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Research Article

h-Stability of Linear Matrix Differential Systems

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This paper investigates the stability problem of linear matrix differential systems and gives some sufficient conditions of h-stability for linear matrix system and its associated perturbed system by using the Kronecker product of matrices. An example is also worked out to illustrate our results.

1. Introduction

The theory of stability in the sense of Lyapunov is well known and is used in the real world. It is obvious that, in applications, asymptotic stability is more important than stability because the desirable feature is to know the size of the region of asymptotic stability. However, when we study the asymptotic stability, it is not easy to work with nonexponential types of stability. In recent years, Medina and Pinto [1, 2] extended the study of exponential stability to a variety of reasonable systems called h-systems. They introduced the notion of h-stability with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential stability and the uniform Lipschitz stability) under some perturbations. Choi et al. [3] investigated h-stability for the nonlinear differential systems by employing the notion of t_∞-similarity and the Lyapunov functions. And then, Choi et al. [4–6] also characterized the h-stability in variation for nonlinear difference systems via n_∞-similarity and the Lyapunov functions and obtained some results related to stability for the perturbations of nonlinear difference systems.

However, as far as the author's scope, there are few discussions and results for matrix differential systems. In this paper, we shall investigate the h-stability problem for linear matrix differential systems by employing the Kronecker product of matrices which can be found in Lakshmikantham and Deo's monograph [7]. Some preliminaries are presented in Section 2. A theorem is given in this section, which is important to complete the main results of this paper. In Section 3, sufficient conditions for the h-stability are given for linear matrix system and its associated perturbed system. An example is also worked out at the end of this paper.

2. Preliminaries

Consider the linear matrix differential equation

$$X' = A(t)X + XB(t), \qquad X(t_0) = X_0 \tag{1}$$

and its associated perturbed system

$$Y' = A(t)Y + YB(t) + R(t, Y), \qquad Y(t_0) = X_0, \tag{2}$$

where $A, B \in C[R^+, R^{n\times n}]$, $R \in C[R^+ \times R^{n\times n}, R^{n\times n}]$, $R(t, 0) \equiv 0$, and $X, Y \in R^{n\times n}$.

Now, we introduce the $\overline{\text{vec}}(\cdot)$ operator which maps an $m \times n$ matrix $P = (p_{ij})$ onto the vector composed of the rows of P

$$\overline{\text{vec}}(P) = (p_{11}, \dots, p_{1n}, p_{21}, \dots, p_{2n}, \dots, p_{m1}, \dots, p_{mn})^T. \tag{3}$$

Let us begin by defining the Kronecker product of matrices.

Definition 1 (see [7]). If $P \in R^{c\times d}$, $Q \in R^{m\times n}$, then the Kronecker product of P and Q, $P \otimes Q \in R^{cm\times dn}$, is defined by the matrix

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1d}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2d}Q \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{c1}Q & p_{c2}Q & \cdots & p_{cd}Q \end{pmatrix}. \tag{4}$$

Among the main properties of this product presented in [8], we recall the following useful ones:

(1) $\overline{\text{vec}}\,(PXQ) = (P \otimes Q^T)\overline{\text{vec}}\,(X)$,

(2) $\overline{\text{vec}}\,(PX + XQ) = (P \otimes I + I \otimes Q^T)\overline{\text{vec}}\,(X)$,

where P, X, Q, and $I \in R^{n\times n}$ and I is an identity matrix.

Then, the equivalent vector differential systems of (1) and (2) can be written as

$$x' = \left(A \otimes I + I \otimes B^T\right) x, \qquad x\left(t_0\right) = x_0, \tag{5}$$

$$y' = \left(A \otimes I + I \otimes B^T\right) y + r\left(t, y\right), \qquad y\left(t_0\right) = x_0, \tag{6}$$

where $x = \overline{\text{vec}}\,(X)$, $y = \overline{\text{vec}}\,(Y)$, $r = \overline{\text{vec}}\,(R)$, $r \in C(R^+ \times R^{n^2}, R^{n^2})$, and $r(t, 0) \equiv 0$.

In order to investigate h-stability of linear matrix equation and its associated perturbed system, we need to consider the following systems and their properties. The techniques and results are similar to those of [7].

Consider the linear differential system

$$x' = P\left(t\right) x, \qquad x\left(t_0\right) = x_0, \tag{7}$$

where P is an $n \times n$ continuous matrix and its perturbation

$$y' = P\left(t\right) y + F\left(t, y\right), \qquad y\left(t_0\right) = x_0, \tag{8}$$

where $F \in C[R^+ \times R^n, R^n]$. Suppose that the solution $x(t, t_0, x_0)$ of (7) exists for all $t \geq t_0$. The fundamental matrix solution $\Phi(t, t_0, x_0)$ of (7) is given by [7]

$$\Phi\left(t, t_0, x_0\right) = \frac{\partial x\left(t, t_0, x_0\right)}{\partial x_0} \tag{9}$$

and $\Phi(t_0, t_0, x_0) = I$.

We are now in a position to give the Alekseev formula, which connects the solutions of (7) and (8).

Lemma 2 (see [7]). *If $x(t, t_0, x_0)$ is the solution of (7) and exists for $t \geq t_0$, any solution $y(t, t_0, x_0)$ of (8), with $y(t_0) = x_0$, satisfies the integral equation*

$$y\left(t, t_0, x_0\right) = x\left(t, t_0, x_0\right) + \int_{t_0}^{t} \Phi\left(t, s, y\left(s, t_0, x_0\right)\right) F\left(s, y\left(s, t_0, x_0\right)\right) ds, \tag{10}$$

for $t \geq t_0$, where $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$.

Lemma 3 (see [7]). *Assume that $x(t, t_0, x_0)$ is the solution of (7) through (t_0, x_0), which exists for $t \geq t_0$, then*

$$x\left(t, t_0, x_0\right) = \left[\int_0^1 \Phi\left(t, t_0, s x_0\right) ds\right] x_0, \tag{11}$$

where $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$.

The following theorem gives an analog of the variation of parameters formula for the solution of (2).

Theorem 4. *Assume that $x(t, t_0, x_0)$ is the solution of (5) for $t \geq t_0$, let*

$$G\left(t, t_0, x_0\right) = A \otimes I + I \otimes B^T. \tag{12}$$

Then one has the following.

(i) $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$ exists and is the fundamental matrix solution of the variational equation

$$\varphi' = G\left(t, t_0, x_0\right) \varphi, \tag{13}$$

such that $\Phi(t_0, t_0, x_0) = I$, and therefore

$$\Phi\left(t, t_0, x_0\right) = W\left(t, t_0\right) \otimes Z^T\left(t, t_0\right), \tag{14}$$

where $W(t, t_0)$ and $Z(t, t_0)$ are solutions of

$$W' = A\left(t\right) W, \qquad W\left(t_0\right) = I, \tag{15}$$

$$Z' = Z B\left(t\right), \qquad Z\left(t_0\right) = I, \tag{16}$$

respectively.

(ii) Any solution of (2) satisfies the integral equation

$$Y\left(t, t_0, X_0\right) = X\left(t, t_0, X_0\right) + \int_{t_0}^{t} W\left(t, s\right) R\left(s, Y\left(s, t_0, X_0\right)\right) Z\left(t, s\right) ds, \tag{17}$$

for $t \geq t_0$.

Proof. (i) It is obvious that $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$ exists and is the fundamental matrix solution of the variational equation

$$\varphi' = G\left(t, t_0, x_0\right) \varphi, \tag{18}$$

such that $\Phi(t_0, t_0, x_0) = I$.

Furthermore, we get

$$\varphi' = G\left(t, t_0, x_0\right) \varphi = \left[A \otimes I + I \otimes B^T\right] \varphi, \tag{19}$$

with the initial value

$$\varphi\left(t_0, t_0, x_0\right) = e, \quad e = \overline{\text{vec}}\,\left(I\right), \tag{20}$$

which has the solution

$$\varphi\left(t, t_0, x_0\right) = \left(W\left(t, t_0\right) \otimes Z^T\left(t, t_0\right)\right) e, \tag{21}$$

where W and Z are the solutions of (15) and (16), respectively, and I is the $n \times n$ identity matrix.

Therefore,

$$\Phi\left(t, t_0, x_0\right) = W\left(t, t_0\right) \otimes Z^T\left(t, t_0\right). \tag{22}$$

(ii) Employing Lemma 2 and substituting for Φ the right-hand side of (22), we get

$$y\left(t, t_0, x_0\right) = x\left(t, t_0, x_0\right) + \int_{t_0}^{t}\left[W\left(t, s\right) \otimes Z^T\left(t, s\right)\right] r\left(s, y\left(s, t_0, x_0\right)\right) ds \tag{23}$$

for $t \geq t_0$, where $y(t, t_0, x_0)$ is any solution of (6).

Now, we define $X(t, t_0, X_0)$, $Y(t, t_0, X_0)$, and $R(t, Y)$ by $x = \overline{\text{vec}}\,(X)$, $y = \overline{\text{vec}}\,(Y)$, and $r = \overline{\text{vec}}\,(R)$. Thus, we have that

$$Y\left(t, t_0, X_0\right) = X\left(t, t_0, X_0\right) + \int_{t_0}^{t} W\left(t, s\right) R\left(s, Y\left(s, t_0, X_0\right)\right) Z\left(t, s\right) ds, \tag{24}$$

for $t \geq t_0$, where $X(t, t_0, X_0)$ is the unique solution of (1) for $t \geq t_0$.

The proof is completed. □

3. Main Results

We firstly give some notions.

Definition 5. A generalized matrix valued norm from $R^{m\times n}$ to R^+ is a mapping $\|\cdot\| : R^{m\times n} \to R^+$ such that

(a) $\|X\| \geq 0$, $\|X\| = 0$ if and only if $X = 0$,

(b) $\|\lambda X\| = |\lambda|\|X\|$, λ is a constant,

(c) $\|X + Y\| \leq \|X\| + \|Y\|$.

Definition 6. The zero solution of (1) is said to be

(hS) h-stability if there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function h on R^+ such that

$$\left\|X\left(t, t_0, X_0\right)\right\| \leq c\left\|X_0\right\| h\left(t\right) h^{-1}\left(t_0\right), \tag{25}$$

for $t \geq t_0 \geq 0$ and $\|X_0\| \leq \delta$, $h^{-1}(t_0) = 1/h(t_0)$.

(hSV) h-stability in variation if there exist $c_1, c_2 \geq 1, \delta > 0$, and a positive bounded continuous function h on R^+ satisfying

$$\begin{aligned} \left\|W\left(t, t_0\right)\right\| &\leq c_1 h\left(t\right) h^{-1}\left(t_0\right), \\ \left\|Z\left(t, t_0\right)\right\| &\leq c_2 h\left(t\right) h^{-1}\left(t_0\right), \end{aligned} \tag{26}$$

provided $\|X_0\| \leq \delta$, where $W(t, t_0)$ and $Z(t, t_0)$ are given in Theorem 4.

Lemma 7 (see [4]). *The linear system*

$$x' = A\left(t\right) x, \qquad x\left(t_0\right) = x_0, \tag{27}$$

is hS if and only if there exist a constant $c \geq 1$ and a positive continuous bounded function h defined on R^+ such that for every x_0 in R^n,

$$\left\|\Phi\left(t, t_0, x_0\right)\right\| \leq c h\left(t\right) h^{-1}\left(t_0\right) \tag{28}$$

for all $t \geq t_0 \geq 0$, where $A(t)$ is an $n \times n$ continuous matrix and $\Phi(t, t_0, x_0)$ is a fundamental matrix of (27).

Theorem 8. *The solution $X = 0$ of* (1) *is hS if and only if the solution $x = 0$ of* (5) *is hS.*

Proof. Necessity. Since the solution $X = 0$ of (1) is hS, there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function h on R^+ such that

$$\left\|X\left(t, t_0, X_0\right)\right\| \leq c\left\|X_0\right\| h\left(t\right) h^{-1}\left(t_0\right) \tag{29}$$

for every $t \geq t_0 \geq 0$, $\|X_0\| \leq \delta$, where $X(t, t_0, X_0)$ is the solution of (1), satisfying

$$X' = A\left(t\right) X + XB\left(t\right), \qquad X\left(t_0\right) = X_0, \tag{30}$$

then we obtain

$$\overline{\text{vec}}\left(X'\right) = \overline{\text{vec}}\left(A\left(t\right) X + XB\left(t\right)\right). \tag{31}$$

It follows that

$$[\overline{\text{vec}}\,(X)]' = \left(A \otimes I + I \otimes B^T\right) \overline{\text{vec}}\,(X), \tag{32}$$

Thus,

$$\begin{aligned} \left\|x\left(t, t_0, x_0\right)\right\| &= \left\|\overline{\text{vec}}\left(X\left(t, t_0, X_0\right)\right)\right\| \\ &= \left\|X\left(t, t_0, X_0\right)\right\| \leq c\left\|x_0\right\| h\left(t\right) h^{-1}\left(t_0\right). \end{aligned} \tag{33}$$

Sufficiency. It can be easily proved by the same method. The proof is completed. □

Theorem 9. *The solution $X = 0$ of* (1) *is hS if and only if there exist a constant $c \geq 1$ and a positive continuous bounded function h defined on R^+ such that for every X_0 in $R^{n\times n}$,*

$$\left\|W\left(t, t_0\right) \otimes Z^T\left(t, t_0\right)\right\| \leq c h\left(t\right) h^{-1}\left(t_0\right) \tag{34}$$

for all $t \geq t_0 \geq 0$.

Proof. Sufficiency. Following Lemma 3 and Theorem 4, we have

$$\begin{aligned} x\left(t, t_0, x_0\right) &= \left[\int_0^1 \Phi\left(t, t_0, s x_0\right) ds\right] x_0, \\ \Phi\left(t, t_0, x_0\right) &= W\left(t, t_0\right) \otimes Z^T\left(t, t_0\right). \end{aligned} \tag{35}$$

It follows that

$$x(t,t_0,x_0) = \left[\int_0^1 W(t,t_0) \otimes Z^T(t,t_0) ds\right] x_0. \quad (36)$$

Hence,

$$\begin{aligned}\|x(t,t_0,x_0)\| &\le \|x_0\| \int_0^1 \left\|W(t,t_0) \otimes Z^T(t,t_0)\right\| ds \\ &\le c\|x_0\| h(t) h^{-1}(t_0), \quad t \ge t_0.\end{aligned} \quad (37)$$

Therefore, the solution $x = 0$ of (5) is hS. By Theorem 8, it implies that the solution $X = 0$ of (1) is hS.

Necessity. If the solution $X = 0$ of (1) is hS, then the solution $x = 0$ of (5) is hS using Theorem 8. By Lemma 7, we have

$$\|\Phi(t,t_0,x_0)\| \le ch(t) h^{-1}(t_0). \quad (38)$$

From Theorem 4, we obtain that

$$\Phi(t,t_0,x_0) = W(t,t_0) \otimes Z^T(t,t_0), \quad (39)$$

Thus,

$$\left\|W(t,t_0) \otimes Z^T(t,t_0)\right\| \le ch(t) h^{-1}(t_0). \quad (40)$$

This completes the proof. □

Corollary 10. *If the zero solution of* (1) *is hSV, then the zero solution of* (1) *is hS.*

Next, we offer sufficient conditions for the h-stability of linear matrix differential systems by using the Lyapunov functions.

Defining the Lyapunov functions

$$\begin{aligned}D^+V_{(5)}(t,x) = \limsup_{\delta\to 0} \frac{1}{\delta} &\times \left[V\left(t+\delta, x+\delta\left(A\otimes I + I\otimes B^T\right)x\right)\right. \\ &\left. -V(t,x)\right],\end{aligned} \quad (41)$$

for $(t,x) \in R^+ \times R^{n^2}$ and for the solution $x(t) = x(t,t_0,x_0)$ of (5),

$$D^+V(t,x) = \limsup_{\delta\to 0} \frac{1}{\delta}\left[V(t+\delta, x(t+\delta)) - V(t,x)\right]. \quad (42)$$

Then, it is well known that

$$D^+V_{(5)}(t,x) = D^+V(t,x), \quad (43)$$

if $V(t,x)$ is the Lipschitzian in x for each $t \in R^+$.

Theorem 11. *Suppose that $h(t)$ is a positive bonded continuously differentiable function on R^+. Furthermore, assume that there exists a function $V(t,x)$ satisfying the following properties:*

(i) *$V \in C(R^+ \times R^{n^2}, R^+)$, and $V(t,x)$ is Lipschitzian in x for each $t \in R^+$,*

(ii) $\|x\| \le V(t,x) \le c\|x\|$, $(t,x) \in R^+ \times R^{n^2}$, $c \ge 1$,

(iii) $D^+V_{(5)}(t,x) \le h'(t)h^{-1}(t)V(t,x)$, $(t,x) \in R^+ \times R^{n^2}$.

Then, the solution $X = 0$ of (1) *is hS.*

Proof. Let $x(t,t_0,x_0)$ be the solution of (5). As a consequence of (iii), we obtain

$$\begin{aligned}V(t, X(t,t_0,x_0)) &\le V(t_0,x_0) \exp \int_{t_0}^t \frac{h'(s)}{h(s)} ds \\ &= V(t_0,x_0) h(t) h^{-1}(t_0).\end{aligned} \quad (44)$$

From the condition (ii), we have

$$\|x(t,t_0,x_0)\| \le c\|x_0\| h(t) h^{-1}(t_0), \quad t \ge t_0 \ge 0. \quad (45)$$

By Theorem 8, we can easily get that the solution $X = 0$ of (1) is hS. The proof is completed. □

Now, we examine the properties of the perturbed linear matrix differential system.

Lemma 12 (see [9]). *Suppose that $k(t,x) \in C(R^+ \times R^n, R^n)$ is strictly increasing in x for $t \ge t_0 \ge 0$ with the property*

$$x(t) - \int_{t_0}^t k(s,x(t)) ds \le y(t) - \int_{t_0}^t k(s,y(t)) ds, \quad t \ge t_0 \ge 0 \quad (46)$$

for $x, y \in C([t_0,\infty), R^n)$. If $x(t_0) < y(t_0)$, then $x(t) < y(t)$ for all $t \ge t_0 \ge 0$.

Theorem 13. *Assume that $X = 0$ of* (1) *is hSV with the nonincreasing function h_1 and h_2. Consider the scalar differential equation*

$$u' = cl(t,u), \qquad u(t_0) = u_0, \quad \text{where } c \ge 1. \quad (47)$$

Suppose that

$$\|R(t,Y)\| \le l(t,\|Y\|), \quad (48)$$

where $l \in C(R^+ \times R^+, R^+)$ is strictly increasing in u for each fixed $t \ge t_0 \ge 0$ with $l(t,0) = 0$.

If $u = 0$ is hS, then the solution $Y = 0$ of (2) *is also hS, whenever $u_0 = c\|Y_0\|$.*

Proof. By Theorem 4, the solutions of (1) and (2) with the same initial values are related by

$$\begin{aligned}Y(t,t_0,Y_0) = X(t,t_0,Y_0) &+ \int_{t_0}^t W(t,s) R(s, Y(s,t_0,Y_0)) Z(t,s) ds.\end{aligned} \quad (49)$$

Then, we have

$$\begin{aligned}\|Y(t,t_0,Y_0)\| \le \|X(t,t_0,Y_0)\| &+ \int_{t_0}^t \|W(t,s)\| \\ &\times \|R(s,Y(s,t_0,Y_0))\| \|Z(t,s)\| ds.\end{aligned} \quad (50)$$

From Corollary 10, it easily follows that

$$\begin{aligned}\|Y(t,t_0,Y_0)\| &\le c_1\|Y_0\| h_1(t) h_1^{-1}(t_0) \\ &\quad + \int_{t_0}^{t} c_2c_3\left[h_2(t) h_2^{-1}(s)\right]^2 \\ &\quad \times \|R(s,Y(s,t_0,Y_0))\| ds \\ &\le c\|Y_0\| + c\int_{t_0}^{t} l(s,\|Y(s)\|)\, ds,\end{aligned} \tag{51}$$

where $c = \max\{c_1, c_2c_3\}$. Since $h_1(t)$ and $h_2(t)$ are nonincreasing, we obtain

$$\begin{aligned}&\|Y(t,t_0,Y_0)\| - c\int_{t_0}^{t} l(s,\|Y(s)\|) ds \\ &\quad \le c\|Y_0\| = u_0 = u(t) - \int_{t_0}^{t} cl(s,u(s)) ds.\end{aligned} \tag{52}$$

By Lemma 12, we have $\|Y(t)\| < u(t)$ for all $t \ge t_0 \ge 0$. Since $u = 0$ of (47) is hS,

$$\begin{aligned}\|Y(t)\| < u(t) &\le c_4u_0h(t)h^{-1}(t_0) \\ &= c_4c\|Y_0\| h(t)h^{-1}(t_0) \\ &= M\|Y_0\| h(t)h^{-1}(t_0), \quad c_4 \ge 1,\ M = c_4c \ge 1.\end{aligned} \tag{53}$$

This completes the proof. □

Theorem 14. *Assume that*

(i) *the zero solution of* (1) *is hSV,*

(ii) $\|R(t,Y(t))\| \le \gamma(t)\|Y(t)\|$ *provided that* $\gamma(t) > 0$ *and* $\int_{t_0}^{\infty}\gamma(t)dt < \infty$ *for* $t_0 \ge 0$.

Then, the solution $Y = 0$ *of* (2) *is hS.*

Proof. By Theorem 4, the solutions of (1) and (2) with the same initial values are related by

$$\begin{aligned}Y(t,t_0,X_0) &= X(t,t_0,X_0) \\ &\quad + \int_{t_0}^{t} W(t,s) R(s,Y(s,t_0,X_0)) Z(t,s) ds.\end{aligned} \tag{54}$$

The assumptions (i) and (ii) yield

$$\begin{aligned}\|Y(t,t_0,X_0)\| &\le \|X(t,t_0,X_0)\| \\ &\quad + \int_{t_0}^{t} \|W(t,s)\| \|R(s,Y(s,t_0,X_0))\| \\ &\quad \times \|Z(t,s)\| ds \le c\|X_0\| h(t)h^{-1}(t_0) \\ &\quad + \int_{t_0}^{t} c_1c_2\left[h_1(t)h_1^{-1}(s)\right]^2 \\ &\quad \times \gamma(s)\|Y(s,t_0,X_0)\| ds.\end{aligned} \tag{55}$$

Then, by Gronwall's inequality, we get

$$\begin{aligned}\|Y(t,t_0,X_0)\| &\le c\|X_0\| h(t)h^{-1}(t_0) \\ &\quad \times \exp\left(\int_{t_0}^{\infty} c_1c_2\left[h_1(t)h_1^{-1}(s)\right]^2\gamma(s)\, ds\right) \\ &\le M\|X_0\| h(t)h^{-1}(t_0),\end{aligned} \tag{56}$$

where $M = \max\{c\exp(\int_{t_0}^{\infty} c_1c_2[h_1(t)h_1^{-1}(s)]^2\gamma(s)ds)\}$.

The proof is completed. □

4. Example

In this section, we give a simple but illustrative example. Consider the matrix differential equation

$$\begin{aligned}X'(t) &= \begin{pmatrix}-1 & 1\\ 0 & -2\end{pmatrix} X(t) + X(t)\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \\ X(0) &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{57}$$

Then, we can obtain the following equations:

$$W' = \begin{pmatrix}-1 & 1\\ 0 & -2\end{pmatrix} W, \qquad W(0) = I, \tag{58}$$

$$Z' = Z\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \qquad Z(0) = I. \tag{59}$$

The solutions of (58) and (59) are

$$W(t) = \begin{pmatrix}e^{-t} & e^{-t} - e^{-2t}\\ 0 & e^{-2t}\end{pmatrix}, \qquad Z(t) = \begin{pmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{pmatrix}, \tag{60}$$

respectively.

Then,

$$W(t)\otimes Z^T(t) = \begin{pmatrix}e^{-2t} & 0 & e^{-2t} - e^{-3t} & 0\\ 0 & e^{-2t} & 0 & e^{-2t} - e^{-3t}\\ 0 & 0 & e^{-3t} & 0\\ 0 & 0 & 0 & e^{-3t}\end{pmatrix}. \tag{61}$$

Thus, we have

$$\left\|W(t)\otimes Z^T(t)\right\| \le ch(t)h^{-1}(0), \tag{62}$$

where $h(t) = e^{-2t}$, $c = 5$, $\|W(t)\otimes Z^T(t)\| = 4e^{-2t}$, $\|D\| = \sum_{i,j}^{m,n}|d_{ij}|$, $D \in R^{m\times n}$, and I is an identity matrix. So, from Theorem 9, we can conclude that the solution $X = 0$ of (57) is hS.

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