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Research Article

A Sum Operator Method for the Existence and Uniqueness of Positive Solutions to a System of Nonlinear Fractional Integral Equations

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This paper is concerned with the existence and uniqueness of positive solutions for a Volterra nonlinear fractional system of integral equations. Our analysis relies on a fixed point theorem of a sum operator. The conditions for the existence and uniqueness of a positive solution to the system are established. Moreover, an iterative scheme is constructed for approximating the solution. The case of quadratic system of fractional integral equations is also considered.

1. Introduction

Fractional calculus has been used for the study of problems in various fields of sciences, such as Abel integral equation and viscoelasticity, analysis of feedback amplifiers, capacitor theory, fractances, generalized voltage dividers, and engineering and biological sciences. In [1], Kilbas et al. give a survey of research in fractional calculus and its applications in mathematical analysis such as ODEs, PDEs, convolution integral equations, and theory of generating equations. Particularly, fractional differential equations have successful applications in nonlinear oscillation analysis of earthquakes, seepage flow in porous media [2], and fluid dynamic models for traffic flow [3], as the fractional derivatives can eliminate the deficiency of continuum traffic flow.

Open problems in this field are finding easy and effective methods for solving the equations. In recent years, many techniques of functional analysis, such as the fixed point theory, the Banach contraction principle, and the Leray-Schauder theory, are applied for solving the nonlinear fractional differential equations [4–11]. Iterative techniques [12–14] and the upper and lower solution method [15, 16] are also introduced to investigate the existence and uniqueness of the solutions to nonlinear fractional order differential equations with various boundary conditions.

Recently, prompted by the applications in physics, the following nonlinear quadratic system of integral equations and its generalizations have provoked some interest:

$$1 = \varphi_i(t) + \lambda_i \varphi_i(t) \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \varphi_i(s)\, ds, \quad \alpha_i \in (0,1),\ i = 1, 2, \ldots, n. \tag{1}$$

Salem [17] applied Krasnoselskii's fixed point theorem to obtain the existence of solutions for the system:

$$x_i(t) = \varphi_i(t) + \lambda_i I^{\alpha_i} \left[f_i(x(t)) + g_i(x(t)) \right], \quad t \in [0,1], \alpha_i \in (0,1), 1 \le i \le n, \tag{2}$$

under the assumptions that $f_i : [0,\infty)^n \to [0,\infty)$ is continuous nondecreasing for all variables, and $g_i : [0,\infty)^n \to [0,\infty)$ is continuous nonincreasing for all variables, where $[0,\infty)^n$ denotes the n-products $[0,\infty) \times [0,\infty) \cdots \times [0,\infty)$ and $x = (x_1, x_2, \ldots, x_n)$. For the physical point of view, only positive solutions are interesting. A simple form of the system (2):

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x(s))\,ds, \tag{3}$$

has been studied in [18, 19].

The aim of this paper is to study the existence and uniqueness of positive solutions for the following Volterra nonlinear fractional system of integral equations:

$$\begin{gathered} x_i(t) = \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha)} \left[f_i(s, x(s)) + g_i(s, x(s))\right] ds, \\ t \in [0,1], \alpha_i \in (0,1), 1 \le i \le n. \end{gathered} \tag{4}$$

Our main interest is to give some alternative answers to the main results of papers [17–19]. By using a fixed point theorem of a sum operator, we not only obtain the existence and uniqueness of positive solutions for the system (4), but also construct some sequences for approximating the unique solution.

2. Basic Definitions and Preliminaries

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proofs of our main results.

Definition 1 (see [1]). The fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{5}$$

provided that the right-hand side is defined pointwisely on $(0, +\infty)$, and $\Gamma(\alpha)$ denotes the gamma function.

Suppose that E is a real Banach space which is partially ordered by a cone $P \subset E$; that is, $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \ge 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

Let $P^{\circ} = \{x \in P \mid x \text{ is an interior point of } P\}$, and then a cone P is said to be solid if P° is nonempty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \le x \le x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \le y$ implies $Ax \le Ay (Ax \ge Ay)$. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2. Let $D = P$ or $D = P^{\circ}$ and γ be a real number with $0 \le \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies

$$A(tx) \ge t^{\gamma} Ax, \quad \forall t \in (0,1), \; x \in D. \tag{6}$$

Definition 3. An operator $A : E \to E$ is said to be homogeneous if it satisfies

$$A(tx) = tAx, \quad \forall t > 0, \; x \in E. \tag{7}$$

An operator $A : P \to P$ is said to be subhomogeneous if it satisfies

$$A(tx) \ge tAx, \quad \forall t \in (0,1), \; x \in P. \tag{8}$$

In the recent paper [20], Zhai and Anderson considered the following sum operator equation:

$$Ax + Bx + Cx = x, \tag{9}$$

where A is an increasing γ-concave operator, B is an increasing subhomogeneous operator, and C is a homogeneous operator. They established the existence and uniqueness of positive solutions for the above equation, and when C is a null operator, they present the following interesting result.

Lemma 4 (see [20]). *Let P be a normal cone in a real Banach space E, let $A : P \to P$ be an increasing γ-concave operator, and let $B : P \to P$ be an increasing subhomogeneous operator. Assume that*

(1) *there is $h > \theta$ such that $Ah \in P_h$, $Bh \in P_h$;*

(2) *there exists a constant $\delta > 0$ such that $Ax \ge \delta Bx$, for all $x \in P$.*

Then the operator equation $Ax + Bx = x$, has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$. for any initial value $y_0 \in P_h$, we have $y_n \to x^*$, as $n \to \infty$.

3. Main Results

In this section, we apply Lemma 4 to study problem (4), and we obtain some new results on the existence and uniqueness of positive solutions.

Now by $C[0,1]$, we mean the Banach space of continuous functions on $[0,1]$ with the usual max-norm $\|\cdot\|$. Also, recall the Banach space of the cartesian product $E =: C[0,1] \times C[0,1] \times \cdots \times C[0,1]$ equipped by the norm $\|x\| =: \max_{1 \le i \le n} \|x_i\|$. Notice that this space can be equipped with a partial order:

$$\begin{gathered} x, y \in E, \; x \le y \Longleftrightarrow x_i(t) \le y_i(t), \\ t \in [0,1], \; i = 1, 2, \ldots, n. \end{gathered} \tag{10}$$

Set $P = \{x \in E \mid x(t) \ge 0, \; t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in E and the normality constant is 1. Take $h(t) = (h_1(t), h_2(t), \ldots, h_n(t))$ and $h_i(t) = t^{\alpha_i}$,

$$P_h = \{x \in P \mid x \sim h\}. \tag{11}$$

Theorem 5. *Assume that*

(S1) *for all i, $f_i, g_i : [0,1] \times [0,\infty)^n \to [0,\infty)$ are continuous and increasing with respect to the arguments x_i, and $g_i(t, 0, 0, \ldots, 0) > 0$ for any $t \in [0,1]$;*

(S2) *for all* i, $g_i(t, x_1, x_2, \ldots, \tau x_i, \ldots, x_n) \geq \tau g_i(t, x_1, x_2, \ldots, x_i, \ldots, x_n)$ *for* $\tau \in (0, 1)$, $t \in [0, 1]$, $x_i \in [0, +\infty)$ *and there exist constants* $\gamma_i \in (0, 1)$ *such that*

$$f_i(t, x_1, x_2, \ldots, \tau x_i, \ldots, x_n) \geq \tau^{\gamma} f_i(t, x_1, x_2, \ldots, x_i, \ldots, x_n) \tag{12}$$

for $\tau \in (0, 1)$, $t \in [0, 1]$, $x_i \in [0, +\infty)$, $i = 1, 2, \ldots, n$;

(S3) *there exists* $\delta_i > 0$ *such that*

$$f_i(t, x_1, x_2, \ldots, x_i, \ldots, x_n) \geq \delta_i g_i(t, x_1, x_2, \ldots, x_i, \ldots, x_n), \quad t \in [0, 1], \; x_i \geq 0, \; i = 1, 2, \ldots, n. \tag{13}$$

Then problem (4) *has a unique positive solution* x^* *in* P_h. *Moreover, for any initial value* $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in P_h$, *constructing successively the sequence*

$$\begin{aligned} x_i^{(m+1)}(t) &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times \left[f_i\left(s, x_1^{(m)}(s), x_2^{(m)}(s), \ldots, x_i^{(m)}(s), \ldots, x_n^{(m)}(s)\right) \right. \\ &\quad \left. + g_i\left(s, x_1^{(m)}(s), x_2^{(m)}(s), \ldots, x_i^{(m)}(s), \ldots, x_n^{(m)}(s)\right)\right] ds, \\ &\quad m = 0, 1, \ldots, \end{aligned} \tag{14}$$

then $x^{(m)} \to x^*$ *as* $m \to \infty$.

Proof. To begin with, we define the following operators $A, B : P \to E$ by

$$\begin{aligned} Ax &= (A_1 x_1, A_2 x_2, \ldots, A_n x_n), \\ Bx &= (B_1 x_1, B_2 x_2, \ldots, B_n x_n), \end{aligned} \tag{15}$$

where

$$\begin{aligned} A_i x_i &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(s, x(s)) \, ds, \\ B_i x_i &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x(s)) \, ds. \end{aligned} \tag{16}$$

Thus x is the positive solution of problem (4) if and only if $x = Ax + Bx$. From (S0) and (S1), we know that $A : P \to P$, $B : P \to P$. In the sequel we check that A, B satisfy all assumptions of Lemma 4.

Firstly, we prove that A, B are two increasing operators. In fact, by (S0) and (S1), for $x, y \in P$ with $x \geq y$, we know that $x_i(t) \geq y_i(t)$, $t \in [0, 1]$, $i = 1, 2, \ldots, n$, and obtain

$$\begin{aligned} A_i x_i &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times f_i(s, x_1(s), x_2(s), \ldots, x_i(s), \ldots, x_n(s)) \, ds \\ &\geq \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times f_i(s, y_1(s), y_2(s), \ldots, y_i(s), \ldots, y_n(s)) \, ds \\ &= A_i y_i; \end{aligned} \tag{17}$$

that is, $Ax \geq Ay$. Similarly, $Bx \geq By$.

Next we show that A is a γ-concave operator and B is a subhomogeneous operator. In fact, for any $\tau \in (0, 1)$ and $x \in P$, by (S1), we obtain

$$\begin{aligned} A_i(\tau x_i)(t) &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times f_i(s, x_1(s), x_2(s), \ldots, \tau x_i(s), \ldots, x_n(s)) \, ds \\ &\geq \tau^{\gamma_i} \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times f_i(s, x_1(s), x_2(s), \ldots, x_i(s), \ldots, x_n(s)) \, ds \\ &= \tau^{\gamma_i} A_i x_i(t). \end{aligned} \tag{18}$$

Consequently, $A(\tau x)(t) \geq \tau^{\gamma} Ax$, where $\gamma = \max_{1 \leq i \leq n} \gamma_i$. Also, for any $\tau \in (0, 1)$ and $x \in P$, by (S0) and (S1), we obtain

$$\begin{aligned} B_i(\tau x_i)(t) &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times g_i(s, x_1(s), x_2(s), \ldots, \tau x_i(s), \ldots, x_n(s)) \, ds \\ &\geq \tau \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times g_i(s, x_1(s), x_2(s), \ldots, x_i(s), \ldots, x_n(s)) \, ds \\ &= \tau B_i x_i(t); \end{aligned} \tag{19}$$

that is, $B(\tau x) \geq \tau Bx$ for $\tau \in (0, 1)$, $x \in P$. So the operator B is a subhomogeneous operator.

Now we show that $Ah \in P_h$, $Bh \in P_h$. In fact, by (S3), we have

$$f_i(s, 1, 1, \ldots, 1) \geq f_i(s, 0, 0, \ldots, 0) \geq \delta_i g_i(s, 0, 0, \ldots, 0) > 0, \tag{20}$$

and thus take

$$M_i = \max_{0\le s\le 1} f_i(s,1,1,\ldots,1), \qquad m_i = \min_{0\le s\le 1} f_i(s,0,0,\ldots,0); \tag{21}$$

then $M_i, m_i > 0$. Let

$$\lambda = \min_{1\le i\le n}\left\{\frac{m_i}{\alpha_i\Gamma(\alpha_i)}\right\}, \qquad \mu = \max_{1\le i\le n}\left\{\frac{M_i}{\alpha_i\Gamma(\alpha_i)}\right\}. \tag{22}$$

It follows from $(S1)$ that

$$\begin{aligned}
A_ih_i(t) &= \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i(s,s^{\alpha_1},s^{\alpha_2},\ldots,s^{\alpha_i},\ldots,s^{\alpha_n})\,ds\\
&\ge \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i(s,0,0,\ldots,0)\,ds\\
&\ge m_i\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}ds\\
&= \frac{m_i}{\alpha_i\Gamma(\alpha_i)}t^{\alpha_i} \ge \lambda t^{\alpha_i} = \lambda h_i(t),\\
A_ih_i(t) &= \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i(s,s^{\alpha_1},s^{\alpha_2},\ldots,s^{\alpha_i},\ldots,s^{\alpha_n})\,ds\\
&\le \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_i(s,1,1,\ldots,1)\,ds\\
&\le M_i\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}ds\\
&= \frac{M_i}{\alpha_i\Gamma(\alpha_i)}t^{\alpha_i} \le \mu t^{\alpha_i} = \mu h_i(t).
\end{aligned} \tag{23}$$

So $\lambda h_i(t) \le A_ih_i(t) \le \mu h_i(t)$, and then $\lambda h(t) \le Ah(t) \le \mu h(t)$, hence $Ah \in P_h$. Similarly, from $g_i(s,0,0,\ldots,0) > 0$ and $(S1)$–$(S3)$, we easily prove $Bh \in P_h$. Hence the condition (1) of Lemma 4 is satisfied.

In the following we show that the condition (2) of Lemma 4 is satisfied. For $x \in P$, from $(S3)$, we have

$$\begin{aligned}
A_ix_i(t) &= \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\\
&\quad\times f_i(s,x_1(s),x_2(s),\ldots,x_i(s),\ldots,x_n(s))\,ds\\
&\ge \delta_i\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\\
&\quad\times g_i(s,x_1(s),x_2(s),\ldots,x_i(s),\ldots,x_n(s))\,ds\\
&= \delta_iB_ix_i(t).
\end{aligned} \tag{24}$$

Take

$$\delta = \min_{1\le i\le n}\delta_i, \tag{25}$$

and then we have $Ax \ge \delta Bx$, $x \in P$. By Lemma 4, the operator equation $Ax+Bx = x$ has a unique solution $x^* \in P_h$; of course, x^* is also a unique solution of problem (4). In addition, by $(S1)$ we know that the unique solution is also positive.

Now for any initial value $x^{(0)} = (x_1^{(0)}, x_2^{(0)},\ldots,x_n^{(0)}) \in P_h$, let us construct successively the sequence

$$x_i^{(m)} = A_ix_i^{(m-1)} + B_ix_i^{(m-1)}, \quad m = 1,2,\ldots, \tag{26}$$

and we have $x_i^{(m)} \to x_i^*$ as $m \to \infty$, and then problem (4) has a unique positive solution $x^{(m)} \to x^*$ in P_h; that is, for any initial value $x^{(0)} = (x_1^{(0)}, x_2^{(0)},\ldots,x_n^{(0)}) \in P_h$, constructing successively the sequence:

$$\begin{aligned}
x_i^{(m+1)}(t) &= \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\\
&\quad\times\Big[f_i\big(s,x_1^{(m)}(s),x_2^{(m)}(s),\ldots,x_i^{(m)}(s),\ldots,x_n^{(m)}(s)\big)\\
&\qquad+g_i\big(s,x_1^{(m)}(s),x_2^{(m)}(s),\ldots,\\
&\qquad\qquad x_i^{(m)}(s),\ldots,x_n^{(m)}(s)\big)\Big]\,ds,\\
&\qquad\qquad m = 0,1,\ldots,
\end{aligned} \tag{27}$$

then $x^{(m)} \to x^*$ as $m \to \infty$. □

Corollary 6. *Assume that*

- $(A1)$ *for all* i, $f_i : [0,1]\times[0,\infty)^n \to [0,\infty)$ *is continuous and increasing with respect to the arguments* x_i, *and* $f_i(t,0,0,\ldots,0) > 0$ *for any* $t \in [0,1]$;
- $(A2)$ *for all* i, $i = 1,2,\ldots,n$, *there exists constant* $\gamma_i \in (0,1)$ *such that*

$$f_i(t,x_1,x_2,\ldots,\tau x_i,\ldots,x_n) \ge \tau^{\gamma} f_i(t,x_1,x_2,\ldots,x_i,\ldots,x_n) \tag{28}$$

for $\tau \in (0,1)$, $t \in [0,1]$, $x_i \in [0,+\infty)$.

Then the problem

$$x_i(t) = \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha)} f_i(s,x(s))\,ds, \qquad t \in [0,1],\ \alpha_i \in (0,1),\ 1 \le i \le n, \tag{29}$$

has a unique positive solution x^* in P_h. Moreover, for any initial value $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in P_h$, constructing successively the sequence

$$x_i^{(m+1)}(t) = \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \times f_i\left(s, x_1^{(m)}(s), x_2^{(m)}(s), \ldots, x_i^{(m)}(s), \ldots, x_n^{(m)}(s)\right) ds, \quad m = 0, 1, \ldots, \tag{30}$$

then $x^{(m)} \to x^*$ as $m \to \infty$.

In what follows, we establish the existence and uniqueness of positive solutions for the following system of quadratic integral equations of the fractional type:

$$\varphi_i(t) = \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha)} \varphi_i^2(s)\, ds + 1, \quad t \in [0,1], \alpha_i \in (0,1), 1 \le i \le n. \tag{31}$$

Corollary 7. *The system* (31) *has a unique positive solution.*

Proof. Let $f_i(x_1, x_2, \ldots, x_n) = (x_i + 1)^2$, and then f_i satisfies $(A1)$ and $(A2)$ of Corollary 6. Thus let $x = (x_1, x_2, \ldots, x_n)$ be the unique positive solution of (29), and then we have

$$x_i(t) = \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha)} (x_i(s) + 1)^2 ds; \tag{32}$$

that is

$$x_i(t) + 1 = \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha)} (x_i(s) + 1)^2 ds + 1. \tag{33}$$

Let $\varphi_i = x_i + 1$, and the $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a unique positive solution of (31). □

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