(gx_{n+1}, gu) - \psi\left(\frac{\varepsilon}{2}\right) \\ &= d(gx_{n+1}, gu) \leq \varepsilon. \end{aligned} \tag{79}$$

So, $u \in A(n,\varepsilon)$. Now, we prove that

$$x_{n+m} \in A(n,\varepsilon), \tag{80}$$

for all $m \geq 1$. From $x_n \in A(n,\varepsilon)$ and $d(gx_{n+1}, Tx_n) = d(A,B)$, we deduce that $x_{n+1} \in A(n,\varepsilon)$; that is (80) holds for $m = 1$. Now, we assume that (80) holds for some $m \geq 1$. From, $x_{n+m} \in A(n,\varepsilon)$ and $d(gx_{n+m+1}, Tx_{n+m}) = d(A,B)$, we deduce that $x_{n+m+1} \in A(n,\varepsilon)$; that is (80) holds for $m+1$ and hence for all $m \geq 1$. Thus, it follows that $\{Tx_n\}$ is a Cauchy sequence. From the completeness of the space X, the sequence $\{Tx_n\}$ converges to some element $y^* \in B$. From $d(gx_{n+1}, gx_{m+1}) \leq d(Tx_n, Tx_m)$, we deduce that $\{gx_n\}$ is also a Cauchy sequence. As A is a complete subspace of X, then there exists $z \in A$ such that $gx_n \to z$. Therefore, we have

$$d(z, y^*) = \lim_{n \to +\infty} d(gx_{n+1}, Tx_n) = d(A,B), \tag{81}$$

and so $z \in A_0$. In light of the fact that A_0 is contained in $g(A_0)$, there is $x^* \in A_0$ such that $z = gx^*$. Since $T(A_0) \subset B_0$, there exists an element $\overline{x} \in A_0$ such that

$$d(g\overline{x}, Tx^*) = d(A,B). \tag{82}$$

In view of the fact that T is a g-weak contraction and (A,B) has the weak P-property and the continuity of ψ at $t = 0$, we get

$$\begin{aligned} d(gx_{n+1}, g\overline{x}) &\leq d(Tx_n, Tx^*) \\ &\leq d(gx_{n+1}, gx^*) - \psi(d(gx_{n+1}, gx^*)). \end{aligned} \tag{83}$$

Letting $n \to +\infty$, it follows that $g\bar{x} = gx^*$. Thus, we conclude that $d(gx^*, Tx^*) = d(A, B)$.

To assert the uniqueness, let us assume that $z^* \in A$ is another element such that $d(gz^*, Tz^*) = d(A, B)$. Then

$$\begin{aligned} d\left(gx^*, gz^*\right) &\leq d\left(Tx^*, Tz^*\right) \\ &\leq d\left(gx^*, gz^*\right) - \psi\left(d\left(gx^*, gz^*\right)\right), \end{aligned} \tag{84}$$

from which it follows that $gx^* = gz^*$ and hence $z^* \in g^{-1}gx^*$. If g is one to one then we deduce the uniqueness. □

Remark 24. From the proof of Theorem 23, we obtain that the method for getting the sequence $\{gx_n\}$, that is the relation $d(gx_n, gx_{n+1}) = d(Tx_{n-1}, Tx_n)$, also gives an iterative algorithm for computing solutions of coincidence equations.

If in Theorem 23 the mapping g is the identity on A, then yields the following result which is a generalization of a result due to Rhoades [25] to non-self-mappings.

Corollary 25. *Let A and B be closed subsets of a complete metric space (X, d) such that $A_0, B_0 \neq \emptyset$ and the pair (A, B) has the weak P-property. Suppose that the mapping $T : A \to B$ satisfies the following conditions:*

(i) *T is a g-weak contraction;*

(ii) *$T(A_0) \subset B_0$.*

Then, there exists a unique element $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to the element x^*.

Example 26. Consider $X = \mathbb{R}^2$ with the usual metric. Let us define

$$\begin{aligned} A &:= \left\{(x, y) \in \mathbb{R}^2 : x = 0,\ y \geq 0\right\}, \\ B &:= \left\{(x, y) \in \mathbb{R}^2 : x = 1,\ y \geq 0\right\}. \end{aligned} \tag{85}$$

Then A and B are nonempty closed subsets of X and $A_0 = A$ and $B_0 = B$. Note that $d(A, B) = 1$. Let $g : A \to A$ and $T : A \to B$ be defined as $g(0, x) = (0, 2x)$ and $T(0, x) = (1, x/(1 + x))$. Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t^2/(1 + t)$, for all $t \geq 0$. Then, T is a g-weak contraction. As (A, B) has the weak P-property and g is one to one, we obtain that $(0, 0) \in A$ is the unique g-best proximity point of T; that is, $d(g(0, 0), T(0, 0)) = d(A, B)$.

The following example shows that the weak P-property in Theorem 23 cannot be relaxed; that is, a g-weakly contractive mapping $T : A \to B$ may not have a g-best proximity point in A if the pair (A, B) does not have the weak P-property, where A and B are nonempty closed subsets of a complete metric space X.

Example 27. Consider $X = \mathbb{R}$ with the usual metric, $A = \{-10, 10\}$ and $B = \{-2, 2\}$. Then A and B are nonempty closed subsets of X with $A_0 = A$ and $B_0 = B$. Note that $d(A, B) = 8$. Let $T : A \to B$ be a mapping given by $T(-10) = 2$ and $T(10) = -2$. It is easy to see that $T : A \to B$ is a contraction mapping with $T(A_0) \subset B_0$ and hence it is g-weakly contractive, where g is the identity mapping. Since $d(x, Tx) = 12 > 8 = d(A, B)$, for all $x \in A$, then T has no g-best proximity points. It is worth noting that the pair (A, B) does not have the weak P-property.

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