

Research Article

Degree of Approximation by Hybrid Operators

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We consider hybrid (Szász-beta) operators, which are a general sequence of integral type operators including beta function, and we give the degree of approximation by these Szász-beta-Durrmeyer operators.

1. Introduction

The Lupaş-Durrmeyer operators were introduced by Sahai and Prasad [1] who studied the asymptotic formula for simultaneous approximation, and many mathematicians have given different results for the Durrmeyer operators (see [2–6]). Now we consider here a sequence of linear positive operators, which was introduced by Gupta et al. [7] as follows. Let n and β be positive integers. For $f \in C[0, \infty)$ satisfying $\int_0^\infty f(t)/(1+t)^{n+\beta+1} dt < \infty$,

$$B_{n,\beta}[f](x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta,k}(t) f(t) dt, \quad (1)$$

$$x \in [0, \infty), \quad n + \beta + 1 > 0,$$

where β is a positive integer,

$$\begin{aligned} s_{n,k}(x) &= e^{-nx} \frac{(nx)^k}{k!}, & b_{n,k}(t) &= \frac{1}{B(k+1, n)} \frac{t^k}{(1+t)^{n+k+1}}, \\ B(k+1, n) &= \frac{\Gamma(k+1) \Gamma(n)}{\Gamma(k+n+1)}. \end{aligned} \quad (2)$$

Let $0 < p \leq \infty$. For a function f on $[0, \infty)$, we define the norm by

$$\|f\|_{L_p([0, \infty))} = \begin{cases} \left(\int_{[0, \infty)} |f(t)|^p dt \right)^{1/p}, & 0 < p < \infty, \\ \sup_{[0, \infty)} |f(t)|, & p = \infty. \end{cases} \quad (3)$$

Recently Jung and Sakai [8] investigated the Lupaş-Durrmeyer operators and studied the circumstances of convergence. Motivated with the idea of Jung and Sakai [8], we give the degree of approximation by Szász-Beta-Durrmeyer operators in this paper.

2. Basic Results

Lemma 1 (cf. [7]). *Let α, m, n , and r be integers with $m \geq 0$, $r \geq 1$, and $n + \alpha > m$:*

$$R_{n,m,r}(\alpha; x) := \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\alpha,k+r}(y) (y-x)^m dy. \quad (4)$$

Then one has

$$(i) R_{n,0,r}(\alpha; x) = 1 \text{ and } R_{n,1,r}(\alpha; x) = ((-\alpha+1)x + r + 1)/(n+\alpha-1),$$

(ii) for $m \geq 1$

$$\begin{aligned} (n + \alpha - m - 1) R_{n,m+1,r}(\alpha; x) &= x R'_{n,m,r}(\alpha; x) \\ &+ ((2m - \alpha + 1)x + m + r + 1) R_{n,m,r}(\alpha; x) \quad (5) \\ &+ mx(x + 2) R_{n,m-1,r}(x), \end{aligned}$$

(iii)

$$R_{n,m,r}(\alpha; x) = O\left(\frac{1}{n^{[(m+1)/2]}}\right) g_{n,m,r}(\alpha; x), \quad (6)$$

where $g_{n,m,r}(\alpha; x)$ is a polynomial of degree $\leq m$ such that the coefficients of $g_{n,m,r}(\alpha; x)$ are bounded independently of n .

Proof. Let $R_{n,m,r}(x) := R_{n,m,r}(\alpha; x)$. Then (i)

$$R_{n,0,r}(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y) dy = \sum_{k=0}^{\infty} s_{n,k}(x) = 1. \quad (7)$$

Using

$$\begin{aligned} \int_0^{\infty} x b_{n,k}(x) dx &= \frac{k+1}{n-1}, \\ \sum_{k=0}^{\infty} k s_{n,k}(x) &= nx, \end{aligned} \quad (8)$$

we see that

$$\begin{aligned} R_{n,1,r}(x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y) (y-x) dy \\ &= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k+r+1}{n+\alpha-1} - x = \frac{(-\alpha+1)x+r+1}{n+\alpha-1}. \end{aligned} \quad (9)$$

(ii) Using $x s'_{n,k}(x) = (k-nx) s_{n,k}(x)$, we obtain

$$\begin{aligned} x(R'_{n,m,r}(x) + m R_{n,m-1,r}(x)) &= \sum_{k=0}^{\infty} x s'_{n,k}(x) \int_0^{\infty} b_{n+\alpha,k+r}(y) (y-x)^m dy \\ &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} (k-nx) b_{n+\alpha,k+r}(y) (y-x)^m dy. \end{aligned} \quad (10)$$

Since we know that

$$y(1+y) b'_{n+\alpha,k+r}(y) = (k+r-(n+\alpha+1)y) b_{n+\alpha,k+r}(y),$$

$$\begin{aligned} k-nx &= (k+r-(n+\alpha+1)y) \\ &- (r-(\alpha+1)x) + (n+\alpha+1)(y-x), \end{aligned} \quad (11)$$

we have

$$\begin{aligned} (k-nx) b_{n+\alpha,k+r}(y) &= y(1+y) b'_{n+\alpha,k+r}(y) - (r-(\alpha+1)x) b_{n+\alpha,k+r}(y) \\ &+ (n+\alpha+1) b_{n+\alpha,k+r}(y) (y-x). \end{aligned} \quad (12)$$

Then substituting (12) into (10), we consider the following:

$$\begin{aligned} x(R'_{n,m,r}(x) + m R_{n,m-1,r}(x)) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} (y(1+y) b'_{n+\alpha,k+r}(y) \\ &- (r-(\alpha+1)x) b_{n+\alpha,k+r}(y) \\ &+ (n+\alpha+1) b_{n+\alpha,k+r}(y) (y-x)) \\ &\times (y-x)^m dy \\ &:= A_1 + A_2 + A_3. \end{aligned} \quad (13)$$

Then since we see

$$\begin{aligned} \int_0^{\infty} y(1+y) b'_{n+\alpha,k+r}(y) (y-x)^m dy &= \int_0^{\infty} ((y-x)^2 + (1+2x)(y-x) + x(1+x)) \\ &\times b'_{n+\alpha,k+r}(y) (y-x)^m dy \\ &= \int_0^{\infty} b'_{n+\alpha,k+r}(y) (y-x)^{m+2} dy \\ &+ (1+2x) \int_0^{\infty} b'_{n+\alpha,k+r}(y) (y-x)^{m+1} dy \\ &+ x(1+x) \int_0^{\infty} b'_{n+\alpha,k+r}(y) (y-x)^m dy \quad (14) \\ &= -(m+2) \int_0^{\infty} b_{n+\alpha,k+r}(y) (y-x)^{m+1} dy \\ &- (1+2x)(m+1) \int_0^{\infty} b_{n+\alpha,k+r}(y) (y-x)^m dy \\ &- x(1+x)m \int_0^{\infty} b_{n+\alpha,k+r}(y) (y-x)^{m-1} dy, \end{aligned}$$

we have

$$\begin{aligned} A_1 &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} y(1+y) b'_{n+\alpha,k+r}(y) (y-x)^m dy \\ &= -(m+2) R_{n,m+1,r}(x) - (1+2x)(m+1) R_{n,m,r}(x) \\ &- x(1+x)m R_{n,m-1,r}(x). \end{aligned} \quad (15)$$

Here the last equation follows from integration by parts. Furthermore, we easily see

$$\begin{aligned} A_2 + A_3 &= - (r - (\alpha + 1)x) R_{n,m,r}(x) \\ &\quad + (n + \alpha + 1) R_{n,m+1,r}(x). \end{aligned} \quad (16)$$

Therefore, we conclude

$$\begin{aligned} x(R'_{n,m,r}(x) + mR_{n,m-1,r}(x)) &= - (m + 2) R_{n,m+1,r}(x) - (1 + 2x)(m + 1) R_{n,m,r}(x) \\ &\quad - x(1 + x)mR_{n,m-1,r}(x) \\ &\quad - (r - (\alpha + 1)x) R_{n,m,r}(x) + (n + \alpha + 1) R_{n,m+1,r}(x) \\ &= (n + \alpha - m - 1) R_{n,m+1,r}(x) \\ &\quad - ((2m - \alpha + 1)x + m + r + 1) R_{n,m,r}(x) \\ &\quad - x(1 + x)mR_{n,m-1,r}(x). \end{aligned} \quad (17)$$

Consequently, (ii) is proved.

(iii) For $m = 1$, (6) holds. Let us assume (6) for $m \geq 1$. We note

$$\begin{aligned} R'_{n,m,r}(x) &= O\left(\frac{1}{n^{[(m+1)/2]}}\right) g'_{n,m,r}(x), \\ g'_{n,m,r}(x) &\in \mathcal{P}_{m-1}. \end{aligned} \quad (18)$$

So, we have, by the assumption of induction,

$$\begin{aligned} (n + \alpha - m - 1) R_{n,m+1,r}(\alpha; x) &= O\left(\frac{1}{n^{[(m+1)/2]}}\right) x g'_{n,m,r}(x) \\ &\quad + ((2m - \alpha + 1)x + m + r + 1) O\left(\frac{1}{n^{[(m+1)/2]}}\right) \\ &\quad \times g_{n,m,r}(x) \\ &\quad + mx(x + 2) O\left(\frac{1}{n^{[m/2]}}\right) g_{n,m-1,r}(x). \end{aligned} \quad (19)$$

Here, if m is even, then

$$\begin{aligned} \left[\frac{m+1}{2}\right] + 1 &= \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \\ \left[\frac{m}{2}\right] + 1 &= \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \end{aligned} \quad (20)$$

and if m is odd, then

$$\begin{aligned} \left[\frac{m+1}{2}\right] + 1 &= \frac{m+1}{2} + 1 = \left[\frac{m+2}{2}\right], \\ \left[\frac{m}{2}\right] + 1 &= \frac{m-1}{2} + 1 = \frac{m+1}{2} = \left[\frac{m+2}{2}\right]. \end{aligned} \quad (21)$$

Hence we have

$$R_{n,m+1,r}(x) = O\left(\frac{1}{n^{[(m+2)/2]}}\right) g_{n,m+1,r}(x), \quad (22)$$

and here we see that $g_{n,m+1,r}(x)$ is a polynomial of degree $\leq m + 1$ such that the coefficients of $g_{n,m+1,r}(x)$ are bounded independently of n . \square

Lemma 2 (cf. [7]). *Let n, β , and r be integers with $r \geq 0$. Let $f \in C^{(r)}[0, \infty)$ satisfy for a positive integer δ*

$$|f^{(r)}(x)| \leq O(1)(x + 1)^\delta. \quad (23)$$

Then one has, for $n + \beta - r > \delta$,

$$\begin{aligned} \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) f^{(r)}(y) dy, \end{aligned} \quad (24)$$

where

$$\lambda_{n,\beta,r} := \frac{(n + \beta - 1)!}{n^r (n + \beta - r - 1)!}. \quad (25)$$

Proof. Using

$$b_{n+\beta-r,k+r}^{(r)}(y) = \frac{(n + \beta - 1)!}{(n + \beta - r - 1)!} \sum_{i=0}^r \binom{r}{i} (-1)^i b_{n+\beta,k+i}(y), \quad (26)$$

we have

$$\begin{aligned} (B_{n,\beta}[f])^{(r)}(x) &= \sum_{k=0}^{\infty} s_{n,k}^{(r)}(x) \int_0^{\infty} b_{n+\beta,k}(y) f(y) dy \\ &= \sum_{i=0}^r \sum_{k=i}^{\infty} \binom{r}{i} (-1)^r (-1)^i n^r s_{n,k-i}(x) \\ &\quad \times \int_0^{\infty} b_{n+\beta,k}(y) f(y) dy \\ &= n^r \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} \sum_{i=0}^r \binom{r}{i} \\ &\quad \times (-1)^r (-1)^i b_{n+\beta,k+i}(y) f(y) dy \\ &= n^r \frac{(n + \beta - r - 1)!}{(n + \beta - 1)!} \sum_{k=0}^{\infty} s_{n,k}(x) \\ &\quad \times \int_0^{\infty} (-1)^r b_{n+\beta-r,k+r}^{(r)}(y) f(y) dy \\ &= \frac{1}{\lambda_{n,\beta,r}} \sum_{k=0}^{\infty} s_{n,k}(x) \\ &\quad \times \int_0^{\infty} b_{n+\beta-r,k+r}(y) f^{(r)}(y) dy. \end{aligned} \quad (27)$$

\square

3. Main Results

Theorem 3. Let $0 < p \leq \infty$, and let δ and r be nonnegative integers. Let n and β be integers with $n + \beta - r > \delta$. Let $f \in C^{(r+1)}[0, \infty)$ satisfy

$$\begin{aligned} |f^{(r)}(x)| &\leq O(1)(x+1)^\delta, \\ |f^{(r+1)}(x)| &\leq O(1)(x+1)^{\delta+2}. \end{aligned} \quad (28)$$

Then one has uniformly, for f and n ,

$$|\lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x)| \leq O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\delta+2}. \quad (29)$$

Proof. Let $|t-x| < \varepsilon$ and $x < \xi < t$. By the second inequality of (28),

$$\begin{aligned} |f^{(r)}(t) - f^{(r)}(x)| &= |t-x| |f^{(r+1)}(\xi)| \\ &\leq O(1) |t-x| (x+1)^{\delta+2}. \end{aligned} \quad (30)$$

Let $\varepsilon = n^{-\nu}$, $0 < \nu < 1$. Then using Lemma 2 and

$$\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) f^{(r)}(x) dy = f^{(r)}(x), \quad (31)$$

we have

$$\begin{aligned} &|\lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x)| \\ &= \left| \sum_{k=0}^{\infty} s_{n,k}(x) \left(\int_{|t-x|<\varepsilon} b_{n+\beta-r,k+r}(t) |f^{(r)}(t) - f^{(r)}(x)| dt \right. \right. \\ &\quad \left. \left. + \int_{|t-x|\geq\varepsilon} b_{n+\beta-r,k+r}(t) |f^{(r)}(t) - f^{(r)}(x)| dt \right) \right| \\ &= E_1 + E_2. \end{aligned} \quad (32)$$

From (30) and Lemma 1, we have

$$\begin{aligned} E_1 &= O(1) \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_{|t-x|<\varepsilon} b_{n+\beta-r,k+r}(t) \right. \\ &\quad \times |t-x| (x+1)^{\delta+2} dt \left. \right| \\ &\leq O(1) \varepsilon |R_{n,0,r}(\beta-r; x)| (x+1)^{\delta+2} = O(1) \varepsilon (x+1)^{\delta+2}. \end{aligned} \quad (33)$$

Next, we estimate E_2 . By the use of the first inequality in (28), we have

$$\begin{aligned} E_2 &= C \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_{|t-x|\geq\varepsilon} b_{n+\beta-r,k+r}(t) \right. \\ &\quad \times (|f^{(r)}(t)| + |f^{(r)}(x)|) dt \left. \right| \\ &\leq C \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_{|t-x|\geq\varepsilon} b_{n+\beta-r,k+r}(t) \right. \\ &\quad \times ((t+1)^\delta + (x+1)^\delta) dt \left. \right|. \end{aligned} \quad (34)$$

Now using $(t+1)^\delta = ((t-x) + x+1)^\delta = \sum_{i=0}^{\delta} \binom{\delta}{i} (t-x)^i (x+1)^{\delta-i}$ and the notation

$$\langle i \rangle = \begin{cases} 1, & i : \text{odd}, \\ 0, & i : \text{even}, \end{cases} \quad (35)$$

we have

$$\begin{aligned} E_2 &\leq C \sum_{k=0}^{\infty} s_{n,k}(x) \int_{|t-x|\geq\varepsilon} b_{n+\beta-r,k+r}(t) \\ &\quad \times \left(\sum_{i=0}^{\delta} \binom{\delta}{i} |t-x|^i \left| \frac{t-x}{\varepsilon} \right|^{\langle i \rangle} (x+1)^{\delta-i} \right) dt \\ &\quad + C \sum_{k=0}^{\infty} s_{n,k}(x) \int_{|t-x|\geq\varepsilon} b_{n+\beta-r,k+r}(t) \left(\frac{t-x}{\varepsilon} \right)^2 (x+1)^\delta dt \\ &= E_{21} + E_{22}. \end{aligned} \quad (36)$$

Then, with $\varepsilon = n^{-\nu}$,

$$\begin{aligned} E_{21} &\leq C \left(\sum_{i=1}^{\delta} \binom{\delta}{i} |R_{n,i+\langle i \rangle,r}(\beta-r; x)| \left(\frac{1}{\varepsilon} \right)^{\langle i \rangle} (x+1)^{\delta-i} \right) \\ &\leq C \sum_{i=1}^{\delta} \binom{\delta}{i} O\left(\frac{n^{\nu\langle i \rangle}}{n^{[(i+\langle i \rangle+1)/2]}}\right) \\ &\quad \times |g_{n,i+\langle i \rangle,r}(\beta-r; x)| (x+1)^{\delta-i} \\ &\leq O\left(\frac{1}{n^{[(i+\langle i \rangle+1)/2]-\nu\langle i \rangle}}\right) (x+1)^{\delta+1} \leq O\left(\frac{1}{n^{1-\nu}}\right) (x+1)^{\delta+1}. \end{aligned} \quad (37)$$

Here for $i \geq 1$, we get

$$\left[\frac{i+\langle i \rangle+1}{2} \right] - \nu \langle i \rangle \geq 1 - \nu, \quad (38)$$

because

$$\begin{aligned} \left[\frac{i + \langle i \rangle + 1}{2} \right] - \nu \langle i \rangle &= \begin{cases} \frac{i+1}{2} - \nu, & i : \text{odd}, \\ \frac{i}{2} - \nu, & i : \text{even}, \end{cases} \\ E_{22} &\leq C |R_{n,2,r}(\beta - r; x)| \left(\frac{1}{\varepsilon} \right)^2 (x+1)^\delta \quad (39) \\ &\leq O\left(\frac{1}{n^{[3/2]}} \right) |g_{n,2,r}(\beta - r; x)| \left(\frac{1}{\varepsilon} \right)^2 (x+1)^\delta \\ &\leq O\left(\frac{n^{2\nu}}{n^{[3/2]}} \right) |g_{n,2,r}(\beta - r; x)| (x+1)^\delta \\ &\leq O\left(\frac{1}{n^{1-2\nu}} \right) (x+1)^{\delta+2}. \end{aligned}$$

Finally we get

$$E_2 \leq O\left(\frac{1}{n^{1-2\nu}} \right) (x+1)^{\delta+2}. \quad (40)$$

From (32),

$$\begin{aligned} &\left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \\ &\leq O\left(\frac{1}{n^\nu} \right) (x+1)^{\delta+2} + O\left(\frac{1}{n^{1-2\nu}} \right) (x+1)^{\delta+2}. \end{aligned} \quad (41)$$

If we put $\nu = 1/3$, then we get

$$\begin{aligned} &\left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \leq O\left(\frac{1}{n^{1/3}} \right) (x+1)^{\delta+2}. \quad (42) \\ &\square \end{aligned}$$

In the following, we let $\phi(t) = 1/(1+t)$, $t \in [0, \infty)$.

Theorem 4. Let r and γ be nonnegative integers. Let n and β be integers with $n + \beta - r > 2\gamma + 1$. Let $f \in C^{(r+2)}[0, \infty)$ satisfy

$$\begin{aligned} &\|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\|_{L_\infty[0,\infty)} < \infty, \\ &\|f^{(r+2)}(x)\phi^{2\gamma}(x)\|_{L_\infty[0,\infty)} < \infty. \end{aligned} \quad (43)$$

Then one has uniformly, for f and n ,

$$\begin{aligned} &\left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \phi^{2(\gamma+1)}(x) \\ &\leq O\left(\frac{1}{n} \right) \left(\|f^{(r+1)}(x)\phi^{2\gamma+1}(x)\|_{L_\infty[0,\infty)} \right. \\ &\quad \left. + \|f^{(r+2)}(x)\phi^{2\gamma}(x)\|_{L_\infty[0,\infty)} \right). \end{aligned} \quad (44)$$

Proof. For $f \in C^{(r+2)}[0, \infty)$, we have

$$\begin{aligned} f^{(r)}(y) &= f^{(r)}(x) + f^{(r+1)}(x)(y-x) \\ &\quad + \int_x^y (y-u) f^{(r+2)}(u) du, \end{aligned} \quad (45)$$

$$\begin{aligned} &\left| \int_x^y (y-u) f^{(r+2)}(u) du \right| \\ &\leq C \|f^{(r+2)}(x)\phi^{2\gamma}(x)\|_{L_\infty[0,\infty)} \\ &\quad \times ((1+x)^{2\gamma} + (1+y)^{2\gamma})(y-x)^2. \end{aligned} \quad (46)$$

From (45), (46), and Lemma 2, we get

$$\begin{aligned} &\lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) \\ &= f^{(r)}(x) + f^{(r+1)}(x) R_{n,1,r}(\beta - r; x) \\ &\quad + \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta-r,k+r}(t) \\ &\quad \times \int_x^t (t-u) f^{(r+2)}(u) du dt, \\ &\left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta-r,k+r}(t) \int_x^t (t-u) f^{(r+2)}(u) du dt \right| \\ &\leq \|f^{(r+2)}(x)\phi^{2\gamma}(x)\|_{L_\infty[0,\infty)} \\ &\quad \times ((1+x)^{2\gamma} |R_{n,2,r}(\beta - r; x)| \\ &\quad + \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta-r,k+r}(t) (1+t)^{2\gamma} (t-x)^2 dt). \end{aligned} \quad (47)$$

Using $(1+t)^{2\gamma} \leq C((t-x)^{2\gamma} + (1+x)^{2\gamma})$, we obtain

$$\begin{aligned} &\left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty b_{n+\beta-r,k+r}(t) (1+t)^{2\gamma} (t-x)^2 dt \right| \\ &\leq C \left(|R_{n,2\gamma+2,r}(\beta - r; x)| + (1+x)^{2\gamma} |R_{n,2,r}(\beta - r; x)| \right). \end{aligned} \quad (49)$$

Therefore, we have

$$\begin{aligned} &\left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \phi^{2(\gamma+1)}(x) \\ &\leq |f^{(r+1)}(x)\phi^{2\gamma+1}(x)| |R_{n,1,r}(\beta - r; x)| \phi(x) \\ &\quad + C \|f^{(r+2)}(x)\phi^{2\gamma}(x)\|_{L_\infty[0,\infty)} (1+x)^{2\gamma} \\ &\quad \times |R_{n,2,r}(\beta - r; x)| \phi^{2\gamma+2}(x) \end{aligned}$$

$$\begin{aligned}
& + C \left\| f^{(r+2)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \\
& \times |R_{n,2(\gamma+1),r}(\beta-r;x)| \phi^{2(\gamma+1)}(x) \\
& \leq O\left(\frac{1}{n}\right) \left| f^{(r+1)}(x) \phi^{2\gamma+1}(x) \right| |g_{n,1,r}(\beta-r;x)| \phi(x) \\
& + O\left(\frac{1}{n}\right) \left\| f^{(r+2)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} (1+x)^{2\gamma} \\
& \times |g_{n,2,r}(\beta-r;x)| \phi^{2(\gamma+1)}(x) \\
& + O\left(\frac{1}{n^{\gamma+1}}\right) \left\| f^{(r+2)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \\
& \times |g_{n,2(\gamma+1),r}(\beta-r;x)| \phi^{2(\gamma+1)}(x).
\end{aligned} \tag{50}$$

For $x \in [0, \infty)$, we have $|g_{n,1,r}(\beta-r;x)|\phi(x) \leq C$, $(1+x)^{2\gamma}|g_{n,2,r}(\beta-r;x)|\phi^{2(\gamma+1)}(x) \leq C$, and $|g_{n,2(\gamma+1),r}(\beta-r;x)|\phi^{2(\gamma+1)}(x) \leq C$. Hence

$$\begin{aligned}
& \left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \phi^{2(\gamma+1)}(x) \\
& \leq O\left(\frac{1}{n}\right) \left(\left\| f^{(r+1)}(x) \phi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} \right. \\
& \quad \left. + \left\| f^{(r+2)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \right).
\end{aligned} \tag{51}$$

□

Let us define the weighted modulus of smoothness by

$$\omega_k(f; \eta; t) := \sup_{0 \leq h \leq t} \left\| \Delta_h^k f(\cdot) \eta(\cdot) \right\|_{L_\infty([0,\infty))}, \quad t \geq 0, k = 1, 2, \tag{52}$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x), \tag{53}$$

$$\Delta_h^2 f(x) = f(x) - 2f(x+h) + f(x+2h). \tag{54}$$

Theorem 5. Let r and γ be nonnegative integers. Let n and β be integers with $n + \beta - r > 2\gamma + 2$. Then one has, for $f \in C^r([0, \infty))$,

$$\begin{aligned}
& \left\| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right) \phi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \leq C \left(\frac{1}{\sqrt{n}} \omega_1 \left(f^{(r)}; \phi^{2\gamma+1}; \frac{1}{\sqrt{n}} \right) + \omega_2 \left(f^{(r)}; \phi^{2\gamma}; \frac{1}{\sqrt{n}} \right) \right).
\end{aligned} \tag{55}$$

To prove Theorem 5, we need the following theorem.

Theorem 6. Let r and γ be nonnegative integers. Let n and β be integers with $n + \beta - r > 2\gamma$. Let $f \in C^{(r)}([0, \infty))$ satisfy

$$\left\| f^{(r)} \phi^{2\gamma} \right\|_{L_\infty([0,\infty))} < \infty. \tag{56}$$

Then one has uniformly, for n, f , and $x \in [0, \infty)$,

$$\left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) \phi^{2\gamma}(x) \right| \leq C \left\| f^{(r)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))}. \tag{57}$$

Proof. Using $(1+y)^{2\gamma} \leq C((1+x)^{2\gamma} + (y-x)^{2\gamma})$, we have

$$\begin{aligned}
& \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) (1+y)^{2\gamma} dy \right| \\
& \leq C \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) \right. \\
& \quad \left. \times ((1+x)^{2\gamma} + (y-x)^{2\gamma}) dy \right| \\
& \leq C \left(R_{n,0,r}(\beta-r;x) \phi^{-2\gamma}(x) + R_{n,2\gamma,r}(\beta-r;x) \right).
\end{aligned} \tag{58}$$

Therefore, by Lemma 1 (6), we have

$$\begin{aligned}
& \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) (1+y)^{2\gamma} dy \right| \\
& \leq C \left(\phi^{-2\gamma}(x) + O\left(\frac{1}{n^\gamma}\right) g_{n,2\gamma,r}(\beta-r;x) \right).
\end{aligned} \tag{59}$$

Since $|g_{n,2\gamma,r}(\beta-r;x)\phi^{2\gamma}(x)|$ is uniformly bounded on $[0, \infty)$, we have with Lemma 2 and (59)

$$\begin{aligned}
& \left| \lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) \phi^{2\gamma}(x) \right| \\
& \leq \left| \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n+\beta-r,k+r}(y) (1+y)^{2\gamma} dy \phi^{2\gamma}(x) \right| \\
& \quad \times \left\| f^{(r)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \\
& \leq C \left\| f^{(r)}(x) \phi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))}.
\end{aligned} \tag{60}$$

Therefore, we have the result. □

The Steklov function $[f]_h(x)$ for $f \in C([0, \infty))$ is defined as follows:

$$\begin{aligned}
[f]_h(x) &:= \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} [2f(x+s+t) - f(x+2(s+t))] ds dt, \\
x &\geq 0, h > 0.
\end{aligned} \tag{61}$$

Then for the Steklov function $[f]_h(x)$ with respect to $f \in C([0, \infty))$, we have the following properties.

Lemma 7 (see [8, Lemma 2.4]). Let $f(x) \in C([0, \infty))$, and let $\eta(x)$ be a positive and nonincreasing function on $[0, \infty)$. Then

(i) $[f]_h(x) \in C^2([0, \infty))$,

(ii)

$$\|([f]_h(x) - f(x))\eta(x)\|_{L_\infty([0, \infty))} \leq \omega_2(f; \eta; \frac{h}{2}), \quad (62)$$

(iii)

$$\begin{aligned} & \| [f]_h'(x) \eta(x) \|_{L_\infty([0, \infty))} \\ & \leq \frac{4}{h} \omega_1(f; \eta; \frac{h}{2}) \frac{\eta(x)}{\eta(x + h/2)} + \frac{1}{h} \omega_1(f; \eta; h) \frac{\eta(x)}{\eta(x + h)}, \end{aligned} \quad (63)$$

(iv)

$$\begin{aligned} & \| [f]_h''(x) \eta(x) \|_{L_\infty([0, \infty))} \\ & \leq \frac{4}{h^2} \left[2\omega_2(f; \eta; \frac{h}{2}) + \frac{1}{4} \omega_2(f; \eta; h) \right]. \end{aligned} \quad (64)$$

Now, we prove Theorem 5.

Proof of Theorem 5. We know that, for $f(x) \in C^r([0, \infty))$,

$$\begin{aligned} [f]_h^{(r)}(x) &= [f^{(r)}]_h(x), \\ [f]_h^{(r+1)}(x) &= [f^{(r)}]'_h(x), \\ [f]_h^{(r+2)}(x) &= [f^{(r)}]''_h(x). \end{aligned} \quad (65)$$

Then first, we split it as follows:

$$\begin{aligned} & \| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq \| \lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \quad + \| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) - [f]_h^{(r)}(x) \right) \\ & \quad \times \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \quad + \| \left([f]_h^{(r)}(x) - f^{(r)}(x) \right) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))}. \end{aligned} \quad (66)$$

Then for the first term, we have, using Theorem 6, (62), and (65),

$$\begin{aligned} & \| \lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq C \| [f^{(r)}(x) - [f^{(r)}]_h(x)] \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq C \omega_2(f^{(r)}; \phi^{2\gamma+2}(x); h). \end{aligned} \quad (67)$$

Here, we suppose $0 < h \leq 1$, and then we know that

$$\begin{aligned} \frac{\phi(x)}{\phi(x+h)} &\leq 2, \\ \frac{\phi(x)}{\phi(x+h/2)} &\leq 2. \end{aligned} \quad (68)$$

For the second term, from Theorem 4, (65), (63), and (64) of Lemma 7,

$$\begin{aligned} & \| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) - [f]_h^{(r)}(x) \right) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq O\left(\frac{1}{n}\right) \left(\| [f]_h^{(r+1)}(x) \phi^{2\gamma+1}(x) \|_{L_\infty([0, \infty))} \right. \\ & \quad \left. + \| [f]_h^{(r+2)}(x) \phi^{2\gamma}(x) \|_{L_\infty([0, \infty))} \right) \\ & \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f^{(r)}; \phi^{2\gamma+1}; h) + \frac{1}{h^2} \omega_2(f^{(r)}; \phi^{2\gamma}; h) \right). \end{aligned} \quad (69)$$

Therefore, we have

$$\begin{aligned} & \| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) - f^{(r)}(x) \right) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f^{(r)}; \phi^{2\gamma+1}; h) + \frac{1}{h^2} \omega_2(f^{(r)}; \phi^{2\gamma}; h) \right) \\ & \quad + \omega_2(f^{(r)}; \phi^{2\gamma+2}; h). \end{aligned} \quad (70)$$

If we let $h = 1/\sqrt{n}$, then

$$\begin{aligned} & \| \left(\lambda_{n,\beta,r}(B_{n,\beta}[f - [f]_h])^{(r)}(x) - f^{(r)}(x) \right) \phi^{2\gamma+2}(x) \|_{L_\infty([0, \infty))} \\ & \leq C \left(\frac{1}{\sqrt{n}} \omega_1(f^{(r)}; \phi^{2\gamma+1}; \frac{1}{\sqrt{n}}) + \omega_2(f^{(r)}; \phi^{2\gamma}; \frac{1}{\sqrt{n}}) \right), \end{aligned} \quad (71)$$

because $\omega_2(f^{(r)}; \phi^{2\gamma+2}; 1/\sqrt{n}) \leq \omega_2(f^{(r)}; \phi^{2\gamma}; 1/\sqrt{n})$. \square

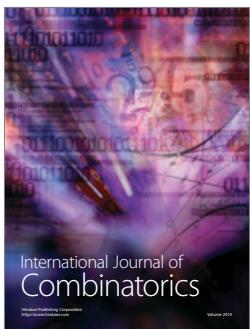
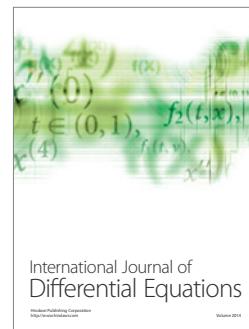
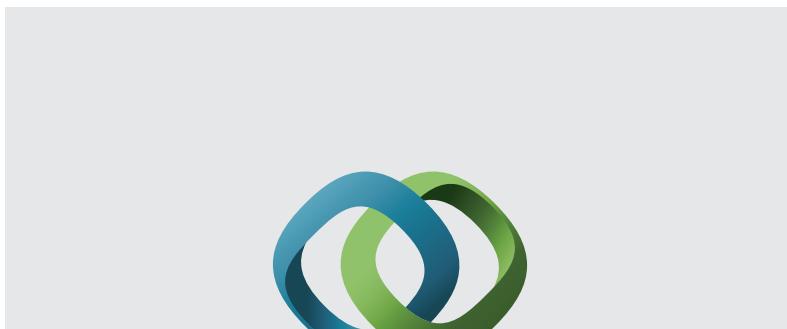
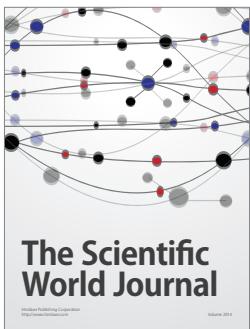
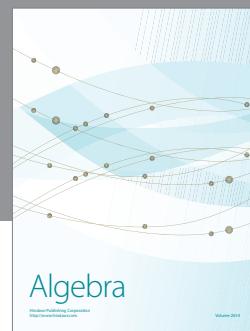
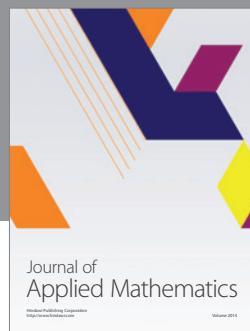
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