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Research Article

Approximate Solutions of Hybrid Stochastic Pantograph Equations with Levy Jumps

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We investigate a class of stochastic pantograph differential equations with Markovian switching and Levy jumps. We prove that the approximate solutions converge to the true solutions in L^2 sense as well as in probability under local Lipschitz condition and generalize the results obtained by Fan et al. (2007), Milošević and Jovanović (2011), and Marion et al. (2002) to cover a class of more general stochastic pantograph differential equations with jumps. Finally, an illustrative example is given to demonstrate our established theory.

1. Introduction

Stochastic delay differential equations (SDDEs) have come to play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. Similar to SDEs, an explicit solution can rarely be obtained for SDDEs. It is necessary to develop numerical methods and to study the properties of these methods. There are many results for the numerical solutions of SDDEs [1–12].

Recently, as a special case of SDDEs, a class of stochastic pantograph delay equations (SPEs) has been received a great deal of attention and various studies have been carried out on the convergence of SPEs [13–16]. However, all equations of the above-mentioned works are driven by white noise perturbations with continuous initial data, and white noise perturbations are not always appropriate to interpret real data in a reasonable way. In real phenomena, the state of stochastic pantograph delay system may be perturbed by abrupt pulses or extreme events. A more natural mathematical framework for these phenomena takes into account other than purely Brownian perturbations. In particular, we incorporate the Levy perturbations with jumps into stochastic pantograph delay system to model abrupt changes.

The study of the convergence of the numerical solutions to SDDEs with jumps is in its infancy [17–20], and there is no research on the numerical solutions to SPEs with Markovian switching and Levy jumps (SPEwMsLJs). In this paper, we study the strong convergence of the Euler method for a class of SPEs with Markovian switching and Levy jumps (SPEwMsLJs). SPEwMsLJs may be regarded as an extension of SPEs with Markovian switching and SPEs with Levy jumps. The main aim is to prove that the Euler approximate solutions converges to the true solutions for SPEwMsLJs in L^2 sense. On the other hand, we study the convergence in probability of the Euler approximate solutions to the true solutions under local Lipschitz condition and some additional conditions in term of Lyapunov-type functions. It should be pointed out that the proof for SPEwMsLJs is certainly not a straightforward generalization of that for SPEs and SPEwMs without Levy jumps. Although the way of analysis follows the ideas of [21], we need to develop several new techniques to deal with Levy jumps. Some known results in Fan et al. [14], Milošević and Jovanović [16], and Marion et al. [21] are generalized to cover a class of more general SPEwMsLJs.

The paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning (4), and the Euler methods is used to produce a numerical solutions. In Section 3, we establish some useful lemmas and prove that

the approximate solutions converge to the true solutions of SPEwMsLJs in L^2 sense. By applying Theorem 4, we study the convergence in probability of the approximate solutions to the true solutions in Section 4. Finally, we give an illustrative example in Section 5.

2. Preliminaries and the Approximate Solution

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual condition; that is, the filtration $(\mathscr{F}_t)$ is continuous on the right and $(\mathscr{F}_0)$ contains all P-null sets. Let $\{W(t), t \geq 0\}$ be a d-dimensional Wiener process defined on the probability space $(\Omega, \mathscr{F}, P)$ adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $D([0,T], R^n)$ denote the family of function f from $[0,T] \to R^n$ that are right continuous and have limits on the left. Also $D([0,T], R^n)$ is equipped with the norm $\|f\| = \sup_{0\leq t\leq T}|f(t)|$, where $|\cdot|$ is the Euclidean norm in R^n; that is, $|x| = \sqrt{x^\top x}$ $(x \in R^n)$. Let $T > 0$, $P \geq 2$, and $\mathscr{L}^P([0,T]; R^n)$ denote the family of all R^n-valued measurable $(\mathscr{F}_t)$-adapted processes $f = \{f(t)\}_{0\leq t\leq T}$ such that $E\sup_{0\leq t\leq T}|f(t)|^P < \infty$. Let $(R^n, \mathscr{B}(R^n))$ be a measurable space and $\pi(du)$ a σ-finite measure on it. Let $p = p(t)$, and let $t \in D_p$ be a stationary $\mathscr{F}_t$-Poisson point process on R^n with characteristic measure π. Denote by $N(dt, du)$ the Poisson counting measure associated with p; that is,

$$N(t, A) = \sum_{s\in D_p, s\leq t} I_A(p(s)). \tag{1}$$

We refer to Mao [3] for the properties of a Wiener process and SDDEs and to Ikeda and Watanabe [22] for the details on Poisson point process.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space $(\Omega, \mathscr{F}, P)$ taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P(r(t+\Delta) = j \mid r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & \text{if } i = j, \end{cases} \tag{2}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while

$$\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}. \tag{3}$$

We assume that Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$ and compensated Poisson random measure $\widetilde{N}(\cdot,\cdot)$. It is known that almost every sample path of $r(\cdot)$ is right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

In this paper, we study the following hybrid stochastic pantograph equations with Levy jump:

$$\begin{aligned} dX(t) = {} & f(X(t), X(qt), r(t))\,dt \\ & + g(X(t), X(qt), r(t))\,dW(t) \\ & + \int_{R^n} h(X(t), X(qt), r(t), u)\,\widetilde{N}(dt, du), \\ & X(0) = X_0, \end{aligned} \tag{4}$$

where $0 < q < 1$ and

$$\begin{aligned} f &: R^+ \times R^n \times S \longrightarrow R^n, \\ g &: R^+ \times R^n \times S \longrightarrow R^{n\times m}, \\ h &: R^+ \times R^n \times S \times R^n \longrightarrow R^n, \end{aligned} \tag{5}$$

$W(t)$ is a standard m-dimensional Brownian motion, and $\widetilde{N}(dt, du)$ is the compensated Poisson random measure given by

$$\widetilde{N}(dt, du) = N(dt, du) - \pi(du)\,dt. \tag{6}$$

Here $\pi(du)$ is the Levy measure associated to N.

Let $C^2(R^n \times S, R_+)$ denote the family of all nonnegative functions $V(x,i)$ on $R^n \times S$ which are continuously twice differentiable in x. For each $V \in C^2(R^n \times S, R_+)$, define an operator LV from $R^n \times R^n \times S$ to R by

$$\begin{aligned} & LV(x, y, i) \\ & \quad \equiv V_x(x,i) f(x, y, i) \\ & \qquad + \frac{1}{2}\operatorname{trace}\left[g^\top(x, y, i) V_{xx}(x,i) g(x, y, i)\right] \\ & \qquad + \int_{R^n} \left[V(x + h(x, y, u), i) - V(x,i) \right. \\ & \qquad\qquad \left. - V_x(x,i) h(x, y, u)\right] \pi(du) \\ & \qquad + \sum_{j=1}^{N} \gamma_{ij} V(x, j), \end{aligned} \tag{7}$$

where

$$\begin{aligned} V_x(x,i) &= \left(\frac{\partial V(x,i)}{\partial x_1}, \ldots, \frac{\partial V(x,i)}{\partial x_n}\right), \\ V_{xx}(x,i) &= \left(\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j}\right)_{n\times n}. \end{aligned} \tag{8}$$

In order to define the Euler approximate solution of (4), we need the property of embedded discrete Markov chain. The following lemma [23] describes this property.

Lemma 1. *For $h > 0$ and $n \geq 0$, then $\{r_n^h,\ n = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(h) = \left(P_{ij}(h)\right)_{N\times N} = e^{h\Gamma}. \tag{9}$$

Given a step size $h > 0$, the discrete Markov chain $\{r_n^h,\ n = 0, 1, 2, \ldots\}$ can be simulated as follows (see Mao and Yuan [24]). Let $r(0) = i_0$ and generate a random number ζ_1 which is uniformly distributed in $[0, 1]$. If $\zeta_1 = 1$, then let $r_1^h = i_1 = N$ or otherwise find the unique integer $i_1 \in S$ for

$$\sum_{j=1}^{i_1-1} P_{i_0,j}(h) \le \zeta_1 < \sum_{j=1}^{i_1} P_{i_0,j}(h) \tag{10}$$

and let $r_1^h = i_1$, where we set $\sum_{i=1}^{0} P_{i_0,j}(h) = 0$ as usual. Generate independently a new random number ζ_2 which is again uniformly distributed in $[0, 1]$. If $\zeta_2 = 1$ then let $r_2^h = i_2 = N$ or otherwise find the unique integer $i_2 \in S$ for

$$\sum_{j=1}^{i_2-1} P_{i_1,j}(h) \le \zeta_2 < \sum_{j=1}^{i_2} P_{i_1,j}(h) \tag{11}$$

and let $r_2^h = i_2$. Repeating this procedure, a trajectory of $\{r_n^h,\ n = 0, 1, 2, \ldots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories.

Now we define the Euler approximate solution to (4) with discrete Markov chain $\{r_n^h = r(nh),\ n = 0, 1, 2, \ldots\}$. For system (4), the discrete approximation is given by the iterative scheme

$$\begin{aligned} Y_{n+1} = {} & Y_n + f\left(Y_n, Y_{[qn]}, r_n^h\right) h \\ & + g\left(Y_n, Y_{[qn]}, r_n^h\right) \Delta W_n \\ & + \int_{R^n} h\left(Y_n, Y_{[qn]}, r_n^h, u\right) \widetilde{N}(h, du), \end{aligned} \tag{12}$$

with initial value $Y_0 = X(0)$, and $[u]$ represents the integer part of u. Here $t_n = nh$ for $n \ge 0$. We have $Y_n \approx X(t_n)$, $Y_{[qn]} \approx X(qt_n)$, $\Delta W_n = W(t_{n+1}) - W(t_n)$, and $\widetilde{N}(h, du) = \widetilde{N}(t_{n+1}, du) - \widetilde{N}(t_n, du)$.

Let us introduce the following notations:

$$Z_1(t) = Y_n, \qquad Z_2(t) = Y_{[qn]}, \qquad \bar{r}(t) = r_n^h \tag{13}$$

for $t \in [t_n, t_{n+1})$. Then we define the continuous Euler approximate solution as follows:

$$\begin{aligned} Y(t) = {} & Y(0) + \int_0^t f\left(Z_1(s), Z_2(s), \bar{r}(s)\right) ds \\ & + \int_0^t g\left(Z_1(s), Z_2(s), \bar{r}(s)\right) dW(s) \\ & + \int_0^t \int_{R^n} h\left(Z_1(s), Z_2(s), \bar{r}(s), u\right) \widetilde{N}(ds, du), \\ & \qquad 0 \le t \le T, \end{aligned} \tag{14}$$

which interpolates the discrete approximation (7).

In order to establish the strong convergence theorem, we suppose the following assumptions are satisfied.

Assumption 2. For each $i \in S$ and $u \in R^n$,

$$\left|f(0,0,i)\right|^2 = \left|g(0,0,i)\right|^2 = \int_{R^n} \left|h(0,0,i,u)\right|^2 \pi(du) = 0. \tag{15}$$

Assumption 3. For every $d \ge 1$, there exists a positive constant K_d such that for all $x_1, y_1, x_2, y_2, u \in R^n$ and $i \in S$, $|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \le d$,

$$\begin{aligned} & \left|f(x_1, y_1, i) - f(x_2, y_2, i)\right|^2 \\ & \quad \vee \left|g(x_1, y_1, i) - g(x_2, y_2, i)\right|^2 \\ & \quad \vee \int_{R^n} \left|h(x_1, y_1, i, u) - h(x_2, y_2, i, u)\right|^2 \pi(du) \\ & \quad \le K_d\left(\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2\right). \end{aligned} \tag{16}$$

3. Strong Convergence of Numerical Solutions

In this section, we will prove that the Euler approximate solutions converge to the true solutions in L^2 sense under the local Lipschitz condition.

Theorem 4. *If Assumptions 2 and 3 hold, then the Euler approximate solutions converge to the true solutions of* (4) *in L^2 sense with order* 1/2*; that is, there exists a positive constant C_d such that*

$$E\left[\sup_{0 \le t \le T \wedge \tau_d} \left|X(t) - Y(t)\right|^2\right] \le C_d\left[h + \circ(h)\right], \tag{17}$$

where $\theta_d = \inf\{t \in [0, T] : |X(t)| \ge d\}$ and $\rho_d = \inf\{t \in [0, T] : |Y(t)| \ge d\}$, and let $\tau_d = \theta_d \wedge \rho_d$.

The proof of Theorem 4 is very technical, so we present some useful lemmas.

Lemma 5. *Under Assumptions 2 and 3, for any $t \in [0, T]$ and $P \ge 2$, there exists a positive constant $C_1(d)$ such that*

$$E\int_0^t \left|Y(s \wedge \tau_d) - Z_1(s \wedge \tau_d)\right|^P ds \le C_1(d)\, h^{P/2} T, \tag{18}$$

where $C_1(d)$ is a positive constant independent of the step size h.

Proof. For any $t \in [0, T \wedge \tau_d]$, there exists an integer n such that $t \in [nh, (n+1)h)$. Then

$$\begin{aligned} & Y(t) - Z_1(t) \\ & \quad = Y(t) - Y_n \\ & \quad = \int_{nh}^t f\left(Z_1(s), Z_2(s), \bar{r}(s)\right) ds \\ & \qquad + \int_{nh}^t g\left(Z_1(s), Z_2(s), \bar{r}(s)\right) dW(s) \\ & \qquad + \int_{nh}^t \int_{R^n} h\left(Z_1(s), Z_2(s), \bar{r}(s), u\right) \widetilde{N}(ds, du). \end{aligned} \tag{19}$$

Using the inequality $|a+b+c|^P \le 3^{P-1}[|a|^P + 3|b|^P + 3|c|^P]$, we get

$$\begin{aligned}
&E\left[\sup_{nh\le t\le(n+1)h}\left|Y(t)-Z_1(t)\right|^P\right] \\
&\le 3^{P-1}\left[E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t} f\left(Z_1(s),\right.\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.Z_2(s),\bar{r}(s)\right)ds\right|^P\right) \\
&\quad + 3^{P-1}E \\
&\quad \times\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t} g\left(Z_1(s),\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.Z_2(s),\bar{r}(s)\right)dW(s)\right|^P\right) \\
&\quad + 3^{P-1}E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t}\int_{R^n} h\left(Z_1(s),\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.Z_2(s),\bar{r}(s),u\right)\right.\right. \\
&\qquad\qquad \left.\left.\left.\times\widetilde{N}(ds,du)\right|^P\right)\right].
\end{aligned} \tag{20}$$

By the *Hölder* inequality and Assumptions 2 and 3, we have

$$\begin{aligned}
&E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t} f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)ds\right|^P\right) \\
&\le E\left(\left|\int_{nh}^{(n+1)h} f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)ds\right|^P\right) \\
&\le h^{P-1}E\int_{nh}^{(n+1)h}\left|f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right|^P ds \\
&\le h^{P-1}E\int_{nh}^{(n+1)h}\left[\left|f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right|^2\right]^{P/2} ds \\
&\le h^{P-1}E\int_{nh}^{(n+1)h}\left[2\left|f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right.\right. \\
&\qquad\qquad \left.-f(0,0),\bar{r}(s)\right|^2 \\
&\qquad\qquad \left.+2\left|f(0,0),\bar{r}(s)\right|^2\right]^{P/2} ds \\
&\le h^{P-1}E\int_{nh}^{(n+1)h}\left[2K_d\left(\left|Z_1(s)\right|^2+\left|Z_2(s)\right|^2\right)\right]^{P/2} ds \\
&\le h^{P-1}2^{P-1}K_d^{P/2}E\int_{nh}^{(n+1)h}\left(\left|Z_1(s)\right|^P+\left|Z_2(s)\right|^P\right)ds.
\end{aligned} \tag{21}$$

By the definition of τ_d, we have $|Y(t)| < d$, $t \in [0,\tau_d \wedge T]$. so we get that $|Z_1(t)|^P < d^P$, $|Z_2(t)|^P < d^P$, and

$$\begin{aligned}
&E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t} f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)ds\right|^P\right) \\
&\qquad \le 2^P d^P K_d^{P/2} h^P.
\end{aligned} \tag{22}$$

By using the Burkholder-Davis-Gundy inequality and Assumptions 2 and 3, we have

$$\begin{aligned}
&E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t} g\left(Z_1(s),Z_2(s),\bar{r}(s)\right)dW(s)\right|^P\right) \\
&\le C_P E\left[\int_{nh}^{(n+1)h}\left|g\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right|^2 ds\right]^{P/2} \\
&\le C_P h^{(P/2)-1}E\left[\int_{nh}^{(n+1)h}\left|g\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right|^P ds\right] \\
&\le C_p 2^P d^P K_d^{P/2} h^{P/2},
\end{aligned} \tag{23}$$

$$\begin{aligned}
&E\left(\sup_{nh\le t\le(n+1)h}\left|\int_{nh}^{t}\int_{R^n} h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right)\widetilde{N}(ds,du)\right|^P\right) \\
&\le C_P E\left[\int_{nh}^{(n+1)h}\int_{R^n}\left|h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right)\right|^2\pi(du)\,ds\right]^{P/2} \\
&\le C_P h^{(P/2)-1}E \\
&\quad\times\int_{nh}^{(n+1)h}\left[\int_{R^n}\left|h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right)\right|^2\pi(du)\right]^{P/2} ds \\
&\le C_P h^{(P/2)-1}E \\
&\quad\times\int_{nh}^{(n+1)h}\left\{\int_{R^n}\left[2\left|h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right)\right.\right.\right. \\
&\qquad\qquad \left.-h(0,0,\bar{r}(s),u)\right|^2 \\
&\qquad\qquad \left.\left.+\left|h(0,0,\bar{r}(s),u)\right|^2\right]\pi(du)\right\}^{P/2} ds \\
&\le C_P h^{(P/2)-1}E\int_{nh}^{(n+1)h}\left[2K_d\left(\left|Z_1(s)\right|^2+\left|Z_2(s)\right|^2\right)\right]^{P/2} ds \\
&\le C_p 2^P d^P K_d^{P/2} h^{P/2}.
\end{aligned} \tag{24}$$

Combing (22)–(24) together, we have

$$\begin{aligned}
&E\left|Y(t)-Z_1(t)\right|^P \\
&\qquad \le 3^{P-1}\left[2^P d^P K_d^{P/2} h^P + 2C_p 2^P d^P K_d^{P/2} h^{P/2}\right], \\
&E\int_0^{t\wedge\tau_d}\left|Y(s)-Z_1(s)\right|^P ds \le C_1(d)\,h^{P/2}T,
\end{aligned} \tag{25}$$

where $C_1(d) = 3^{P-1}[2^P d^P K_d^{P/2} + 2C_p 2^P d^P K_d^{P/2}]$. The proof is complete. □

Lemma 6. *Under Assumptions 2 and 3, for any $t \in [0,T]$ and $P \geq 2$, then*

$$E\int_0^t \left|Y\left(q\left(s\wedge\tau_d\right)\right)-Z_2\left(s\wedge\tau_d\right)\right|^P ds \leq C_2\left(d\right)h^{P/2}T, \quad (26)$$

where $C_2(d)$ is a positive constant independent of the stepsize h.

The proof of this lemma is similar to that of Lemma 5.

Lemma 7. *Under Assumptions 2 and 3, for any $t \in [0,T]$ and $P \geq 2$, then*

$$\begin{aligned}
&E\int_0^t \left|f\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),r\left(s\wedge\tau_d\right)\right)\right.\\
&\qquad\left.-f\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),\bar{r}\left(s\wedge\tau_d\right)\right)\right|^P ds\\
&\vee E\int_0^t \left|g\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),r\left(s\wedge\tau_d\right)\right)\right.\\
&\qquad\left.-g\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),\bar{r}\left(s\wedge\tau_d\right)\right)\right|^P ds\\
&\vee E\int_0^t\int_{R^n} \left|h\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),r\left(s\wedge\tau_d\right),u\right)\right.\\
&\qquad\left.-h\left(Z_1\left(s\wedge\tau_d\right),Z_2\left(s\wedge\tau_d\right),\bar{r}\left(s\wedge\tau_d\right),u\right)\right|^P\\
&\qquad\times\pi\left(du\right)ds \leq C_3\left(d\right)\left[h^{P/2}+\circ\left(h^{P/2}\right)\right],
\end{aligned} \quad (27)$$

where $C_3(d)$ is a positive constant independent of the stepsize h.

The proof of this lemma is similar to that of [16, 24].

Proof of Theorem 4. Combining (4) and (14), one has

$$\begin{aligned}
&X\left(t\right)-Y\left(t\right)\\
&\quad=\int_0^t\left[f\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\left.-f\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right]ds\\
&\quad+\int_0^t\left[g\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\left.-g\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right]dW\left(s\right)\\
&\quad+\int_0^t\int_{R^n}\left[h\left(X\left(s\right),X\left(qs\right),r\left(s\right),u\right)\right.\\
&\qquad\left.-h\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right),u\right)\right]\widetilde{N}\left(ds,du\right).
\end{aligned} \quad (28)$$

Then applying the generalized Itô's formula, we can show that

$$\begin{aligned}
&\left|X\left(t\right)-Y\left(t\right)\right|^2\\
&\quad=2\int_0^t\left|X\left(s\right)-Y\left(s\right)\right|\\
&\qquad\times\left|f\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\qquad\left.-f\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right|ds\\
&\quad+\int_0^t\left|g\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)-g\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right|^2ds\\
&\quad+\int_0^t\int_{R^n}\left|h\left(X\left(s\right),X\left(qs\right),r\left(s\right),u\right)\right.\\
&\qquad\qquad\left.-h\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right),u\right)\right|^2\pi\left(du\right)ds\\
&\quad+2\int_0^t\left|X\left(s\right)-Y\left(s\right)\right|\\
&\qquad\times\left|g\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\qquad\left.-g\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right|dW\left(s\right)\\
&\quad+\int_0^t\int_{R^n}\left[2\left|X\left(s\right)-Y\left(s\right)\right|\right.\\
&\qquad\times\left|h\left(X\left(s\right),X\left(qs\right),r\left(s\right),u\right)\right.\\
&\qquad\qquad\left.\left.-h\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right),u\right)\right|\right]\widetilde{N}\left(ds,du\right)\\
&\quad+\int_0^t\int_{R^n}\left|h\left(X\left(s\right),X\left(qs\right),r\left(s\right),u\right)\right.\\
&\qquad\qquad\left.-h\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right),u\right)\right|^2\widetilde{N}\left(ds,du\right).
\end{aligned} \quad (29)$$

Hence, for any $t \in [0,T]$, we get

$$\begin{aligned}
&E\sup_{0\leq t\leq T\wedge\tau_d}\left|X\left(t\right)-Y\left(t\right)\right|^2\\
&\quad\leq E\int_0^{T\wedge\tau_d}\left|X\left(s\right)-Y\left(s\right)\right|^2ds\\
&\qquad+E\int_0^{\tau_d\wedge T}\left|f\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\qquad\left.-f\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right|^2ds\\
&\qquad+E\int_0^{\tau_d\wedge T}\left|g\left(X\left(s\right),X\left(qs\right),r\left(s\right)\right)\right.\\
&\qquad\qquad\left.-g\left(Z_1\left(s\right),Z_2\left(s\right),\bar{r}\left(s\right)\right)\right|^2ds\\
&\qquad+E\int_0^{\tau_d\wedge T}\int_{R^n}\left|h\left(X\left(s\right),X\left(qs\right),r\left(s\right),u\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad -h\left(Z_1(s), Z_2(s), \bar{r}(s), u\right)\Big|^2 \pi(du)\, ds \\
&+ 2E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \left|X(s\wedge\tau_d) - Y(s\wedge\tau_d)\right| \\
&\qquad \times \left|g\left(X(s\wedge\tau_d), X\left(q(s\wedge\tau_d)\right), r(s)\right)\right. \\
&\qquad\quad \left. - g\left(Z_1(s), Z_2(s), \bar{r}(s)\right)\right| dW(s) \\
&+ E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \int_{R^n} \left[2\left|X(s) - Y(s)\right|\right. \\
&\qquad \times \left|h\left(X(s), X(qs), r(s), u\right)\right. \\
&\qquad\quad \left. -h\left(Z_1(s), Z_2(s), \bar{r}(s), u\right)\right|\Big] \\
&\qquad \times \widetilde{N}(ds, du) \\
&+ E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \int_{R^n} \left|h\left(X(s), X(qs), r(s), u\right)\right. \\
&\qquad\quad \left. -h\left(Z_1(s), Z_2(s), \bar{r}(s), u\right)\right|^2 \\
&\qquad \times \widetilde{N}(ds, du).
\end{aligned} \tag{30}
$$

By Assumption 3 and Lemmas 5–7, we have

$$
\begin{aligned}
&E \int_0^{\tau_d\wedge T} \left|f\left(X(s), X(qs), r(s)\right)\right. \\
&\qquad \left. -f\left(Z_1(s), Z_2(s), \bar{r}(s)\right)\right|^2 ds \\
&\le 2E \int_0^{\tau_d\wedge T} \left|f\left(X(s), X(qs), r(s)\right)\right. \\
&\qquad \left. -f\left(Z_1(s), Z_2(s), r(s)\right)\right|^2 ds \\
&\quad + 2E \int_0^{\tau_d\wedge T} \left|f\left(Z_1(s), Z_2(s), r(s)\right)\right. \\
&\qquad \left. -f\left(Z_1(s), Z_2(s), \bar{r}(s)\right)\right|^2 ds \\
&\le 2K_d E \int_0^{\tau_d\wedge T} \left(\left|X(s) - Z_1(s)\right|^2\right. \\
&\qquad \left. + \left|X(qs) - Z_2(s)\right|^2\right) ds + 2C_3(d)\, h \\
&\le 2K_d E \\
&\quad \times \int_0^{\tau_d\wedge T} \left(2\left|X(s) - Y(s)\right|^2 + \left|Y(s) - Z_1(s)\right|^2\right) \\
&\qquad + \left(2\left|X(qs) - Y(qs)\right|^2 + 2\left|Y(qs) - Z_2(s)\right|^2\right) ds \\
&\quad + 2C_3(d)\left[h + \circ(h)\right] \\
&\le 4K_d\left[C_1(d) + C_2(d)\right] hT + 2C_3(d)\left[h + \circ(h)\right] \\
&\quad + 8K_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} \left|X(u) - Y(u)\right|^2 ds.
\end{aligned} \tag{31}
$$

Similarly, by Assumption 3 and Lemmas 5–7, we obtain

$$
\begin{aligned}
&E \int_0^{\tau_d\wedge T} \left|g\left(X(s), X(qs), r(s)\right)\right. \\
&\qquad \left. -g\left(Z_1(s), Z_2(s), \bar{r}(s)\right)\right|^2 ds \\
&\le 4K_d\left[C_1(d) + C_2(d)\right] hT + 2C_3(d)\left[h + \circ(h)\right] \\
&\quad + 8K_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} \left|X(u) - Y(u)\right|^2 ds,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&E \int_0^{\tau_d\wedge T} \int_{R^n} \left|h\left(X(s), X(qs), r(s), u\right)\right. \\
&\qquad \left. -h(Z_1(s), Z_2(s), \bar{r}(s), u)\right|^2 \pi(du)\, ds \\
&\le 4K_d\left[C_1(d) + C_2(d)\right] hT + 2C_3(d)\left[h + \circ(h)\right] \\
&\quad + 8K_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} \left|X(u) - Y(u)\right|^2 ds.
\end{aligned} \tag{33}
$$

On the other hand, by the Burkholder-Davis-Gundy inequality, Young's inequality, and Lemmas 5–7, we have for any $\varepsilon_1 > 0$

$$
\begin{aligned}
&E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \left|X(s) - Y(s)\right| \\
&\qquad \times \left|g\left(X(s), X(qs), r(s)\right)\right. \\
&\qquad\quad \left. - g\left(Z_1(s), Z_2(s), \bar{r}(s)\right)\right| dW(s) \\
&\le 3E\left[\sup_{0\le t\le \tau_d\wedge T} \left|X(t) - Y(t)\right|\right. \\
&\qquad \times \left(\int_0^{\tau_d\wedge T} \left|g\left(X(t), X(qt), r(t)\right)\right.\right. \\
&\qquad\qquad \left.\left. -g\left(Z_1(t), Z_2(t), \bar{r}(t)\right)\right|^2 dt\right)\Bigg]^{1/2} \\
&\le 3\left[\varepsilon_1 E \sup_{0\le t\le \tau_d\wedge T} \left|X(t) - Y(t)\right|^2\right]^{1/2} \\
&\quad \times \left[\frac{1}{\varepsilon_1} E\left(\int_0^{\tau_d\wedge T} \left|g\left(X(t), X(qt), r(t)\right)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -g\left(Z_1(t), Z_2(t), \bar{r}(t)\right)\right|^2 dt\right)\right]^{1/2} \\
&\le 6\varepsilon_1 E \sup_{0\le t\le \tau_d\wedge T} \left|X(t) - Y(t)\right|^2 \\
&\quad + \frac{6}{\varepsilon_1} E \int_0^{\tau_d\wedge T} \left|g\left(X(t), X(qt), r(t)\right)\right. \\
&\qquad \left. -g\left(Z_1(t), Z_2(t), \bar{r}(t)\right)\right|^2 dt
\end{aligned}
$$

$$\begin{aligned}
&\le 6\varepsilon_1 E \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|^2 \\
&\quad + \frac{24}{\varepsilon_1} K_d \left[C_1(d)+C_2(d)\right] hT \\
&\quad + \frac{12}{\varepsilon_1} C_3(d)\left[h+\circ(h)\right] \\
&\quad + \frac{48}{\varepsilon_1} K_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} |X(u)-Y(u)|^2 ds.
\end{aligned} \tag{34}$$

We have for any $\varepsilon_2 > 0$

$$\begin{aligned}
&E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \int_{R^n} \left[2|X(s)-Y(s)| \left|h(X(s),X(qs),r(s),u) \right.\right. \\
&\qquad \left.\left. -h(Z_1(s),Z_2(s),\bar{r}(s),u)\right|\right] \widetilde{N}(ds,du) \\
&\le CE\left(\sum_{t\in D_p, t\le \tau\wedge T} |X(t)-Y(t)|^2 \right. \\
&\qquad \times \left|h(X(t),X(qt),r(t),p_t) \right. \\
&\qquad \left.\left. -h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^2\right)^{1/2} \\
&\le CE \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|\, E \\
&\quad \times \left(\sum_{t\in D_p, t\le \tau_d\wedge T} \left|h(X(t),X(qt),r(t),p_t) \right.\right. \\
&\qquad \left.\left. -h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^2\right)^{1/2} \\
&\le CE\left[\varepsilon_2 \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|^2\right]^{1/2} \\
&\quad \times E\left[\frac{1}{\varepsilon_2} \sum_{t\in D_p, t\le \tau_d\wedge T} \left|h(X(t),X(qt),r(t),p_t) \right.\right. \\
&\qquad \left.\left. -h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^2\right]^{1/2} \\
&\le C\varepsilon_2 E \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|^2 \\
&\quad + \frac{C}{\varepsilon_2} E\left(\sum_{t\in D_p, t\le \tau_d\wedge T} \left|h(X(t),X(qt),r(t),p_t) \right.\right. \\
&\qquad \left.\left. - h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^2\right) \\
&\le C\varepsilon_2 E \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|^2 \\
&\quad + \frac{C}{\varepsilon_2} E \int_0^{\tau_d\wedge T} \int_{R^n} \left|h(X(t),X(qt),r(t),u) \right. \\
&\qquad \left. -h(Z_1(t),Z_2(t),\bar{r}(t),u)\right|^2 \pi(du)\, dt \\
&\le C\varepsilon_2 E \sup_{0\le t\le \tau_d\wedge T} |X(t)-Y(t)|^2 \\
&\quad + \frac{4C}{\varepsilon_2} K_d \left[C_1(d)+C_2(d)\right] hT + \frac{2C}{\varepsilon_2} C_3(d)\left[h+\circ(h)\right] \\
&\quad + \frac{8C}{\varepsilon_2} K_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} |X(u)-Y(u)|^2 ds,
\end{aligned} \tag{35}$$

where C is some constant that may change from line to line. Similar, we have

$$\begin{aligned}
&E \sup_{0\le t\le \tau_d\wedge T} \int_0^t \int_{R^n} \left|h(X(s),X(qs),r(s),u) \right. \\
&\qquad \left. -h(Z_1(s),Z_2(s),\bar{r}(s),u)\right|^2 \widetilde{N}(ds,du) \\
&\le CE\left(\sum_{t\in D_p, t\le \tau_d\wedge T} \left|h(X(t),X(qt),r(t),p_t) \right.\right. \\
&\qquad \left.\left. -h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^4\right)^{1/2} \\
&\le CE\left(\sum_{s\in D_p, s\le \tau_d\wedge T} \left|h(X(t),X(qt),r(t),p_t) \right.\right. \\
&\qquad \left.\left. -h(Z_1(t),Z_2(t),\bar{r}(t),p_t)\right|^2\right) \\
&\le CE \int_0^{\tau_d\wedge T} \int_{R^n} \left|h(X(t),X(qt),r(t),u) \right. \\
&\qquad \left. -h(Z_1(t),Z_2(t),\bar{r}(t),u)\right|^2 \pi(du)\, dt \\
&\le 4CK_d \left[C_1(d)+C_2(d)\right] hT + 2CC_3(d)\left[h+\circ(h)\right] \\
&\quad + 8CK_d \int_0^{\tau_d\wedge T} E \sup_{0\le u\le s} |X(u)-Y(u)|^2 ds.
\end{aligned} \tag{36}$$

Substituting (31)–(36) into (30), we obtain that

$$\begin{aligned} &E\left[\sup_{0\le t\le \tau_d\wedge T}|X(t)-Y(t)|^2\right] \\ &\quad\le (6\varepsilon_1+C\varepsilon_2)E\left[\sup_{0\le t\le \tau_d\wedge T}|X(t)-Y(t)|^2\right] \\ &\qquad+\left(10+\frac{24}{\varepsilon_1}+2C\frac{1+\varepsilon_2}{\varepsilon_2}\right)K_d\left[C_1(d)+C_2(d)\right]Th \\ &\qquad+\left(4+\frac{12}{\varepsilon_1}\right)C_3(d)\left[h+\circ(h)\right] \\ &\qquad+\left[1+\left(20+\frac{48}{\varepsilon_1}+4C\frac{1+\varepsilon_2}{\varepsilon_2}\right)K_d\right] \\ &\qquad\times\int_0^T E\sup_{0\le u\le \tau\wedge s}|X(u)-Y(u)|^2 ds. \end{aligned} \tag{37}$$

By choosing $\varepsilon_1>0$, $\varepsilon_2>0$ sufficiently small and letting $6\varepsilon_1+C\varepsilon_2=2/3$, we have

$$\begin{aligned} &E\left[\sup_{0\le t\le \tau\wedge T}|X(t)-Y(t)|^2\right] \\ &\quad\le 3\left(10+\frac{24}{\varepsilon_1}+2C\frac{1+\varepsilon_2}{\varepsilon_2}\right)K_d\left[C_1(d)+C_2(d)\right]Th \\ &\qquad+3\left(4+\frac{12}{\varepsilon_1}\right)C_3(d)\left[h+\circ(h)\right] \\ &\qquad+3\left[1+\left(20+\frac{48}{\varepsilon_1}+4C\frac{1+\varepsilon_2}{\varepsilon_2}\right)K_d\right] \\ &\qquad\times\int_0^T E\sup_{0\le u\le \tau\wedge s}|X(u)-Y(u)|^2 ds. \end{aligned} \tag{38}$$

Therefore, we apply Gronwall's inequality to get

$$\begin{aligned} &E\left[\sup_{0\le t\le \tau\wedge T}|X(t)-Y(t)|^2\right] \\ &\le\left\{3\left(10+\frac{24}{\varepsilon_1}+2C\frac{1+\varepsilon_2}{\varepsilon_2}\right)K_d\left[C_1(d)+C_2(d)\right]Th\right. \\ &\qquad\left.+3\left(4+\frac{12}{\varepsilon_1}\right)C_3(d)\left[h+\circ(h)\right]\right\}e^{3[1+(20+(48/\varepsilon_1)+4C((1+\varepsilon_2)/\varepsilon_2))K_d]}. \end{aligned} \tag{39}$$

This completes the proof. □

Remark 8. Under the local Lipschitz condition, Theorem 4 not only tells us the strong convergence of the approximate solutions to the true solutions but also tells us the rate of the convergence with order 1/2 by (39).

Remark 9. When $r(t)\equiv 0$ or $h\equiv 0$, (4) reduces to which was studied by Fan et al. [14], Xiao and Zhang [15], and Milošević and Jovanović [16]. Our results in the present paper generalized and improved the results in [14–16].

4. Convergence of Numerical Solutions in Probability

In this section, by applying Theorem 4, we will show the convergence in probability of the approximate solutions to the true solutions under local Lipschitz condition. Before we give the convergence theorem, we need some additional conditions based on Lyapunov-type functions.

Assumption 10. For $x\in R^n$ and $i\in S$, there exist a positive function $V\in C^2(R^n\times S;R^+)$, $K>0$, and two constants $\lambda_1>\lambda_2\ge 0$ such that

$$\lim_{|x|\to\infty}V(x,i)=\infty, \tag{40}$$

$$LV(x,y,i)\le K\left[1-\lambda_1V(x,i)+\lambda_2V(y,i)\right]. \tag{41}$$

Assumption 11. There exists a positive constant L_d such that, for all $x,y\in R^n$ and $i\in S$ with $|x|\vee|y|\le d$,

$$\begin{aligned} &\left|V(x,i)-V(y,i)\right|\vee\left|V_x(x,i)-V_x(y,i)\right| \\ &\qquad\vee\left|V_{xx}(x,i)-V_{xx}(y,i)\right|\le L_d\left|x-y\right|. \end{aligned} \tag{42}$$

Now, let us state our convergence theorem.

Theorem 12. *Let the assumptions of Theorem 4 hold. Also assume that there exists a C^2 function $V:R^n\times S\to R_+$ satisfying (40)–(42). Then the Euler approximate solutions converges to the true solutions of (4) in the sense of the probability. That is,*

$$\lim_{h\to 0}\sup_{0\le t\le T}|Y(t)-X(t)|^2=0,\quad \textit{in probability}. \tag{43}$$

Proof. The proof is rather technical, and we divide it into three steps.

Step 1. We assume the existence of the nonnegative Lyapunov function $V(x,i)$ satisfying (40). Applying the Itô's formula, $V(X(t),r(t))$ yields

$$\begin{aligned} &dV(X(t),r(t)) \\ &\quad=LV(X(t),X(qt),r(t))\,dt \\ &\qquad+V_x(X(t),r(t))\,g(X(t),X(qt),r(t))\,dW(t) \\ &\qquad+\int_{R^n}\left[V(X(t)+h(X(t),X(qt),r(t),u),r(t))\right. \\ &\qquad\qquad\left.-V(X(t),r(t))\right]\widetilde{N}(dt,du). \end{aligned} \tag{44}$$

Integrating from 0 to $t\wedge\theta_d$ and taking expectations gives

$$\begin{aligned} &EV(X(t\wedge\theta_d),r(t\wedge\theta_d)) \\ &\quad=V(X_0,r_0)+E\int_0^{t\wedge\theta_d}LV(X(s),X(qs),r(s))\,ds. \end{aligned} \tag{45}$$

By (41), we have

$$\begin{aligned} &EV\left(X\left(t\wedge\theta_d\right),r\left(t\wedge\theta_d\right)\right)\\ &\quad\le V\left(X_0,r_0\right)+KE\int_0^{t\wedge\theta_d}\left[1-\lambda_1V\left(X\left(s\right),r\left(s\right)\right)\right.\\ &\qquad\qquad\left.+\lambda_2V\left(X\left(qs\right),r\left(s\right)\right)\right]ds \quad (46)\\ &\quad\le V\left(X_0,r_0\right)+KT\\ &\qquad+K\left(\lambda_1+\lambda_2\right)\int_0^{t\wedge\theta}\sup_{0\le u\le s}EV\left(X\left(u\right),r\left(u\right)\right)ds. \end{aligned}$$

Thus, for any $t_1\in[0,T]$, it follows that

$$\begin{aligned} &\sup_{0\le t\le t_1}EV\left(X\left(t\wedge\theta_d\right),r\left(t\wedge\theta_d\right)\right)\\ &\quad\le V\left(X_0,r_0\right)+KT\\ &\qquad+\left(\lambda_1+\lambda_2\right)K\int_0^{t_1}\sup_{0\le t\le s}EV\left(X\left(t\wedge\theta_d\right),r\left(t\wedge\theta_d\right)\right)ds. \end{aligned} \quad (47)$$

Using the Gronwall inequality, we obtain that

$$\begin{aligned} &\sup_{0\le t\le T}EV\left(x\left(t\wedge\theta_d\right),r\left(t\wedge\theta_d\right)\right)\\ &\qquad\le\left[V\left(X_0,r_0\right)+KT\right]e^{(\lambda_1+\lambda_2)KT}. \end{aligned} \quad (48)$$

Let $\mathscr{V}_d=\inf\{V(x,i):|x|\ge d\}$. By (40), we have $\lim_{d\to\infty}\mathscr{V}_d=\infty$. Noting that $|X(\theta_d)|=d$, as $\theta_d<T$, we derive from (48) that

$$\begin{aligned} &\left[V\left(X_0,r_0\right)+KT\right]e^{(\lambda_1+\lambda_2)KT}\\ &\quad\ge\sup_{0\le t\le T}EV\left(x\left(t\wedge\theta_d\right),r\left(t\wedge\theta_d\right)\right)\\ &\quad\ge E\left[V\left(X\left(\theta_d\right),r\left(\theta_d\right)\right)I_{\{\theta_d<T\}}\left(\omega\right)\right]\\ &\quad\ge\mathscr{V}_dP\left(\theta_d<T\right). \end{aligned} \quad (49)$$

That is,

$$P\left(\theta_d<T\right)\le\frac{\left[V\left(X_0,r_0\right)+KT\right]e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}. \quad (50)$$

Recall that $\mathscr{V}_d\to\infty$ as $d\to\infty$. For a given T, X_0, and $r(0)$, it follows that

$$\frac{\left[V\left(X_0,r_0\right)+KT\right]e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\longrightarrow 0, \quad (51)$$

as $d\to\infty$. Let

$$\frac{\varepsilon}{3}=\frac{\left[V\left(X_0,r_0\right)+KT\right]e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\in(0,1). \quad (52)$$

Thus we have

$$P\left(\theta_d<T\right)\le\frac{\varepsilon}{3}. \quad (53)$$

Step 2. We will give the estimate of $P(\rho_d<T)$. By (14), applying the Itô's formula to $V(Y(t),r(t))$ yields

$$\begin{aligned} &dV\left(Y\left(t\right),r\left(t\right)\right)\\ &\quad=V_x\left(Y\left(t\right),r\left(t\right)\right)f\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)\\ &\qquad+\frac{1}{2}\operatorname{trace}\left[g^{\top}\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)\right.\\ &\qquad\qquad\left.\times V_{xx}\left(Y\left(t\right),r\left(t\right)\right)g\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)\right]dt\\ &\qquad+V_x\left(Y\left(t\right),r\left(t\right)\right)g\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)dW\left(t\right)\\ &\qquad+\int_{R^n}\left[V\left(Y\left(t\right)+h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right),r\left(t\right)\right)\right.\\ &\qquad\qquad-V\left(Y\left(t\right),r\left(t\right)\right)-V_x\left(Y\left(t\right),r\left(t\right)\right)\\ &\qquad\qquad\left.\times h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right)\right]\Pi\left(du\right)dt\\ &\qquad+\int_{R^n}\left[V\left(Y\left(t\right)+h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right),r\left(t\right)\right)\right.\\ &\qquad\qquad\left.-V\left(Y\left(t\right),r\left(t\right)\right)\right]\widetilde{N}\left(dt,du\right)\\ &\qquad+\sum_{j=1}^{N}\gamma_{r(t)j}V\left(Y\left(t\right),j\right)dt. \end{aligned} \quad (54)$$

By (7), we have

$$\begin{aligned} &dV\left(Y\left(t\right),r\left(t\right)\right)\\ &\quad=LV\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)dt\\ &\qquad+\left[V_x\left(Y\left(t\right),r\left(t\right)\right)-V_x\left(Z_1\left(t\right),\bar{r}\left(t\right)\right)\right]\\ &\qquad\times f\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)dt\\ &\qquad+\frac{1}{2}\operatorname{trace}\left[g^{\top}\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)\right.\\ &\qquad\qquad\times\left[V_{xx}\left(Y\left(t\right),r\left(t\right)\right)-V_{xx}\left(Z_1\left(t\right),\bar{r}\left(t\right)\right)\right]\\ &\qquad\qquad\left.\times g\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)\right]dt\\ &\qquad+\int_{R^n}\left\{\left[V\left(Y\left(t\right)+h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right),r\left(t\right)\right)\right.\right.\\ &\qquad\qquad\left.-V\left(Z_1\left(t\right)+h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right),\bar{r}\left(t\right)\right)\right]\\ &\qquad\qquad-\left[V\left(Y\left(t\right),r\left(t\right)\right)-V\left(Z_1\left(t\right),\bar{r}\left(t\right)\right)\right]\\ &\qquad\qquad-\left[V_x\left(Y\left(t\right),r\left(t\right)\right)-V_x\left(Z_1\left(t\right),\bar{r}\left(t\right)\right)\right]\\ &\qquad\qquad\left.\times h\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right),u\right)\right\}\Pi\left(du\right)dt\\ &\qquad+V_x\left(Y\left(t\right),r\left(t\right)\right)g\left(Z_1\left(t\right),Z_2\left(t\right),\bar{r}\left(t\right)\right)dW\left(t\right) \end{aligned}$$

$$+\int_{R^n}\left[V\left(Y(t)+h\left(Z_1(t),Z_2(t),\bar{r}(t),u\right),r(t)\right)\right.$$
$$\left.-V\left(Y(t),r(t)\right)\right]\widetilde{N}(dt,du)$$
$$+\sum_{j=1}^{N}\gamma_{r(t)j}V\left(Y(t),j\right)dt-\sum_{j=1}^{N}\gamma_{\bar{r}(t)j}V\left(Z_1(t),j\right)dt. \tag{55}$$

Integrating from 0 to $\rho_d\wedge t$, taking expectations, and by (41), we have

$$\begin{aligned}
&EV\left(Y\left(t\wedge\rho_d\right),r\left(t\wedge\rho_d\right)\right)\\
&\quad\le V\left(X_0,r_0\right)+KT+\lambda_1KE\int_0^{\rho_d\wedge t}V\left(Y(s),r(s)\right)ds\\
&\quad+\lambda_2KE\int_0^{\rho_d\wedge t}V\left(Y(qs),r(s)\right)ds\\
&\quad-\lambda_1KE\int_0^{\rho_d\wedge t}\left[V\left(Z_1(s),\bar{r}(s)\right)+V\left(Y(s),r(s)\right)\right]ds\\
&\quad+\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5,
\end{aligned} \tag{56}$$

where

$$\begin{aligned}
\mathcal{I}_1&=\lambda_2KE\int_0^{\rho_d\wedge t}\left|V\left(Z_2(s),\bar{r}(s)\right)-V\left(Y(qs),r(s)\right)\right|ds,\\
\mathcal{I}_2&=E\int_0^{\rho_d\wedge t}\left|V_x\left(Y(s),r(s)\right)-V_x\left(Z_1(s),\bar{r}(s)\right)\right|\\
&\qquad\times f\left(Z_1(s),Z_2(s),\bar{r}(s)\right)ds,\\
\mathcal{I}_3&=\frac{1}{2}E\int_0^{\rho_d\wedge t}\left|V_{xx}\left(Y(s),r(s)\right)-V_{xx}\left(Z_1(s),\bar{r}(s)\right)\right|\\
&\qquad\times\left|g\left(Z_1(s),Z_2(s),\bar{r}(s)\right)\right|^2ds,\\
\mathcal{I}_4&=E\int_0^{\rho_d\wedge t}\int_{R^n}\left[\left|V\left(Y(s)+h\left(Z_1(s),Z_2(s),\right.\right.\right.\right.\\
&\qquad\left.\left.\bar{r}(s),u\right),r(s)\right)\\
&\qquad-V\left(Z_1(s)\right.\\
&\qquad\left.+h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right),\bar{r}(s)\right)\Big|\\
&\qquad+\left|V\left(Y(s),r(s)\right)-V\left(Z_1(s),\bar{r}(s)\right)\right|\\
&\qquad+\left|V_x\left(Y(s),r(s)\right)-V_x\left(Z_1(s),\bar{r}(s)\right)\right|\\
&\qquad\times\left|h\left(Z_1(s),Z_2(s),\bar{r}(s),u\right)\right|\Big]\pi(du)\,ds,\\
\mathcal{I}_5&=E\int_0^{\rho_d\wedge t}\sum_{j=1}^{N}\left|\gamma_{r(s)j}V\left(Y(s),j\right)\right|ds\\
&\qquad-E\int_0^{\rho_d\wedge t}\sum_{j=1}^{N}\left|\gamma_{\bar{r}(s)j}V\left(Z_1(s),j\right)\right|ds.
\end{aligned} \tag{57}$$

By (42) and Young's inequality, we have

$$\begin{aligned}
\mathcal{I}_1&\le\lambda_2KE\int_0^{\rho_d\wedge t}\left|V\left(Z_2(s),\bar{r}(s)\right)-V\left(Y(qs),\bar{r}(s)\right)\right|ds\\
&\quad+\lambda_2KE\int_0^{\rho_d\wedge t}\left|V\left(Y(qs),\bar{r}(s)\right)-V\left(Y(qs),r(s)\right)\right|ds\\
&\le\lambda_2KL_dE\int_0^{\rho_d\wedge t}\left|Z_2(s)-Y(qs)\right|ds\\
&\quad+\lambda_2KE\int_0^{T}\left|V\left(Y(qs),\bar{r}(s)\right)-V\left(Y(qs),r(s)\right)\right|ds\\
&\le\lambda_2KL_d\int_0^{\rho_d\wedge t}\left(E\left|Z_2(s)-Y(qs)\right|^2\right)^{1/2}ds+\lambda_2\mathcal{J}.
\end{aligned} \tag{58}$$

Let $N=[T/h]$, the integer part of T/h, and let I_G be the indicator function of the set G. Then

$$\begin{aligned}
\mathcal{J}&\le K\sum_{n=0}^{N}E\int_{nh}^{(n+1)h}\left|V\left(Y_{[qn]},r_n^h\right)-V\left(Y_{[qn]},r(s)\right)\right|ds\\
&\le K\sum_{n=0}^{N}E\int_{nh}^{(n+1)h}\left|V\left(Y_{[qn]},r_n^h\right)\right|\\
&\quad+\left|f\left(Y_{[qn]},r(s)\right)\right|I_{\{r(s)\ne r_n^h\}}ds.
\end{aligned} \tag{59}$$

By setting $V_1=\sup_{|x|\le R,i\in S}V(x,i)$ and using the Markov property, we have

$$\begin{aligned}
\mathcal{J}&\le 2KV_1\sum_{n=0}^{N}E\int_{nh}^{(n+1)h}I_{\{r(s)\ne r_n^h\}}ds\\
&=2KV_1\sum_{n=0}^{N}\int_{nh}^{(n+1)h}E\left[E\left[I_{\{r(s)\ne r_n^h\}}\right]\mid r_n^h\right]ds\\
&=2KV_1\sum_{n=0}^{N}\int_{nh}^{(n+1)h}E\left[\sum_{i\in S}I_{\{r_n^h=i\}}P\left(r(s)\ne i\mid r_n^h=i\right)\right]ds\\
&=2KV_1\\
&\quad\times\sum_{n=0}^{N}\int_{nh}^{(n+1)h}E\left[\sum_{i\in S}I_{\{r_n^h=i\}}\sum_{j\ne i}\left(\gamma_{ij}(s-nh)+\circ(s-nh)\right)\right]\\
&\qquad\times ds\\
&\le 2KV_1\sum_{n=0}^{N}\int_{nh}^{(n+1)h}E\left[\left(\max_{1\le i\le N}(-\gamma_{ii})h+\circ(h)\right)\sum_{i\in S}I_{\{r_n^h=i\}}\right]ds\\
&=2KV_1\left[\max_{1\le i\le N}(-\gamma_{ii})h+\circ(h)\right]T.
\end{aligned} \tag{60}$$

Inserting (60) into (58), we have, by Lemma 5,

$$\begin{aligned}\mathcal{I}_1 &\le \lambda_2 K L_d \int_0^{\rho_d\wedge t} \left(E\left|Z_2(s)-Y(qs)\right|^2\right)^{1/2} ds \\ &\quad + 2\lambda_2 K V_1 \max_{1\le i\le N}(-\gamma_{ii})\, h + \circ(h)\, T \\ &\le \lambda_2 K L_d \sqrt{C_2(d)}\, T h^{1/2} + 2\lambda_2 K V_1 T \max_{1\le i\le N}(-\gamma_{ii})\,[h+\circ(h)]\,.\end{aligned} \tag{61}$$

Similarly, by Assumptions 2, 3, and 11, Lemma 5, and Markov property, we have

$$\begin{aligned}\mathcal{I}_2 &\le E\int_0^{\rho_d\wedge t} \left|V_x(Y(s),r(s)) - V_x(Z_1(s),r(s))\right| \\ &\qquad \times \left|f(Z_1(s),Z_2(s),\bar{r}(s))\right| ds \\ &\quad + E\int_0^{\rho_d\wedge t} \left|V_x(Z_1(s),r(s)) - V_x(Z_1(s),\bar{r}(s))\right| \\ &\qquad \times \left|f(Z_1(s),Z_2(s),\bar{r}(s))\right| ds \\ &\le 2\sqrt{K_d}\, d E\int_0^{\rho_d\wedge t} \left|V_x(Y(s),r(s)) - V_x(Z_1(s),r(s))\right| ds \\ &\quad + 2\sqrt{K_d}\, d E\int_0^{\rho_d\wedge t} \left|V_x(Z_1(s),r(s)) \right. \\ &\qquad \left. - V_x(Z_1(s),\bar{r}(s))\right| ds \\ &\le 2\sqrt{K_d}\, d L_d E\int_0^{\rho_d\wedge t} \left|Y(s)-Z_1(s)\right| ds \\ &\quad + 2d\sqrt{K_d} E\int_0^{T} \left|V_x(Z_1(s),r(s)) - V_x(Z_1(s),\bar{r}(s))\right| ds \\ &\le 2d\sqrt{K_d} L_d \sqrt{C_1(d)}\, T h^{1/2} \\ &\quad + 4\sqrt{K_d}\, d V_2 T \max_{1\le i\le N}(-\gamma_{ii})\,[h+\circ(h)]\,,\end{aligned} \tag{62}$$

where $V_2 = \sup_{|x|\le R, i\in S} V_x(x,i)$. For the term $\mathcal{I}_3$ in (56), by Assumptions 2, 3, and 11 and Lemma 5, we have

$$\begin{aligned}\mathcal{I}_3 &\le \frac{1}{2}E\int_0^{\rho_d\wedge t} \left|V_{xx}(Y(s),r(s)) - V_{xx}(Z_1(s),r(s))\right| \\ &\qquad \times \left|g(Z_1(s),Z_2(s),\bar{r}(s))\right|^2 ds \\ &\quad + \frac{1}{2}E\int_0^{\rho_d\wedge t} \left|V_{xx}(Z_1(s),r(s)) - V_{xx}(Z_1(s),\bar{r}(s))\right| \\ &\qquad \times \left|g(Z_1(s),Z_2(s),\bar{r}(s))\right|^2 ds \\ &\le 2d^2 K_d E\int_0^{\rho_d\wedge t} \left|V_{xx}(Y(s),r(s)) - V_{xx}(Z_1(s),r(s))\right| ds \\ &\quad + 2d^2 K_d E\int_0^{\rho_d\wedge t} \left|V_{xx}(Z_1(s),r(s)) \right. \\ &\qquad \left. - V_{xx}(Z_1(s),\bar{r}(s))\right| ds \\ &\le 2d^2 K_d L_d E\int_0^{\rho_d\wedge t} \left|Y(s)-Z_1(s)\right| ds \\ &\quad + 2d^2 K_d E\int_0^{T} \left|V_{xx}(Z_1(s),r(s)) \right. \\ &\qquad \left. - V_{xx}(Z_1(s),\bar{r}(s))\right| ds \\ &\le 2d^2 K_d L_d \sqrt{C_1(d)}\, T h^{1/2} \\ &\quad + 4d^2 K_d V_3 T \max_{1\le i\le N}(-\gamma_{ii})\,[h+\circ(h)]\,,\end{aligned} \tag{63}$$

where $V_3 = \sup_{|x|\le R, i\in S} V_{xx}(x,i)$. For the term $\mathcal{I}_4$ in (56), by Assumptions 2, 3, and 11 and Lemma 5, we have

$$\begin{aligned}\mathcal{I}_4 &\le E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V(Y(s) \right. \\ &\qquad + h(Z_1(s),Z_2(s),\bar{r}(s),u),r(s)) \\ &\qquad - V(Z_1(s) + h(Z_1(s),Z_2(s),\bar{r}(s),u), \\ &\qquad \left. r(s))\right| \pi(du)\, ds \\ &\quad + E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V(Z_1(s) \right. \\ &\qquad + h(Z_1(s),Z_2(s),\bar{r}(s),u),r(s)) \\ &\qquad - V(Z_1(s) \\ &\qquad \left. + h(Z_1(s),Z_2(s),\bar{r}(s),u),\bar{r}(s))\right| \\ &\qquad \times \pi(du)\, ds \\ &\quad + E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V(Y(s),r(s)) - V(Z_1(s),r(s))\right| \\ &\qquad \times \pi(du)\, ds \\ &\quad + E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V(Z_1(s),r(s)) - V(Z_1(s),\bar{r}(s))\right| \\ &\qquad \times \pi(du)\, ds \\ &\quad + E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V_x(Y(s),r(s)) - V_x(Z_1(s),r(s))\right| \\ &\qquad \times \left|h(Z_1(s),Z_2(s),\bar{r}(s),u)\right| \pi(du)\, ds \\ &\quad + E\int_0^{\rho_d\wedge t}\int_{R^n} \left|V_x(Z_1(s),r(s)) - V_x(Z_1(s),\bar{r}(s))\right| \\ &\qquad \times \left|h(Z_1(s),Z_2(s),\bar{r}(s),u)\right| \pi(du)\, ds\end{aligned}$$

$$
\begin{aligned}
&\le 2L_d E\int_0^{\rho_d\wedge t}\int_{R^n}\left|Y(s)-Z_1(s)\right|\pi(du)\,ds\\
&\quad+2d\sqrt{K_d\int_{R^n}\pi(du)}\\
&\quad\times E\int_0^{\rho_d\wedge t}\left|V_x(Y(s),r(s))-V_x(Z_1(s),r(s))\right|ds\\
&\quad+\int_{R^n}\pi(du)\,E\\
&\quad\times\int_0^{\rho_d\wedge t}\left|V(Z_1(s),r(s))-V(Z_1(s),\bar{r}(s))\right|ds\\
&\quad+2d\sqrt{K_d\int_{R^n}\pi(du)}\\
&\quad\times E\int_0^{\rho_d\wedge t}\left|V_x(Z_1(s),r(s))-V_x(Z_1(s),\bar{r}(s))\right|ds\\
&\quad+E\int_0^{\rho_d\wedge t}\int_{R^n}\left|V(Z_1(s)\right.\\
&\qquad\qquad\left.+h(Z_1(s),Z_2(s),\bar{r}(s),u),r(s))\right.\\
&\qquad-V(Z_1(s)\\
&\qquad\qquad\left.+h(Z_1(s),Z_2(s),\bar{r}(s),u),\bar{r}(s))\right|\\
&\qquad\times\pi(du)\,ds\\
&\le 2\left[\int_{R^n}\pi(du)+2d\sqrt{K_d\int_{R^n}\pi(du)}\right]\\
&\quad\times\sqrt{C_1(d)}L_dTh^{1/2}\\
&\quad+2\left[\int_{R^n}\pi(du)V_1+2d\sqrt{K_d\int_{R^n}\pi(du)}V_2\right]\\
&\quad\times T\max_{1\le i\le N}(-\gamma_{ii})\left[h+\circ(h)\right]+\mathscr{L},
\end{aligned}
\tag{64}
$$

where

$$
\begin{aligned}
\mathscr{L}\le{}&E\int_0^{\rho_d\wedge t}\int_{R^n}\left|V(Z_1(s)+h(Z_1(s),Z_2(s),\bar{r}(s),u),r(s))\right.\\
&\qquad\left.-V(Z_1(s),\bar{r}(s))\right|\pi(du)\,ds\\
&+E\int_0^{\rho_d\wedge t}\int_{R^n}\left|V(Z_1(s),r(s))\right.\\
&\qquad\left.-V(Z_1(s),\bar{r}(s))\right|\pi(du)\,ds\\
&+E\int_0^{\rho_d\wedge t}\int_{R^n}\left|V(Z_1(s),\bar{r}(s))\right.\\
&\qquad\left.-V(Z_1(s)+h(Z_1(s),Z_2(s),\bar{r}(s),u),\right.\\
&\qquad\qquad\left.\bar{r}(s))\right|\pi(du)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 4dL_d\sqrt{K_d\int_{R^n}\pi(du)T}\\
&\quad+2V_1T\int_{R^n}\pi(du)\max_{1\le i\le N}(-\gamma_{ii})\left[h+\circ(h)\right].
\end{aligned}
\tag{65}
$$

Inserting (65) into (64), we have

$$
\begin{aligned}
\mathscr{I}_4\le{}&2\left[\int_{R^n}\pi(du)+2d\sqrt{K_d\int_{R^n}\pi(du)}\right]\sqrt{C_1(d)}L_dTh^{1/2}\\
&+4dL_d\sqrt{K_d\int_{R^n}\pi(du)T}\\
&+4\left[\int_{R^n}\pi(du)V_1+d\sqrt{K_d\int_{R^n}\pi(du)}V_2\right]\\
&\times T\max_{1\le i\le N}(-\gamma_{ii})\left[h+\circ(h)\right].
\end{aligned}
\tag{66}
$$

For the term $\mathscr{I}_5$, by Assumption 11, equation (56), and Lemma 5, we have

$$
\begin{aligned}
\mathscr{I}_5\le{}&E\int_0^{\rho_d\wedge t}\sum_{j=1}^N\left|\gamma_{r(s)j}\right|\left|V(Y(s),j)-V(Z_1(s),j)\right|ds\\
&+E\int_0^{\rho_d\wedge t}\sum_{j=1}^N\left|\gamma_{r(s)j}-\gamma_{\bar{r}(s)j}\right|\left|V(Z_1(s),j)\right|ds\\
\le{}&N\max_{1\le i\le N}(-\gamma_{ii})L_d\sqrt{C_1(d)}Th^{1/2}+2V_1NT\max_{1\le i\le N}(-\gamma_{ii}).
\end{aligned}
\tag{67}
$$

Combing these inequalities and (53), we derive that, for $t_1\in[0,T]$,

$$
\begin{aligned}
&\sup_{0\le t\le t_1}EV(Y(t\wedge\rho_d),r(t\wedge\rho_d))\\
&\quad\le V(X_0,r_0)\\
&\qquad+\left[K+4dL_d\sqrt{K_d\int_{R^n}\pi(du)}+2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T\\
&\qquad+(\lambda_1+\lambda_2)K\\
&\qquad\times\int_0^{t_1}\sup_{0\le t\le s}EV(Y(t\wedge\rho_d))\,ds+M_1h^{1/2}+M_2\left[h+\circ(h)\right],
\end{aligned}
\tag{68}
$$

where

$$M_1 = L_d\sqrt{C_1(d)}T\left[2d\sqrt{K_d} + 2d^2K_d + 2\int_{R^n}\pi(du) + 4d\sqrt{K_d\int_{R^n}\pi(du)} + N\max_{1\le i\le N}(-\gamma_{ii})\right] + \lambda_2 KL_d\sqrt{C_2(d)}T,$$

$$M_2 = \left[\left(2\lambda_2 K + 4\int_{R^n}\pi(du)\right)V_1 + \left(4d\sqrt{K_d} + 4d\sqrt{K_d\int_{R^n}\pi(du)}\right)V_2 + 4d^2K_dV_3\right]T\max_{1\le i\le N}(-\gamma_{ii}). \tag{69}$$

For arbitrary $0 \le t_1 \le T$, by the Gronwall inequality, we get

$$\begin{aligned}&\sup_{0\le t\le T} EV(Y(t\wedge\rho_d), r(t\wedge\rho_d))\\ &\le \Bigg\{V(X_0, r_0) + \left[K + 4dL_d\sqrt{K_d\int_{R^n}\pi(du)} + 2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T + M_1h^{1/2} + M_2[h + \circ(h)]\Bigg\}e^{(\lambda_1+\lambda_2)KT}.\end{aligned} \tag{70}$$

Noting that $|Y(\rho_d)| = d$, as $\rho_d < T$, we derive from (70) that

$$\begin{aligned}&\Bigg\{V(X_0, r_0) + \left[K + 4dL_d\sqrt{K_d\int_{R^n}\pi(du)} + 2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T + M_1h^{1/2} + M_2[h + \circ(h)]\Bigg\}e^{(\lambda_1+\lambda_2)KT}\\ &\quad\ge EV(Y(t\wedge\rho_d), r(t\wedge\rho_d))\\ &\quad\ge E\left[V(Y(\rho_d), r(\rho_d))I_{\{\rho_d<T\}}(w)\right]\\ &\quad\ge \mathscr{V}_d P(\rho_d < T).\end{aligned} \tag{71}$$

So we have

$$\begin{aligned}P(\rho_d < T) &\le \frac{\left\{V(X_0, r_0) + \left[K + 4dL_d\sqrt{K_d\int_{R^n}\pi(du)}\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\\ &\quad + \frac{\left\{\left[2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\\ &\quad + \frac{\left\{M_1h^{1/2} + M_2[h + \circ(h)]\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}.\end{aligned} \tag{72}$$

Step 3. Let $\epsilon, \delta \in (0, 1)$ be arbitrarily small. By setting

$$\overline{\Omega} = \left\{w : \sup_{0\le t\le T}|Y(t) - X(t)|^2 \ge \delta\right\} \tag{73}$$

and using Theorem 4, we have

$$\begin{aligned}C_d[h + \circ(h)] &\ge E\left[\sup_{0\le t\le T}|Y(t) - X(t)|^2 I_{\tau_d\ge T}(w)\right]\\ &\ge E\left[\sup_{0\le t\le T}|Y(t) - X(t)|^2 I_{\tau_d\ge T}(w)I_{\overline{\Omega}}(w)\right]\\ &\ge \delta E\left[I_{\tau_d\ge T}(w)I_{\overline{\Omega}}(w)\right] = \delta P\left(\{\tau_d \ge T\}\cap\overline{\Omega}\right)\\ &\ge \delta\left[P\left(\overline{\Omega}\right) - P(\tau_d < T)\right].\end{aligned} \tag{74}$$

Combing (53) and (72) together, one gets

$$\begin{aligned}P(\tau_d < T) &\le P(\rho_d < T) + P(\theta_d < T)\\ &\le \frac{\varepsilon}{3} + \frac{\left\{V(X_0, r_0) + \left[K + 4dL_d\sqrt{K_d\int_{R^n}\pi(du)}\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\\ &\quad + \frac{\left\{\left[2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}\\ &\quad + \frac{\left\{M_1h^{1/2} + M_2[h + \circ(h)]\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}.\end{aligned} \tag{75}$$

Hence on using (74), we conclude that

$$\begin{aligned} &P\left(\overline{\Omega}\right) \\ &= P\left(\sup_{0\le t\le T}|Y(t)-X(t)|^2 \ge \delta\right) \\ &\le \frac{\left\{V(X_0,r_0)+\left[K+4dL_d\sqrt{K_d\int_{R^n}\pi(du)}\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d} \\ &\quad+\frac{\left\{\left[2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d} \\ &\quad+\frac{\left\{M_1h^{1/2}+M_2[h+\circ(h)]\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d} \\ &\quad+\frac{C_d}{\delta}[h+\circ(h)]+\frac{\varepsilon}{3}. \end{aligned} \tag{76}$$

Recalling that $\mathscr{V}_d \to \infty$ as $d \to \infty$, we can choose d sufficiently large for

$$\begin{aligned} &\frac{\left\{V(X_0,r_0)+\left[K+4dL_d\sqrt{K_d\int_{R^n}\pi(du)}\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d} \\ &+\frac{\left\{\left[2V_1N\max_{1\le i\le N}(-\gamma_{ii})\right]T\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}<\frac{\varepsilon}{3} \end{aligned} \tag{77}$$

and then choose h sufficiently small for

$$\frac{\left\{M_1h^{1/2}+M_2[h+\circ(h)]\right\}e^{(\lambda_1+\lambda_2)KT}}{\mathscr{V}_d}+\frac{C_d}{\delta}[h+\circ(h)]<\frac{\varepsilon}{3} \tag{78}$$

to obtain

$$P\left(\sup_{0\le t\le T}|Y(t)-X(t)|^2\ge\delta\right)<\varepsilon. \tag{79}$$

The proof of Theorem 12 is now complete. □

5. An Example

In this section, we construct one example to demonstrate the effectiveness of this theory. Let $r(t)$ be a right-continuous Markov chain taking values in $S=\{1,2\}$ with the generator

$$\Gamma=\left(\gamma_{ij}\right)_{2\times2}=\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix} \tag{80}$$

Let $\widetilde{N}(dt,du)$ be a compensated Poisson random measures and is given by $\pi(du)dt=\lambda f(u)du\,dt$, where $\lambda=2$ and

$$f(u)=\frac{1}{\sqrt{2\pi u}}e^{-(lnu)^2/2},\quad 0\le u<\infty \tag{81}$$

is the density function of a lognormal random variable. Of course $\widetilde{N}(dt,du)$ and $r(t)$ are assumed to be independent. Consider a linear stochastic pantograph delay equations with Markovian switching and pure jumps

$$\begin{aligned} dX(t) &= a(r(t))X(t)\,dt \\ &\quad+\int_{|u|\le1}ub(r(t))X(0.5t)\widetilde{N}(dt,du). \end{aligned} \tag{82}$$

Here $a(1)=-3$, $a(2)=-1$, $b(1)=1/\sqrt{10}$, and $b(2)=1/\sqrt{6}$. Then (82) can be regarded as the result of the two equations

$$\begin{aligned} dX(t) &= -3X(t)\,dt+\frac{1}{\sqrt{10}}\int_{|u|\le1}uX(0.5t)\widetilde{N}(dt,du), \\ dX(t) &= -X(t)\,dt+\frac{1}{\sqrt{6}}\int_{|u|\le1}uX(0.5t)\widetilde{N}(dt,du), \end{aligned} \tag{83}$$

switching among each other according to the movement of the Markov chain $r(t)$. Obviously, (82) satisfies Assumptions 2 and 3. Given the stepsize h, we can have the Euler method

$$\begin{aligned} Y_{n+1} &= Y_n+a\left(r_n^h\right)Y_nh \\ &\quad+\int_{t_n}^{t_{n+1}}\int_{|u|\le1}ub\left(r_n^h\right)Y_{[0.5n]}\widetilde{N}(dt,du), \end{aligned} \tag{84}$$

with $Y_0=X(0)$. Let $\overline{Z}_1(t)=Y_n$, $\overline{Z}_2(t)=Y_{[0.5n]}$, and $\overline{r}(t)=r_n^h$, for $t\in[t_n,t_{n+1})$. Then we define the continuous Euler approximate solution

$$\begin{aligned} Y(t) &= Y(0)+\int_0^t a(\overline{r}(s))\overline{Z}_1(s)\,ds \\ &\quad+\int_0^t\int_{|u|\le1}ub(\overline{r}(s))\overline{Z}_2(s)\widetilde{N}(ds,du). \end{aligned} \tag{85}$$

Since the conditions of Theorem 4 are satisfied, then the approximate solution (85) will converge to the true solution of (82) in the sense of the L^2-norm. To examine the convergence in the sense of the probability, we construct a function

$$V(x,i)=\begin{cases}\beta_1x^2, & \text{if } i=1,\\ \beta_2x^2, & \text{if } i=2.\end{cases} \tag{86}$$

Then

$$\begin{aligned} &LV(x,y,i) \\ &=\begin{cases}(2\beta_2-8\beta_1)x^2+\dfrac{1}{10}\displaystyle\int_{|u|\le1}u^2\pi(du)\beta_1y^2, & \text{if } i=1,\\ (2\beta_1-4\beta_2)x^2+\dfrac{1}{6}\displaystyle\int_{|u|\le1}u^2\pi(du)\beta_2y^2, & \text{if } i=2.\end{cases} \end{aligned} \tag{87}$$

From the properties of the lognormal distribution, we have

$$LV(x,y,1)\le-\left(8-\frac{2\beta_2}{\beta_1}\right)\beta_1x^2+\frac{1}{5}e^2\beta_1y^2, \tag{88}$$

$$LV(x,y,2)\le-\left(4-\frac{2\beta_1}{\beta_2}\right)\beta_2x^2+\frac{1}{3}e^2\beta_2y^2. \tag{89}$$

If $6/(12 - e^2) < (\beta_2/\beta_1) < 4 - (1/10)e^2$, then it follows that Assumptions 10 and 11 are satisfied. Consequently, the approximate solution (85) will converge to the true solution of (82) for any $t \in [0, T]$ in the sense of Theorem 12.

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