Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 750315, 13 pages
http://dx.doi.org/10.1155/2013/750315

Research Article

Blow-up Phenomena and Persistence Properties of Solutions to the Two-Component DGH Equation

Panpan Zhai, Zhengguang Guo, and Weiming Wang

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China

Correspondence should be addressed to Weiming Wang; weimingwang2003@163.com

Received 11 April 2013; Accepted 22 May 2013

Academic Editor: Carlo Bianca

This paper is concerned with blow-up phenomena and persistence properties for an integrable two-component Dullin-Gottwald-Holm shallow water system. We give sufficient conditions on the initial data which guarantee blow-up phenomena of solutions in finite time for both periodic and nonperiodic cases, respectively. Furthermore, the persistence properties of solutions to the system are investigated.

1. Introduction

In 2001, Dullin et al. [1] derived a new equation by using the method of asymptotic analysis and a near-identity normal form transformation in water wave theory to describe the unidirectional propagation of surface waves in a shallow water regime, it reads as follows:

$$m_t + 2\omega u_x + u m_x + 2m u_x = -\gamma u_{xxx}, \quad x \in \mathbb{R},\ t > 0. \tag{1}$$

Since (1) was derived by Dullin et al., we call it DGH equation. Let $m = u - \alpha^2 u_{xx}$; then we can rewrite (1) as the following form:

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3u u_x + \gamma u_{xxx} = \alpha^2 \left(2u_x u_{xx} + u u_{xxx}\right), \tag{2}$$

where α^2 and $\gamma/2\omega$ are squares of length scales, the constant $\omega = \sqrt{gh}/2$ is the critical shallow water speed for undisturbed water at rest at spatial infinity, h is the mean fluid depth, and g is the gravitational constant. This equation is connected with two separately integrable soliton equations for shallow water waves.

When $\alpha^2 = 0$, (2) becomes the KdV equation as follows:

$$u_t + 2\omega u_x + 3u u_x = -\gamma u_{xxx}. \tag{3}$$

Bourgain proved that solutions to the KdV equation are global as long as the initial data is square integrable [2], which is also shown in [3]. Another remarkable property is that it is integrable and the solitary waves are nonlinearly stable, and when $\omega = 0$, there exists a smooth soliton solution [4].

Instead, taking $\gamma = 0$ in (2), it turns out to be the Camassa-Holm equation:

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3u u_x = \alpha^2 \left(2u_x u_{xx} + u u_{xxx}\right), \tag{4}$$

which was derived by Camassa and Holm in [5] by approximating directly the Hamiltonian for Euler's equations in the shallow water regime in 1993. Many researches have been carried out on the Camassa-Holm equation in recent years. Local well-posedness has been proven by several authors; see [6–9]. Blow-up phenomena have been investigated in [10–15]. Besides, in [11], global solutions were also discussed. In [16], global existence of weak solutions was proved, but uniqueness was obtained only under a priori assumption. Furthermore, the authors in [17] studied the persistence properties of solution to the Camassa-Holm equation. In [18–20], the soliton solution was studied.

It is very interesting that (2) still preserves the bi-Hamiltonian structure and complete integrability. Indeed, (2) can be rewritten in two compatible Hamiltonian forms in terms of $m = u - \alpha^2 u_{xx}$ [1] as follows,

$$m_t = -B_2 \frac{\delta E}{\delta m} = -B_1 \frac{\delta F}{\delta m}, \tag{5}$$

where

$$B_1 = \partial_x - \alpha^2\partial_x^3,$$
$$B_2 = \partial_x (m+\omega) + (m+\omega)\partial_x + \gamma\partial_x^3,$$
$$E(u) = \frac{1}{2}\int \left(u^2 + \alpha^2 u_x^2\right) dx, \tag{6}$$
$$F(u) = \frac{1}{2}\int \left(u^3 + \alpha^3 u u_x^2 + 2\omega u^2 - \gamma u_x^2\right) dx.$$

$E(u)$ and $F(u)$ are two conserved quantities.

In [21], Tian et al. studied the well-posedness of the Cauchy problem and the scattering problem of the DGH equation. The issue of passing to the limit as the dispersive parameter tends to zero for the solution of the DGH equation was investigated, and the convergence of the solutions to DGH equation as $\alpha^2 \to 0$ was studied. Besides, data of the scattering problem for the equation could be explicitly expressed. The blow-up phenomenon of solutions to the DGH equation was the subject of [21–24].

In this paper, we are interested in the following Cauchy problem for an integrable two-component DGH equation. It reads as

$$\begin{aligned} &u_t - \alpha^2 u_{txx} + 2\omega u_x + 3uu_x + \gamma u_{xxx} \\ &\quad = \alpha^2\left(2u_x u_{xx} + uu_{xxx}\right) - \sigma\overline{\rho}\,\overline{\rho}_x, \quad x\in\mathbb{R},\ t>0, \\ &\quad \overline{\rho}_t + \left(\overline{\rho}u\right)_x = 0, \quad x\in\mathbb{R},\ t>0, \\ &\quad u(0,x) = u_0(x), \quad x\in\mathbb{R}, \\ &\quad \overline{\rho}(0,x) = \overline{\rho}_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{7}$$

The variable $u(x,t)$ describes the horizontal velocity of the fluid, and $\overline{\rho}(x,t)$ denotes the horizontal deviation of the surface from equilibrium, all measured in dimensionless units. This model can be derived by Constantin and Ivanov's approach [25] from shallow water theory, which includes the two-component Camassa-Holm system [26–32] as its special case with $\alpha = 1$ and $\gamma = 0$ in (7). From a geometric point of view, (7) is the model for geodesic motion on the semidirect product Lie group of diffeomorphisms acting on densities, with respect to the H^1-norm of velocity and the L^2-norm on the density. Mathematically, (7) admits not only breaking-wave solutions but also global in time solutions. In view of the context of hydrodynamics, we have the assumptions $u \to 0$, $\overline{\rho} \to 1$ as $x \to \infty$, at any instant t. For convenience of later discussion, letting $\rho = \overline{\rho} - 1$, $\alpha > 0$, and $\sigma = 1$, we have $\rho \to 0$ as $x \to \infty$. Then it follows that

$$\begin{aligned} &u_t - \alpha^2 u_{txx} + 2\omega u_x + 3uu_x + \gamma u_{xxx} - 2\alpha^2 u_x u_{xx} \\ &\quad - \alpha^2 u u_{xxx} + \rho\rho_x + \rho_x = 0, \quad x\in\mathbb{R},\ t>0, \\ &\quad \rho_t + \left(\rho u\right)_x + u_x = 0, \quad x\in\mathbb{R},\ t>0, \\ &\quad u(0,x) = u_0(x), \quad x\in\mathbb{R}, \\ &\quad \rho(0,x) = \rho_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{8}$$

Define $\Lambda = (1-\alpha^2\partial_x^2)^{-1}$; then the operator Λ can be expressed by

$$\Lambda f = G * f = \int_{\mathbb{R}} G(x-y) f(y)\, dy, \tag{9}$$

where $G(x)$ is the Green function for Λ and $*$ denotes the convolution. It is easy to get $G * (1/2) = 1/2$. For the periodic case, $G(x) = \cosh((x/\alpha) - ([x]/\alpha) - (1/2\alpha))/2\alpha\sinh(1/2\alpha)$ is the Green function for Λ in the unit circle. While, for the nonperiodic case, the Green function for Λ in $\mathbb{R}$ is $G(x) = (1/2\alpha)e^{-|x/\alpha|}$. Using this identity, we can rewrite (1) as the following nonlocal form:

$$\begin{aligned} &u_t + u_x\left(u - \frac{\gamma}{\alpha^2}\right) \\ &\quad = -\partial_x G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \left(2\omega + \frac{\gamma}{\alpha^2}\right)u + \frac{1}{2}\rho^2 + \rho\right), \\ &\qquad\qquad x\in\mathbb{R},\ t>0, \\ &\quad \rho_t + \left(\rho u\right)_x + u_x = 0, \quad x\in\mathbb{R},\ t>0, \\ &\quad u(0,x) = u_0(x), \quad x\in\mathbb{R}, \\ &\quad \rho(0,x) = \rho_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{10}$$

We note that less deep results exist yet for this model except for the recent ones [33, 34]. Our motivation here is to explore some new blow-up conditions especially for the initial data were added on some different quantities. This paper is organized as follows. In Section 2, we recall some preliminary results on local well-posedness and blow-up scenario and present some useful conversation laws. In Section 3, we study the blow-up phenomenon of strong solutions in both periodic and nonperiodic cases. In Section 4, persistence properties of solutions to the system are studied.

2. Preliminaries

We can use Kato's theory [35] to establish the following local well-posedness theorem for (10); its proof is referred to in the discussions of [33, 34], so we omit it here.

Theorem 1. *Assume an initial data* $(u_0,\rho_0)\in H^s\times H^{s-1}$, $s\geq 2$. *Then there exists a maximal* $T = T(\|(u_0,\rho_0)\|_{H^s\times H^{s-1}}) > 0$ *and a unique solution*

$$(u,\rho)\in C\left([0,T); H^s\times H^{s-1}\right)\cap C^1\left([0,T); H^{s-1}\times H^{s-2}\right) \tag{11}$$

of system (10). *Moreover, the solution* (u,ρ) *depends continuously on the initial value* (u_0,ρ_0), *and the maximal time of existence* $T>0$ *is independent of* s.

Next, we will give some useful conserved quantities, which are important to discuss the blow-up criteria.

Proposition 2. *Suppose* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$; *then the solution* (u, ρ) *of the system* (10) *guaranteed by Theorem 1 satisfies*

$$\int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2 + \rho^2\right) dx = \int_{\mathbb{R}} \left(u_0^2 + \alpha^2 u_{0x}^2 + \rho_0^2\right) dx. \quad (12)$$

Proof. Consider

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2 + \rho^2\right) dx \\ &= \int_{\mathbb{R}} \left(2uu_t + 2\alpha^2 u_x u_{xt} + 2\rho\rho_t\right) dx \\ &= \int_{\mathbb{R}} \left[2u\left(u_t - \alpha^2 u_{xxt}\right) + 2\rho\rho_t\right] dx. \end{aligned} \quad (13)$$

From the first equation of (10), we have

$$\begin{aligned} &\int_{\mathbb{R}} 2u\left(u_t - \alpha^2 u_{xxt}\right) dx \\ &= \int_{\mathbb{R}} 2u\left(-2\omega u_x - 3uu_x - \gamma u_{xxx}\right. \\ &\qquad \left. + 2\alpha^2 u_x u_{xx} + \alpha^2 u u_{xxx} - \rho\rho_x - \rho_x\right) dx. \end{aligned} \quad (14)$$

Similarly, from the second equation of (10), we can get

$$\int_{\mathbb{R}} 2\rho\rho_t dx = \int_{\mathbb{R}} -2\rho\left((\rho u)_x + u_x\right) dx. \quad (15)$$

So we can obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2 + \rho^2\right) dx \\ &= \int_{\mathbb{R}} \left[2u\left(-2\omega u_x - 3uu_x - \gamma u_{xxx} + 2\alpha^2 u_x u_{xx}\right.\right. \\ &\qquad \left.\left. + \alpha^2 u u_{xxx} - \rho\rho_x - \rho_x\right) - 2\rho\left((\rho u)_x + u_x\right)\right] dx \\ &= \int_{\mathbb{R}} \left[-2\omega\left(u^2\right)_x - 2\left(u^3\right)_x + \gamma\left(u_x^2\right)_x\right. \\ &\qquad \left. + 2\alpha^2\left(u^2 u_{xx}\right)_x - 2(\rho u)_x - 2(\rho\rho u)_x\right] dx \\ &= 0. \end{aligned} \quad (16)$$

This completes the proof. □

We also note that the quantities $\int_{\mathbb{R}} u(x,t)dx$ and $\int_{\mathbb{R}} \rho(x,t)dx$ are also conserved for $\rho_0(x) \in L^1$, $u_0(x) \in L^1$. These conservation laws will play important roles in considering the behavior of solutions. In what follows, we denote $\int_{\mathbb{R}}(u^2 + \alpha^2 u_x^2 + \rho^2)dx$ by $\|(u,\rho)\|^2_{H^1_\alpha \times L^2}$.

Moreover, using the techniques in [36], one can get the following criterion for finite time wave breaking to (10).

Theorem 3 (see [34]). *Let* $(u_0, \rho_0) \in H^s \times H^{s-1}$ *with* $s \geq 2$, *and let* $T > 0$ *be the maximal time of existence of the solution* (u, ρ) *to the model* (10) *with initial data* (u_0, ρ_0). *Then the corresponding solution* (u, ρ) *blows up in finite time if and only if*

$$\lim_{t \uparrow T} \left\{\inf_{x \in \mathbb{R}} u_x(t,x)\right\} = -\infty. \quad (17)$$

We will then need to introduce the standard particle trajectory method for later use. Now consider the following initial value problem:

$$\begin{aligned} q_t &= u(t,q), \quad t \in [0,T), \\ q(0,x) &= x, \quad x \in \mathbb{R}, \end{aligned} \quad (18)$$

where $u \in C^1([0,T), H^{s-1})$ is the first component of the solution (u, ρ) to system (10) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $(s \geq 2)$, and $T > 0$ is the maximal time of existence. By direct calculation, we have

$$q_{tx}(t,x) = u_x(t, q(t,x))\, q_x(t,x). \quad (19)$$

Then,

$$q_x(t,x) = \exp\left(\int_0^t u_x(\tau, q(\tau,x))\, d\tau\right) > 0, \quad t > 0,\ x \in \mathbb{R}, \quad (20)$$

which means that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0,T)$. Consequently, the L^∞-norm of any function $v(t,\cdot)$ is preserved under the family of the diffeomorphism $q(t,\cdot)$; that is,

$$\|v(t,\cdot)\|_{L^\infty} = \|v(t, q(t,\cdot))\|_{L^\infty}, \quad t \in [0,T). \quad (21)$$

Similarly,

$$\begin{aligned} \inf_{x \in \mathbb{R}} v(t,x) &= \inf_{x \in \mathbb{R}} v(t, q(t,x)), \quad t \in [0,T), \\ \sup_{x \in \mathbb{R}} v(t,x) &= \sup_{x \in \mathbb{R}} v(t, q(t,x)), \quad t \in [0,T). \end{aligned} \quad (22)$$

3. Blow-Up Criteria

In this section, we establish sufficient conditions to guarantee the formation of singularities for the corresponding solutions to (10) in periodic and nonperiodic cases. These sufficient conditions are different from each other.

Hereinafter, we investigate the blow-up phenomenon with periodic setting; that is, $x \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$. First, we introduce the following lemmas for later use.

Lemma 4 (see [23]). *For all* $f \in H^1(\mathbb{S})$, *the following inequality holds:*

$$G * \left(f^2 + \frac{\alpha^2}{2} f_x^2\right)(x) \geq C_0 f^2(x), \quad (23)$$

with

$$C_0 = \frac{1}{2} + \frac{\arctan(\sinh(1/2\alpha))}{2\sinh(1/2\alpha) + 2\arctan(\sinh(1/2\alpha))\sinh^2(1/2\alpha)}. \tag{24}$$

Moreover, C_0 is the optimal constant obtained by the function

$$\begin{aligned} f_0 = {} & \left(1 + \arctan\left(\sinh\left(\frac{x}{\alpha} - \frac{[x]}{\alpha} - \frac{1}{2\alpha}\right)\right)\right. \\ & \times \sinh\left(\frac{x}{\alpha} - \frac{[x]}{\alpha} - \frac{1}{2\alpha}\right)\Big) \\ & \times \left(1 + \arctan\left(\sinh\left(\frac{1}{2\alpha}\right)\right)\sinh\left(\frac{1}{2\alpha}\right)\right)^{-1}. \end{aligned} \tag{25}$$

Lemma 5 (see [23]). *For all $f \in H^1(\mathbb{S})$, the following inequality holds:*

$$\max_{x\in\mathbb{S}} f^2(x) \le C_1 \int_{\mathbb{S}} \left(f^2 + \alpha^2 f_x^2\right) dx, \tag{26}$$

where

$$C_1 = \frac{\cosh(1/2\alpha)}{2\alpha\sinh(1/2\alpha)}. \tag{27}$$

Moreover, C_1 is the minimum value, so in this sense, C_1 is the optimal constant which is obtained by the associated Green function

$$G(x) = \frac{\cosh((x/\alpha) - ([x]/\alpha) - (1/2\alpha))}{2\alpha\sinh(1/2\alpha)}. \tag{28}$$

Lemma 6 (see [37]). *Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \le -Cy^2(t) + K \tag{29}$$

with the constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{K/C}$, then the solution to the previous equation goes to $-\infty$ in finite time.

Lemma 7 (see [24]). *Assume $f(x) \in H^s(\mathbb{S})$, $s > 2$. If $\int_{\mathbb{S}} f(x)dx = 0$, then*

$$\begin{aligned} \|f(x)\|^2_{L^\infty(\mathbb{S})} &\le \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)\,dx, \\ \int_{\mathbb{S}} f^2(x)\,dx &\le \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)\,dx, \\ \int_{\mathbb{S}} f^2(x) f_x^2(x)\,dx &\le \frac{1}{12}\left(\int_{\mathbb{S}} f_x^2(x)\,dx\right)^2. \end{aligned} \tag{30}$$

Now we give some sufficient conditions on the initial data to guarantee finite time wave breaking for the periodic case.

Theorem 8. *Assume that the initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \ge 2$, satisfies*

$$\int_{\mathbb{S}} u_{0x}^3 dx < -\frac{\|(u_0,\rho_0)\|^2_{H^1_\alpha\times L^2}}{\alpha^3}\sqrt{3A + 6C_1\|(u_0,\rho_0)\|^2_{H^1_\alpha\times L^2}}, \tag{31}$$

where C_1 is defined as in Lemma 5, and constant A is determined later. Moreover, for convenience, we suppose that $2\omega + \gamma/\alpha^2 = 0$. Then the corresponding strong solution to (10) blows up in finite time.

Proof. By assumption and Theorem 1, there exists a solution (u, ρ) to system (10) corresponding to the initial value $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \ge 2$. Suppose that the statement is not true; that is, the corresponding solution does not blow up in finite time, and then due to Theorem 3, there exists $M_1 \ge 0$, such that

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} u_x(t,x) \ge -M_1. \tag{32}$$

Let

$$M(t) = u_x(t, q(t,x)), \qquad \gamma(t) = \rho(t, q(t,x)), \qquad t \in [0,T), \tag{33}$$

with $q(t,x)$ determined in (18). Then we have the following inequality for each $x \in \mathbb{R}$ and $t \in [0,T)$:

$$M(t) \ge -M_1. \tag{34}$$

Since

$$\gamma'(t) = -\gamma(t)M(t), \tag{35}$$

we have, by solving it,

$$|\gamma(t)| = |\gamma(0)|\, e^{\int_0^t -M(\tau)d\tau}. \tag{36}$$

So we can get

$$|\rho(t, q(t,x))| = |\gamma(t)| = |\gamma(0)|\, e^{\int_0^t -M(\tau)d\tau} \le \|\rho_0\|_{L^\infty} e^{M_1 t}, \tag{37}$$

then we have

$$\|\rho(t,\cdot)\|_{L^\infty} \le \|\rho_0\|_{L^\infty} e^{M_1 t}. \tag{38}$$

Therefore, the following inequality holds:

$$\left\|(\rho+1)^2\right\|_{L^\infty} \le \left(\|\rho\|_{L^\infty} + 1\right)^2 \le \left(\|\rho_0\|_{L^\infty} e^{M_1 t} + 1\right)^2 \doteq A. \tag{39}$$

With the notation of $\partial_x^2(G * f) = (1/\alpha^2)(G * f - f)$, differentiating both sides of the first equation of (10) with respect to variable x, we obtain

$$\begin{aligned} & u_{tx} + u_x^2 + uu_{xx} + 2\omega u_{xx} \\ & \quad = \frac{1}{\alpha^2}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) \\ & \qquad - \frac{1}{\alpha^2} G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right). \end{aligned} \tag{40}$$

We note that $G * (1/2) = 1/2$; then (40) can be rewritten as follows:

$$\begin{aligned} &u_{tx} + u_x^2 + uu_{xx} + 2\omega u_{xx} \\ &\quad = \frac{1}{\alpha^2}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right) \\ &\qquad - \frac{1}{\alpha^2}G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right). \end{aligned} \tag{41}$$

Multiplying u_x^2 on both sides and integrating by parts with respect to x, one obtains

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\ &\quad = -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 u^2 dx \\ &\qquad + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx \\ &\qquad - \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right)dx \\ &\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 u^2 dx \\ &\qquad + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx, \end{aligned} \tag{42}$$

where we have used the following identity:

$$\int_{\mathbb{S}} u_x^2 u u_{xx} dx = -\frac{1}{3}\int_{\mathbb{S}} u_x^4 dx. \tag{43}$$

Since

$$\left(\int_{\mathbb{S}} u_x^3 dx\right)^2 \le \frac{1}{\alpha^2}\left(\int_{\mathbb{S}} u_x^4 dx\right)\left(\int_{\mathbb{S}} \alpha^2 u_x^2 dx\right), \tag{44}$$

it follows that

$$\int_{\mathbb{S}} u_x^4 dx \ge \frac{\alpha^2}{\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2}\left(\int_{\mathbb{S}} u_x^3 dx\right)^2. \tag{45}$$

We can also get

$$\begin{aligned} &\frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 u^2 dx \le \frac{3}{\alpha^4}C_1\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^4, \\ &\left|\frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx\right| \\ &\quad = \frac{3}{2\alpha^2}\int_{\mathbb{S}} u_x^2(\rho+1)^2 dx \\ &\quad \le \frac{3}{2\alpha^4}\|(\rho+1)^2\|_{L^\infty}\int_{\mathbb{S}} \alpha^2 u_x^2 dx \\ &\quad \le \frac{3}{2\alpha^4}\|(\rho+1)^2\|_{L^\infty}\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2. \end{aligned} \tag{46}$$

So we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\ &\quad \le -\frac{\alpha^2}{2\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2}\left(\int_{\mathbb{S}} u_x^3 dx\right)^2 + \frac{3}{\alpha^4}C_1\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^4 \\ &\qquad + \frac{3}{2\alpha^4}\|(\rho+1)^2\|_{L^\infty}\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2 \\ &\quad \le -\frac{\alpha^2}{2\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2}\left(\int_{\mathbb{S}} u_x^3 dx\right)^2 + \frac{3}{\alpha^4}C_1\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^4 \\ &\qquad + \frac{3}{2\alpha^4}A\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2. \end{aligned} \tag{47}$$

Due to the theory of Riccati type differential equations in Lemma 6, if (31) holds, then we have

$$\int_{\mathbb{S}} u_x^3 dx \longrightarrow -\infty, \tag{48}$$

as t tends to some T_0.

On the other hand, the majorization

$$\begin{aligned} \int_{\mathbb{S}} u_x^3 dx &\ge \inf u_x(x,t)\int_{\mathbb{S}} u_x^2 dx \\ &> c(\alpha)\inf u_x(x,t)\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2 \end{aligned} \tag{49}$$

implies that

$$\lim_{t\uparrow T_0}\inf u_x(x,t) = -\infty. \tag{50}$$

It contradicts the assumption

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} u_x(t,x) \ge -M_1. \tag{51}$$

By Theorem 3, we know that the solution must blow up in finite time. □

Next, we find if the initial velocity has zero mean and initial energy is sufficiently large, wave breaking can occur.

Theorem 9. *Suppose $z_0 = (u_0,\rho_0) \in H^s \times H^{s-1}$, $s > 5/2$, and α satisfies $0 < \sinh(1/2\alpha)/\alpha < 3$. If the initial energy satisfies $\int_{\mathbb{S}}(u_0^2 + \alpha^2 u_{0x}^2 + \rho_0^2)dx > C_0$ for some positive constant C_0, and then $\int_{\mathbb{S}} u_0(x)dx = 0$ holds for nontrival z_0. Besides, assume that $2\omega + \gamma/\alpha^2 = 0$ just for convenience. Then the solution to the system (10) with initial value z_0 blows up in finite time.*

Proof. From the first equation of (10), we know that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\ &\quad = -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 u^2 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}\rho^2 + \rho\right)dx \\ &\qquad - \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)dx. \end{aligned} \tag{52}$$

On the other hand, we know that $\int_{\mathbb{S}} u(x,t)dx = 0$ in view of the hypothesis, and the following inequality holds:

$$\frac{1}{2\alpha\sinh(1/2\alpha)} \le G(x) \le \frac{\cosh(1/2\alpha)}{2\alpha\sinh(1/2\alpha)}. \tag{53}$$

Using Lemma 7 and (53), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\
&\quad = -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 u^2 dx \\
&\qquad + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx \\
&\qquad - \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2 G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right)dx \\
&\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{1}{4\alpha^2}\left(\int_{\mathbb{S}} u_x^2 dx\right)^2 \\
&\qquad + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx \\
&\qquad - \frac{3}{2}\int_{\mathbb{S}} u_x^2\left(G * u_x^2\right)dx \\
&\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx \\
&\qquad + \left(\frac{1}{4\alpha^2} - \frac{3}{4\alpha\sinh(1/2\alpha)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2.
\end{aligned} \tag{54}$$

Similar to Theorem 8, assume that the solution (u,ρ) of system (10) does not blow up; that is, there is $M_1 \ge 0$, such that

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} u_x(t,x) \ge -M_1, \tag{55}$$

then

$$\|\rho(t,\cdot)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}\exp(M_1 t) \doteq \widetilde{\kappa}. \tag{56}$$

So we can get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\
&\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3}{\alpha^2}\int_{\mathbb{S}} u_x^2\left(\frac{1}{2}(\rho+1)^2\right)dx \\
&\qquad + \left(\frac{1}{4\alpha^2} - \frac{3}{4\alpha\sinh(1/2\alpha)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2 \\
&\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{3(\widetilde{\kappa}+1)^2}{2\alpha^2}\int_{\mathbb{S}} u_x^2 dx \\
&\qquad + \left(\frac{1}{4\alpha^2} - \frac{3}{4\alpha\sinh(1/2\alpha)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2.
\end{aligned} \tag{57}$$

If $0 < \sinh(1/2\alpha)/\alpha < 3$, then $(1/4\alpha^2) - (3/4\alpha\sinh(1/2\alpha)) < 0$. Since $\int_{\mathbb{S}}(u_0^2 + \alpha^2 u_{0x}^2 + \rho_0^2)dx > C_0$ for some constant C_0, due to Lemma 7, there is some $\delta^\star > 0$ such that

$$\begin{aligned}
&\frac{3(\widetilde{\kappa}+1)^2}{2\alpha^2}\int_{\mathbb{S}} u_x^2 dx \le \delta^\star\left(\int_{\mathbb{S}} u_x^2 dx\right)^2, \\
&\left(\frac{3}{4\alpha\sinh(1/2\alpha)} - \frac{1}{4\alpha^2} - \delta^\star\right) > 0.
\end{aligned} \tag{58}$$

Moreover, Lemma 7 also implies that

$$\int_{\mathbb{S}} u_x^2 dx \ge \frac{12}{12\alpha^2+1}\|u(x)\|_{H_\alpha^1(\mathbb{S})}^2. \tag{59}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \\
&\quad \le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx - \frac{144}{(12\alpha^2+1)^2} \\
&\qquad \times\left(\frac{3}{4\alpha\sinh(1/2\alpha)} - \frac{1}{4\alpha^2} - \delta^\star\right)\|u(x)\|_{H_\alpha^1(\mathbb{S})}^4.
\end{aligned} \tag{60}$$

Note that $\|u(x)\|_{H_\alpha^1(\mathbb{S})}$ is bounded. In view of Hölder's inequality, there exists

$$\int_{\mathbb{S}} u_x^4 dx \ge \left(\int_{\mathbb{S}} u_x^3 dx\right)^{4/3}. \tag{61}$$

For simplicity of notations, we denote by $\varphi(t)$ and μ the following quantities:

$$\int_{\mathbb{S}} u_x^3 dx, \quad \frac{144}{(12\alpha^2+1)^2}\left(\frac{3}{4\alpha\sinh(1/2\alpha)} - \frac{1}{4\alpha^2} - \delta^\star\right), \tag{62}$$

respectively. Therefore we have

$$\frac{d\varphi(t)}{dt} \le -\frac{1}{2}\varphi^{4/3}(t) - \mu\|u(x)\|_{H_\alpha^1(\mathbb{S})}^4. \tag{63}$$

First, we can easily get $\varphi(t) \le \varphi(0) - \mu\|u\|_{H_\alpha^1(\mathbb{S})}^4 t$, and it is not difficult to find that there exists a time t_0 such that $\varphi(t_0) < 0$. Then for all $t > t_0$, we have

$$\frac{d\varphi(t)}{dt} \le -\frac{1}{2}\varphi^{4/3}(t), \quad \text{with } \varphi(t_0) < 0. \tag{64}$$

Solving this inequality yields

$$\varphi(t) \le \left(\varphi^{-1/3}(t_0) + \frac{1}{6}(t-t_0)\right)^{-3}, \tag{65}$$

which goes to $-\infty$ as t tends to $-6\varphi^{-1/3}(t_0) + t_0$; that is, there exists a time $T \le -6\varphi^{-1/3}(t_0) + t_0$ such that

$$\lim_{t\uparrow T}\int_{\mathbb{S}} u_x^3 dx = -\infty. \tag{66}$$

Since

$$\int_{\mathbb{S}} u_x^3 dx \geq \inf u_x(x,t) \int_{\mathbb{S}} u_x^2 dx \geq c(\alpha) \inf u_x(x,t) \|u\|_{H_\alpha^1(\mathbb{S})}^2, \tag{67}$$

it shows that

$$\lim_{t\uparrow T} \inf_{x\in\mathbb{R}} u_x(x,t) = -\infty. \tag{68}$$

Then it contradicts the assumption

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} u_x(x,t) > -M_1. \tag{69}$$

By Theorem 3, we know that the solution must blow up in finite time. This finishes the proof. □

From now on, we give some blow-up criteria for the nonperiodic case.

Theorem 10. *Assume that the initial data* $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, *satisfy*

$$u_0'(x_0) < -\sqrt{2K_0}, \tag{70}$$

where K_0 *is a constant determined later. And assume that* $2\omega + \gamma/\alpha^2 = 0$ *just for convenience. Then* T *is finite and the slope of* u *tends to negative infinity as* t *goes to* T *while* u *is uniformly bounded on* $[0,T)$.

Proof. Let (u,ρ) be the solution to system (10) with the initial data (u_0,ρ_0) and let T be the maximal existence time of solution. Defining

$$m(t) = \inf_{x\in\mathbb{R}} [u_x(t,x)] = u_x(t,\xi(t)), \qquad \gamma(t) = \rho(t,\xi(t)), \tag{71}$$

one can get that m is almost everywhere differentiable on $[0,T)$ with

$$\frac{dm}{dt}(t) = u_{tx}(t,\xi(t)). \tag{72}$$

Note that $u_{xx}(t,\xi(t)) = 0$, and with (40), we have

$$\begin{aligned} m_t = & -m^2 + \frac{1}{\alpha^2}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) \\ & - \frac{1}{\alpha^2} G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right), \end{aligned} \tag{73}$$

and with the relation $1/2 = G * (1/2)$, we can get

$$\begin{aligned} m_t = & -m^2 + \frac{1}{\alpha^2}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right) \\ & - \frac{1}{\alpha^2} G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}(\rho+1)^2\right) \\ \leq & -\frac{1}{2}m^2 + \frac{1}{\alpha^2}\left(u^2 + \frac{1}{2}(\rho+1)^2\right). \end{aligned} \tag{74}$$

Since u and ρ are both bounded, then there exists a $K_0 > 0$, so that $(1/\alpha^2)(u^2 + (1/2)(\rho+1)^2) < K_0$, and thus we have

$$m_t \leq -\frac{1}{2}m^2 + K_0. \tag{75}$$

Note that if $m(0) < -\sqrt{2K_0}$, then $m(t) < -\sqrt{2K_0}$ for all $t \in [0,T)$. So there exists a T such that

$$\lim_{t\uparrow T} m(t) = -\infty. \tag{76}$$

This completes the proof of the theorem. □

The authors in [25] claimed that the corresponding solution to the two-component Camassa-Holm equation exists globally in time provided that some initial data satisfy smallness conditions. However, we find surprisingly the smallness of initial energy can also lead to breaking wave solutions. This is stated as follows.

Theorem 11. *Let* $(u_0,\rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s > 5/2$ *and let* $T > 0$ *be the maximal time of existence of the solution* (u,ρ) *to* (10) *with initial data* (u_0,ρ_0). *Assume there is some* $x_0 \in \mathbb{R}$ *such that* $\rho_0(x_0) = -1$, *and there holds* $0 < \gamma < 1 - 2\omega\alpha^2$ *for suitable* α *and* γ. *Moreover,*

$$\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2 < \frac{\alpha^5\left(1 - 2\omega\alpha^2 - \gamma\right)}{2\alpha^4 + 2\omega\alpha^2 + \gamma}. \tag{77}$$

Then the corresponding solution to (10) *blows up in finite time in the following sense, and there exists a* T_0 *with*

$$\begin{aligned} & 0 < T_0 \\ & \leq 2 + \frac{4\alpha^7 + 4\alpha^7 |u_{0,x}(x_0)|}{\alpha^5 - 2\omega\alpha^7 - \gamma\alpha^5 - (2\alpha^4 + 2\omega\alpha^2 + \gamma)\|(u_0,\rho_0)\|_{H_\alpha^1\times L^2}^2}, \end{aligned} \tag{78}$$

such that

$$\lim_{t\uparrow T_0} \inf\left(\inf_{x\in\mathbb{R}} u_x(x,t)\right) = -\infty. \tag{79}$$

Proof. As in Theorem 8, we define

$$M(t) = \inf_{x\in\mathbb{R}} [u_x(t,x)] = u_x(t,\xi(t)), \quad t \in [0,T), \tag{80}$$

so that

$$u_{xx}(t,\xi(t)) = 0, \quad \text{a.e. } t \in [0,T). \tag{81}$$

We recall that $q(t,\cdot)$ defined by (18) is a diffeomorphism of the line for any $t \in [0,T)$, so there exists an $x(t) \in \mathbb{R}$ such that

$$q(t,x(t)) = \xi(t), \quad t \in [0,T). \tag{82}$$

Differentiating the first equation of (10), we can get the expression of M_t as follows:

$$\begin{aligned} &M_t\left(t,q\left(t,x\left(t\right)\right)\right)\\ &\quad=-M^2+\frac{1}{\alpha^2}\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\left(\rho+1\right)^2\right)\\ &\qquad-\frac{1}{\alpha^2}G*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\left(\rho+1\right)^2\right)\\ &\qquad-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u\\ &\quad=-\frac{1}{2}M^2+\frac{1}{\alpha^2}u^2+\frac{1}{2\alpha^2}\left(\rho+1\right)^2\\ &\qquad-\frac{1}{\alpha^2}G*\left(u^2+\frac{\alpha^2}{2}u_x^2\right)-\frac{1}{2\alpha^2}G*\left(\rho+1\right)^2\\ &\qquad-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u. \end{aligned} \tag{83}$$

Besides, along the trajectory of $q(t,x(t))$, we have

$$\frac{d\left(\left(\rho\left(t,q\right)+1\right)q_x\right)}{dt}=q_x\left(\rho_t+\rho_xu+\rho u_x+u_x\right)=0. \tag{84}$$

Then choose $\xi(0)=x_0$ and $\rho(\xi(0))=\rho_0(x_0)=-1$, which implies that

$$\rho\left(t,\xi\left(t\right)\right)\equiv-1. \tag{85}$$

Therefore we can obtain

$$\begin{aligned} M_t&=-\frac{1}{2}M^2+\frac{1}{\alpha^2}u^2-\frac{1}{\alpha^2}G*\left(u^2+\frac{\alpha^2}{2}u_x^2\right)\\ &\quad-\frac{1}{2\alpha^2}G*\left(\rho+1\right)^2-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u\\ &\le-\frac{1}{2}M^2+\frac{1}{\alpha^2}u^2-\frac{1}{2\alpha^2}u^2-\frac{1}{2\alpha^2}G*\left(\rho+1\right)^2\\ &\quad-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u\\ &=-\frac{1}{2}M^2+\frac{1}{2\alpha^2}u^2-\frac{1}{2\alpha^2}G*\left(\rho+1\right)^2\\ &\quad-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u, \end{aligned} \tag{86}$$

where we use the fact

$$G*\left(u^2+\frac{\alpha^2}{2}u_x^2\right)\ge\frac{1}{2}u^2. \tag{87}$$

We define

$$f=\frac{1}{2\alpha^2}u^2-\frac{1}{2\alpha^2}G*\left(\rho+1\right)^2-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u. \tag{88}$$

For any $x\in\mathbb{R}$, then

$$\begin{aligned} u^2\left(x\right)&=\int_{-\infty}^{x_0}uu_xdx+\int_{\infty}^{x_0}uu_xdx\\ &\le\frac{1}{2\alpha}\left(\int_{-\infty}^{x_0}\left(u^2+\alpha^2u_x^2\right)dx+\int_{\infty}^{x_0}\left(u^2+\alpha^2u_x^2\right)dx\right)\\ &=\frac{1}{2\alpha}\|u\|_{H_\alpha^1}^2\le\frac{1}{2\alpha}\|(u,\rho)\|_{H_\alpha^1\times L^2}^2. \end{aligned} \tag{89}$$

We also have

$$\begin{aligned} &\left|G*\rho\right|\le\|G\|_{L^2}\|\rho\|_{L^2}=\frac{1}{2\sqrt{\alpha}}\|\rho\|_{L^2}\le\frac{1}{4}+\frac{1}{4\alpha}\|(u,\rho)\|_{H_\alpha^1\times L^2}^2,\\ &\left|\partial_x^2G*u\right|=\left|G_x*u_x\right|\le\|G_x\|_{L^2}\|u_x\|_{L^2}\\ &\qquad=\frac{1}{2\sqrt{\alpha^3}}\|u_x\|_{L^2}\le\frac{1}{4}+\frac{1}{4\alpha^5}\|(u,\rho)\|_{H_\alpha^1\times L^2}^2. \end{aligned} \tag{90}$$

Thus, we can get

$$\begin{aligned} f&=\frac{1}{2\alpha^2}u^2-\frac{1}{2\alpha^2}G*\rho^2-\frac{1}{\alpha^2}G*\rho\\ &\quad-\frac{1}{2\alpha^2}-\left(2\omega+\frac{\gamma}{\alpha^2}\right)\partial_x^2G*u\\ &\le\frac{1}{2\alpha^2}u^2+\frac{1}{\alpha^2}\left|G*\rho\right|-\frac{1}{2\alpha^2}+\left(2\omega+\frac{\gamma}{\alpha^2}\right)\left|\partial_x^2G*u\right|\\ &\le\frac{1}{4\alpha^3}\|(u,\rho)\|_{H_\alpha^1\times L^2}^2+\frac{1}{4\alpha^3}\|(u,\rho)\|_{H_\alpha^1\times L^2}^2\\ &\quad+\frac{1}{4\alpha^5}\left(2\omega+\frac{\gamma}{\alpha^2}\right)\|(u,\rho)\|_{H_\alpha^1\times L^2}^2-\frac{1}{4\alpha^2}+\frac{1}{4}\left(2\omega+\frac{\gamma}{\alpha^2}\right)\\ &\doteq-a<0. \end{aligned} \tag{91}$$

Therefore, when $\|(u_0,\rho_0)\|_{H^1\times L^2}^2<(\alpha^5-2\omega\alpha^7-\gamma\alpha^5)/(2\alpha^4+2\omega\alpha^2+\gamma)$ holds, we always have $f<0$. In this case, we can obtain

$$M'\left(t\right)\le-\frac{1}{2}M^2-a<0,\quad t\in\left[t,T\right), \tag{92}$$

so $M(t)$ is decreasing strictly in such interval. Assume that the solution (u,ρ) of (10) does not blow up; that is, it exists globally in time; that is, $T=\infty$. Then integrating (92) over $[0,t_1]$ with $t_1=(1+|u_{0,x}(x_0)|)/a$ yields

$$M\left(t_1\right)=M\left(0\right)-\int_0^{t_1}M'\left(t\right)dt\le\left|u_{0,x}\left(x_0\right)\right|-at_1\le-1. \tag{93}$$

We can also find that $M'(t)\le-(1/2)M^2$, which leads to

$$-\frac{d}{dt}\left(\frac{1}{M\left(t\right)}\right)=\frac{1}{M^2\left(t\right)}\frac{d}{dt}M\left(t\right)\le-\frac{1}{2},\quad t\in\left[t_1,T\right). \tag{94}$$

Integrating both sides, we have

$$-\frac{1}{M(t)}-1\le-\frac{1}{M(t)}+\frac{1}{M(t_1)}\le-\frac{1}{2}(t-t_1), \tag{95}$$

where we use the result $M(t_1)\le-1$ obtained previously. Solve the inequality

$$M(t)\le\frac{2}{(t-t_1)-2}\longrightarrow-\infty,\quad \text{as } t\longrightarrow t_1+2. \tag{96}$$

That implies that $T\le t_1+2<\infty$, which leads to a contradiction as $t\to\infty$. This completes the proof. □

4. Persistence Properties

Attention now is turned to determining the persistence properties of solutions to the two-component DGH equation. We show that certain decay properties of the initial data persist in corresponding solution in later time as long as it exists. Precisely, we prove that the corresponding solution (u,ρ) and its first order spatial derivatives preserve the exponential decay as their initial values do. This is a very important investigation since we can obtain the detailed asymptotic behavior of solution from the initial values. The main idea comes from the recent works of Zhou et al. [17, 38]. Now, we state our result as follows.

Theorem 12. *Assume that for some $T>0$ and $s>5/2$, $(u,\rho)\in C([0,T];H^s\times H^{s-1})$ is a strong solution of the initial value problem to the system* (10) *and $2\omega+\gamma/\alpha^2=0$ just for convenience. If $u_0=u(x,0)$, $\rho_0=\rho(x,0)$ satisfy*

$$|u_0(x)|,|\partial_x u_0(x)|,|\rho_0(x)|,|\partial_x\rho_0(x)|\sim O\left(e^{-\theta x}\right)\quad \text{as } x\longrightarrow\infty, \tag{97}$$

for some $\theta\in(0,1)$. Then

$$|u(x,t)|,|u_x(x,t)|,|\rho(x,t)|,|\rho_x(x,t)|\sim O\left(e^{-\theta x}\right)\quad \text{as } x\longrightarrow\infty \tag{98}$$

uniformly in the time interval $[0,T]$.

Notation. Consider

$$\begin{aligned} &u(x,t)\sim O\left(e^{-\theta x}\right),\quad \text{as } x\longrightarrow\infty,\ \text{if}\lim_{x\to\infty}\frac{|u(x)|}{e^{-\theta x}}=L,\\ &u(x,t)\sim o\left(e^{-\theta x}\right),\quad \text{as } x\longrightarrow\infty,\ \text{if}\lim_{x\to\infty}\frac{|u(x)|}{e^{-\theta x}}=0. \end{aligned} \tag{99}$$

Proof. First, we figure out the estimates on $\|u(x,t)\|_{L^\infty}$, $\|u_x(x,t)\|_{L^\infty}$, $\|\rho(x,t)\|_{L^\infty}$, and $\|\rho_x(x,t)\|_{L^\infty}$. Then we use the weight function to obtain the desired result.

Step 1. Estimate for $\|u(x,t)\|_{L^\infty}$, $\|\rho(x,t)\|_{L^\infty}$.

Multiplying the first equation of (10) by u^{2n-1} with $n\in\mathbb{Z}$ and then integrating both sides with respect to x-variable, we can get

$$\begin{aligned} &\int_{\mathbb{R}}u^{2n-1}u_t dx\\ &\quad+\int_{\mathbb{R}}u^{2n-1}uu_x dx+\int_{\mathbb{R}}u^{2n-1}2\omega u_x dx\\ &\quad+\int_{\mathbb{R}}u^{2n-1}\partial_x p*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\rho^2+\rho\right)dx=0. \end{aligned} \tag{100}$$

The first term of the previous identity is

$$\int_{\mathbb{R}}u^{2n-1}u_t dx=\frac{1}{2n}\frac{d}{dt}\|u(t)\|_{L^{2n}}^{2n}=\|u(t)\|_{L^{2n}}^{2n-1}\frac{d}{dt}\|u(t)\|_{L^{2n}}, \tag{101}$$

and the estimate of the second term is

$$\left|\int_{\mathbb{R}}u^{2n-1}uu_x dx\right|\le\|u_x(t)\|_{L^\infty}\|u(t)\|_{L^{2n}}^{2n}. \tag{102}$$

In view of Hölder's inequality, we can obtain the following estimate for the third and the last terms in (100):

$$\begin{aligned} &\int_{\mathbb{R}}2\omega u^{2n-1}u_x dx\le 2\omega\|u(t)\|_{L^{2n}}^{2n-1}\|u_x(t)\|_{L^{2n}},\\ &\int_{\mathbb{R}}u^{2n-1}\partial_x p*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\rho^2+\rho\right)dx\\ &\quad\le\|u(t)\|_{L^{2n}}^{2n-1}\left\|\partial_x p*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\rho^2+\rho\right)\right\|_{L^{2n}}. \end{aligned} \tag{103}$$

Putting all the previous inequalities into (100) yields

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{L^{2n}}&\le\|u_x(t)\|_{L^\infty}\|u(t)\|_{L^{2n}}+2\omega\|u_x(t)\|_{L^{2n}}\\ &\quad+\left\|\partial_x p*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\rho^2+\rho\right)\right\|_{L^{2n}}. \end{aligned} \tag{104}$$

Thus, according to the Sobolev embedding theorem, there exists a constant

$$M=\sup_{t\in[0,T]}\|u(x,t)\|_{H^s}, \tag{105}$$

such that applying Gronwall's inequality gives us

$$\begin{aligned} &\|u(t)\|_{L^{2n}}\\ &\quad\le\left(\|u(0)\|_{L^{2n}}+\int_0^t 2\omega\|u_x(\tau)\|_{L^{2n}}\right.\\ &\qquad\left.+\int_0^t\left\|\partial_x p*\left(u^2+\frac{\alpha^2}{2}u_x^2+\frac{1}{2}\rho^2+\rho\right)\right\|_{L^{2n}}d\tau\right)e^{Mt}. \end{aligned} \tag{106}$$

For any $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we know that

$$\lim_{q\to\infty}\|f\|_{L^q} = \|f\|_{L^\infty}. \tag{107}$$

Thus, taking the limits in (106), we get

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le \left(\|u(0)\|_{L^\infty} + \int_0^t 2\omega\|u_x(\tau)\|_{L^\infty}d\tau \right.\\ &\quad \left. + \int_0^t \left\|\partial_x p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^\infty} d\tau\right) e^{Mt}.\end{aligned} \tag{108}$$

Similarly, for the second equation of (10), we have

$$\begin{aligned}&\int_{\mathbb{R}} \rho^{2n-1}\rho_t dx + \int_{\mathbb{R}} \rho^{2n-1}\rho_x u dx\\ &\quad + \int_{\mathbb{R}} \rho^{2n-1}\rho u_x dx + \int_{\mathbb{R}} \rho^{2n-1}u_x dx = 0,\end{aligned} \tag{109}$$

and using the same method, one can get

$$\frac{d}{dt}\|\rho(t)\|_{L^{2n}} \le \|u_x\|_{L^\infty}\|\rho(t)\|_{L^{2n}} + \|u\|_{L^\infty}\|\rho_x\|_{L^{2n}} + \|u_x\|_{L^{2n}}. \tag{110}$$

Using the Gronwall's inequality, one gets

$$\begin{aligned}\|\rho(t)\|_{L^{2n}} &\le \left(\|\rho(0)\|_{L^{2n}} + \int_0^t \left(\|u\|_{L^\infty}\|\rho_x\|_{L^{2n}} + \|u_x\|_{L^{2n}}\right) d\tau\right) e^{Mt}.\end{aligned} \tag{111}$$

Taking the limits, one gets

$$\begin{aligned}\|\rho(t)\|_{L^\infty} &\le \left(\|\rho(0)\|_{L^\infty} + \int_0^t \left(\|u\|_{L^\infty}\|\rho_x\|_{L^\infty} + \|u_x\|_{L^\infty}\right) d\tau\right) e^{Mt}.\end{aligned} \tag{112}$$

Step 2. Estimate for $\|u_x(x,t)\|_{L^\infty}$, $\|\rho_x(x,t)\|_{L^\infty}$.

We will establish an estimate on $\|u_x(x,t)\|_{L^\infty}$ using the same method. Differentiating the first equation (10) with respect to x-variable produces the following equation:

$$\begin{aligned}&u_{xt} + uu_{xx} + u_x^2 + 2\omega u_{xx}\\ &\quad + \partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) = 0.\end{aligned} \tag{113}$$

Then, multiplying (113) by u_x^{2n-1} with $n \in \mathbb{Z}^+$, integrating the result in the x-variable, and considering the second term in the previous identity with integration by parts, one can get

$$\int_{\mathbb{R}} uu_{xx}u_x^{2n-1}dx = \int_{\mathbb{R}} u\left(\frac{u_x^{2n}}{2n}\right)_x dx = -\frac{1}{2n}\int_{\mathbb{R}} u_x u_x^{2n} dx, \tag{114}$$

so we have

$$\begin{aligned}&\int_{\mathbb{R}} u_x^{2n-1}u_{xt}dx - \frac{1}{2n}\int_{\mathbb{R}} u_x u_x^{2n}dx + \int_{\mathbb{R}} u_x^{2n-1}u_x^2 dx\\ &\quad + \int_{\mathbb{R}} u_x^{2n-1}2\omega u_{xx}dx\\ &\quad + \int_{\mathbb{R}} u_x^{2n-1}\partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) dx = 0.\end{aligned} \tag{115}$$

Similarly, we can get the following inequality:

$$\begin{aligned}\frac{d}{dt}\|u_x(t)\|_{L^{2n}} &\le 2\|u_x(t)\|_{L^\infty}\|u_x(t)\|_{L^{2n}} + 2\omega\|u_{xx}(t)\|_{L^{2n}}\\ &\quad + \left\|\partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^{2n}}.\end{aligned} \tag{116}$$

Using Gronwall's inequality, we can similarly get

$$\begin{aligned}\|u_x(t)\|_{L^{2n}} &\le e^{2Mt}\left(\|u_x(0)\|_{L^{2n}} + \int_0^t 2\omega\|u_{xx}(\tau)\|_{L^{2n}}d\tau\right.\\ &\quad \left. + \int_0^t \left\|\partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^{2n}} d\tau\right).\end{aligned} \tag{117}$$

Taking the limits, one gets

$$\begin{aligned}\|u_x(t)\|_{L^\infty} &\le e^{2Mt}\left(\|u_x(0)\|_{L^\infty} + \int_0^t 2\omega\|u_{xx}(\tau)\|_{L^\infty}d\tau\right.\\ &\quad \left. + \int_0^t \left\|\partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^\infty} d\tau\right).\end{aligned} \tag{118}$$

Differentiating the second equation of (10), one has

$$\rho_{tx} + \rho_{xx}u + 2\rho_x u_x + \rho u_{xx} + u_{xx} = 0, \tag{119}$$

and using the same method, we can obtain

$$\begin{aligned}&\int_{\mathbb{R}} \rho_x^{2n-1}\rho_{tx}dx + \int_{\mathbb{R}} \rho_x^{2n-1}\rho_{xx}udx + 2\int_{\mathbb{R}} \rho_x^{2n-1}\rho_x u_x dx\\ &\quad + \int_{\mathbb{R}} \rho_x^{2n-1}\rho u_{xx}dx + \int_{\mathbb{R}} \rho_x^{2n-1}u_{xx}dx = 0.\end{aligned} \tag{120}$$

In a similar way, we should deal with the second term as follows:

$$\left|\int_{\mathbb{R}} \rho_x^{2n-1}\rho_{xx}udx\right| = \left|\int_{\mathbb{R}} \left(\frac{\rho_x^{2n}}{2n}\right)_x udx\right| = \left|\frac{1}{2n}\int_{\mathbb{R}} \rho_x^{2n}u_x dx\right|, \tag{121}$$

thus we can get the estimates of $\|\rho_x(x,t)\|_{L^\infty}$ as follows:

$$\|\rho_x(t)\|_{L^\infty} \le e^{3Mt}\left(\|\rho_x(0)\|_{L^\infty} + \int_0^t \left(\|u_{xx}\|_{L^\infty}\|\rho\|_{L^\infty} + \|u_{xx}\|_{L^\infty}\right) d\tau\right). \tag{122}$$

In what follows, we use the weight function to get the desired result. We shall now introduce the weight function $\varphi_N(x)$ with $N \in \mathbb{Z}^+$, which is independent of t as follows:

$$\varphi_N(x) = \begin{cases} 1, & x \le 0, \\ e^{\theta x}, & x \in (0,N), \\ e^{\theta N}, & x \ge N. \end{cases} \tag{123}$$

Note that $0 \le \varphi_N'(x) \le \varphi_N(x)$. Multiplying the first equation of (10) and (106) by φ_N, we have

$$\begin{aligned} &\varphi_N u_t + \varphi_N u u_x + 2\omega\varphi_N u_x \\ &\quad + \varphi_N \partial_x p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) = 0, \\ &\varphi_N u_{xt} + \varphi_N u u_{xx} + \varphi_N u_x^2 + 2\omega\varphi_N u_{xx} \\ &\quad + \varphi_N \partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right) = 0. \end{aligned} \tag{124}$$

In order to get the estimate of $u_x\varphi_N$, we need to eliminate the second derivatives as follows:

$$\begin{aligned} &\left|\int_{\mathbb{R}} \varphi_N u u_{xx} (u_x\varphi_N)^{2n-1} dx\right| \\ &\quad = \left|\int_{\mathbb{R}} u(u_x\varphi_N)^{2n-1}\left((u_x\varphi_N)_x - u_x\varphi_N'\right) dx\right| \\ &\quad = \left|\int_{\mathbb{R}} u\left(\frac{(u_x\varphi_N)^{2n}}{2n}\right)_x dx - \int_{\mathbb{R}} u u_x \varphi_N' (u_x\varphi_N)^{2n-1} dx\right| \\ &\quad \le 2\left(\|u(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty}\right)\|u_x\varphi_N\|_{L^{2n}}^{2n}, \end{aligned} \tag{125}$$

where we use the fact $0 \le \varphi_N'(x) \le \varphi_N(x)$. Hence, as in the weightless case, we get the following inequality in view of the estimates of $\|u(x,t)\|_{L^\infty}$ and $\|u_x(x,t)\|_{L^\infty}$:

$$\begin{aligned} &\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} \\ &\quad \le e^{2Mt}\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right) \\ &\qquad + e^{2Mt}\int_0^t \left\|\varphi_N\partial_x p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^\infty} d\tau \\ &\qquad + e^{2Mt}\int_0^t \left\|\varphi_N\partial_x^2 p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \frac{1}{2}\rho^2 + \rho\right)\right\|_{L^\infty} d\tau. \end{aligned} \tag{126}$$

Similarly, we get estimates for $\rho\varphi_N$ and $\rho_x\varphi_N$ as follows:

$$\begin{aligned} &\|\rho\varphi_N\|_{L^\infty} + \|\rho_x\varphi_N\|_{L^\infty} \\ &\quad \le e^{2Mt}\left(\|\rho(0)\varphi_N\|_{L^\infty} + \|\rho_x(0)\varphi_N\|_{L^\infty}\right). \end{aligned} \tag{127}$$

On the other hand, a simple calculation shows that there exists $C > 0$, depending only on $\theta \in (0,1)$ such that for any $N \in \mathbb{Z}^+$,

$$\varphi_N(x)\int_{\mathbb{R}} e^{-|(x-y)/\alpha|}\frac{1}{\varphi_N(y)}dy \le C. \tag{128}$$

Therefore for any appropriate function g, one can get

$$\begin{aligned} &\left|\varphi_N\partial_x G(x) * g^2(x)\right| \\ &\quad = \left|\frac{1}{2\alpha}\varphi_N(x)\int_{\mathbb{R}} e^{-|(x-y)/\alpha|} g^2(y)\,dy\right| \\ &\quad = \frac{1}{2\alpha}\varphi_N(x)\int_{\mathbb{R}} e^{-|(x-y)/\alpha|}\frac{1}{\varphi_N(y)}\varphi_N(y)\,g(y)\,g(y)\,dy \\ &\quad \le \frac{1}{2\alpha}\left(\varphi_N(x)\int_{\mathbb{R}} e^{-|(x-y)/\alpha|}\frac{1}{\varphi_N(y)}dy\right)\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty} \\ &\quad \le C'\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty}, \end{aligned} \tag{129}$$

and using the same method,

$$\left|\varphi_N\partial_x^2 G(x) * g^2(x)\right| \le C'\|g\varphi_N\|_{L^\infty}\|g\|_{L^\infty}. \tag{130}$$

Besides,

$$\begin{aligned} &\left|\varphi_N\partial_x^2 p * \rho\right| \\ &\quad = \left|\varphi_N p_x * \rho_x\right| \le |\varphi_N|\,\|p_x\|_{L^2}\|\rho_x\|_{L^2} \le C'\|\varphi_N\rho_x\|_{L^\infty}. \end{aligned} \tag{131}$$

In a similar way, we have

$$\left|\varphi_N\partial_x p * \rho\right| \le C'\|\varphi_N\rho\|_{L^\infty}. \tag{132}$$

Then we can get the following estimate:

$$\begin{aligned} &\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} + \|\rho(t)\varphi_N\|_{L^\infty} + \|\rho_x(t)\varphi_N\|_{L^\infty} \\ &\quad \le \kappa_0\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right. \\ &\qquad \left. + \|\rho(0)\varphi_N\|_{L^\infty} + \|\rho_x(0)\varphi_N\|_{L^\infty}\right) \\ &\quad + \kappa_0\left(\int_0^t \left(\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty}\right.\right. \\ &\qquad \left.\left. + \|\rho(t)\varphi_N\|_{L^\infty} + \|\rho_x(t)\varphi_N\|_{L^\infty}\right) dt\right). \end{aligned} \tag{133}$$

Using Gronwall's inequality of integral form, one can get

$$\begin{aligned} &\|u(t)\varphi_N\|_{L^\infty} + \|u_x(t)\varphi_N\|_{L^\infty} + \|\rho(t)\varphi_N\|_{L^\infty} + \|\rho_x(t)\varphi_N\|_{L^\infty} \\ &\quad \le \kappa_0\left(\|u(0)\varphi_N\|_{L^\infty} + \|u_x(0)\varphi_N\|_{L^\infty}\right. \\ &\qquad \left. + \|\rho(0)\varphi_N\|_{L^\infty} + \|\rho_x(0)\varphi_N\|_{L^\infty}\right). \end{aligned} \tag{134}$$

Taking the limit as N goes to infinity, we can obtain

$$\begin{aligned}&\left|u(x,t)e^{\theta x}\right| + \left|u_x(x,t)e^{\theta x}\right| + \left|\rho(x,t)e^{\theta x}\right| + \left|\rho_x(x,t)e^{\theta x}\right| \\ &\quad \le \kappa_0\left(\left\|u(0)e^{\theta x}\right\| + \left\|u_x(0)e^{\theta x}\right\| \right. \\ &\qquad \left. + \left\|\rho(0)e^{\theta x}\right\| + \left\|\rho_x(0)e^{\theta x}\right\|\right).\end{aligned} \tag{135}$$

This completes the proof. □

Acknowledgments

This work was partially supported by Zhejiang Provincial Natural Science Foundation of China under LQ12A01009 and LY12A01014 and NSFC under Grant no. 11226172.

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