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Research Article

Common Fixed Points of Generalized Cyclic Meir-Keeler-Type Contractions in Partially Ordered Metric Spaces

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The purpose of this paper is to prove some common point theorems for the generalized cyclic Meir-Keeler-type (α, φ, A, B)-contraction in partially ordered metric spaces. Our results generalize many recent common point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, let D be a subset of X, and let $f : D \to X$ be a map. We say that f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1}$$

The well-known Banach fixed point theorem asserts that if $D = X$, f is contractive, and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasicontraction if there exists $k < 1$ such that

$$\begin{aligned} &d(fx, fy) \\ &\quad\leq k \cdot \max \{d(x, y), d(x, fx), d(y, fy), \\ &\qquad d(x, fy), d(y, fx)\}, \end{aligned} \tag{2}$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

The following definitions and results will be needed in the sequel. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $f : A \cup B \to A \cup B$ is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. In 2003, Kirk et al. [3, 4] proved the following fixed point theorem.

Theorem 1 (see [3, 4]). *Let A and B be two nonempty closed subsets of a complete metric space (X, d), and suppose that $f : A \cup B \to A \cup B$ satisfies*

(i) *$f(A) \subset B$ and $f(B) \subset A$,*
(ii) *$d(fx, fy) \leq k \cdot d(x, y)$ for all $x \in A$, $y \in B$, and $k \in (0, 1)$.*

Then $A \cap B$ is nonempty, and f has a unique fixed point in $A \cap B$.

Recently, many authors proved some fixed point theorems for cyclic maps satisfying various contractive conditions (see, [5–20]).

Let X be a nonempty set, and let $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d. Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered metric space. Two elements $x, y \in X$ are said to be comparable if either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. Altun et al. [21] introduced the notion of weakly increasing mappings and proved some existing theorems.

Definition 2 (see [21]). Let $(X, \sqsubseteq)$ be a partially ordered set and $f, g : X \to X$. Then f, g are said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ for all $x \in X$.

And the following definition was introduced in [22].

Definition 3 (see [22]). Let $(X, \sqsubseteq)$ be a partially ordered set, let A, B be closed subsets of X with $X = A \cup B$, and let $f, g : X \to X$. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \sqsubseteq gfx$ for all $x \in A$ and $gx \sqsubseteq fgx$ for all $x \in B$.

In this paper, we introduce the new notion of generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction. The purpose of this paper is to prove some common point theorems for the generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction in partially ordered metric spaces. Our results generalize many recent common point theorems in the literature.

2. Main Results

In the sequel, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing, continuous function in each coordinate;

(ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t, 0, 2t) \leq t$, $\psi(t, t, t, 2t, 0) \leq t$, $\psi(0, 0, t, t, 0) \leq t$, and $\psi(t, 0, 0, t, t) \leq t$;

(ψ_2) $\psi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

We start with the following definition.

Definition 4 (see [23]). Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1. \tag{3}$$

Definition 5. Let A, B be two nonempty subsets of a set X with $X = A \cup B$, let $f : A \to B$, $g : B \to A$ with $f(A) \subset B$ and $g(B) \subset A$, and let $\alpha : X \times X \to \mathbb{R}^+$. Then the pair (f, g) is called α-admissible if the following conditions hold:

(1) $\alpha(fx, fx) \geq 1, \forall x \in A \Rightarrow \alpha(gfx, gfx) \geq 1$,

(2) $\alpha(gy, gy) \geq 1, \forall y \in B \Rightarrow \alpha(fgy, fgy) \geq 1$.

In 1969, Meir and Keeler [24] introduced the following notion of Meir-Keeler-type contraction in a metric space (X, d).

Definition 6. Letting (X, d) be a metric space, $f : X \to X$. Then f is called a Meir-Keeler-type contraction whenever for each $\eta > 0$, there exists $\gamma > 0$ such that

$$\eta \leq d(x, y) < \eta + \gamma \Longrightarrow d(fx, fy) < \eta. \tag{4}$$

We now state the new notions of generalized cyclic Meir-Keeler-type (ψ, A, B)-contractions and generalized Meir-Keeler-type (α, ψ, A, B)-contractions in partially ordered metric spaces as follows.

Definition 7. Let $(X, \sqsubseteq, d)$ be a partially ordered metric space, let A, B be two nonempty subsets of X with $X = A \cup B$, and let $f : A \to B$, $g : B \to A$ with $f(A) \subset B$ and $g(B) \subset A$. Then the pair (f, g) is called a generalized cyclic Meir-Keeler-type (ψ, A, B)-contraction; if for any comparable elements $x, y \in X$ with $x \in A$ and $y \in B$, we have that for each $\eta > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\eta \leq \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) \\ &\quad < \eta + \delta \\ &\quad \Longrightarrow d(fx, gy) < \eta, \end{aligned} \tag{5}$$

where $\psi \in \Psi$.

Definition 8. Let $(X, \sqsubseteq, d)$ be a partially ordered metric space, let A, B be two nonempty subsets of X with $X = A \cup B$, and let $f : A \to B$, $g : B \to A$ with $f(A) \subset B$ and $g(B) \subset A$, and $\alpha : X \times X \to \mathbb{R}^+$. Then (f, g) is called a generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction if the following conditions hold:

(1) the pair (f, g) is α-admissible;

(2) for any comparable elements $x, y \in X$ with $x \in A$ and $y \in B$, we have that for each $\eta > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\eta \leq \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) \\ &\quad < \eta + \delta \\ &\quad \Longrightarrow \alpha(fx, fx)\alpha(gy, gy)d(fx, gy) < \eta, \end{aligned} \tag{6}$$

where $\psi \in \Psi$.

Remark 9. Note that if f is a generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction, then we have that for any comparable elements $x, y \in X$ with $x \in A$ and $y \in B$,

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy)d(fx, gy) \\ &\quad \leq \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)). \end{aligned} \tag{7}$$

Further, if

$$\psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) = 0, \tag{8}$$

then $d(fx, gy) = 0$.

On the other hand, if

$$\psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)) > 0, \tag{9}$$

then

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy)d(fx, gy) \\ &\quad < \psi(d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)). \end{aligned} \tag{10}$$

We now state our first main result for the generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction as follows.

Theorem 10. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let A, B be nonempty closed subsets of X with $X = A \cup B$, let $\alpha : X \times X \to \mathbb{R}^+$, and let $f, g : X \to X$ be two mappings such that the pair (f, g) is a generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction and (A, B)-weakly increasing. Suppose that the following conditions hold:*

(i) *f or g is continuous;*

(ii) *there exists $x_0 \in A$ with $\alpha(fx_0, fx_0) \geq 1$;*

(iii) *if $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \nu$, then $\alpha(f\nu, f\nu) \geq 1$ and $\alpha(g\nu, g\nu) \geq 1$.*

Then f and g have a common fixed point in X.

Proof. By (ii), there exists $x_0 \in X$ with $\alpha(fx_0, fx_0) \geq 1$. Since $f(A) \subset B$ and the pair (f, g) is α-admissible, there exists $x_1 \in B$ such that

$$x_1 = fx_0, \quad \alpha(gx_1, gx_1) = \alpha(gfx_0, gfx_0) \geq 1. \tag{11}$$

Since $g(B) \subset A$ and the pair (f, g) is α-admissible, there exists $x_2 \in A$ such that

$$x_2 = gx_1, \quad \alpha(fx_2, fx_2) = \alpha(fgx_1, fgx_1) \geq 1. \tag{12}$$

Continuing this process, we construct the sequence $\{x_n\}$ in X such that

$$\begin{gathered} x_{2n+1} = fx_{2n}, \qquad x_{2n+2} = gx_{2n+1}, \\ x_{2n} \in A, \ x_{2n+1} \in B, \end{gathered} \tag{13}$$

and for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{gathered} \alpha(x_{2n+1}, x_{2n+1}) = \alpha(fx_{2n}, fx_{2n}) \geq 1, \\ \alpha(x_{2n+2}, x_{2n+2}) = \alpha(gx_{2n+1}, gx_{2n+1}) \geq 1. \end{gathered} \tag{14}$$

Since the pair (f, g) is (A, B)-weakly increasing, we have that

$$x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2 \sqsubseteq fgx_1 = fx_2 = x_3 \sqsubseteq \cdots, \tag{15}$$

and so we conclude that for all $n \in \mathbb{N} \cup \{0\}$,

$$gfx_{2n} = gx_{2n+1} = x_{2n+2} \sqsubseteq fgx_{2n+1} = fx_{2n+2} = x_{2n+3}. \tag{16}$$

Step 1. We will show that $\{x_n\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$.

Case 1. Suppose that $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$ in the inequality (16). Since x_{2n} and x_{2n+1} are comparable in X with $x_{2n} \in A$ and $x_{2n+1} \in B$, by the Remark 9, we have

$$\begin{aligned} &d(x_{2n+1}, x_{2n+2}) \\ &\quad = d(fx_{2n}, gx_{2n+1}) \\ &\quad \leq \alpha(fx_{2n}, fx_{2n})\,\alpha(gx_{2n+1}, gx_{2n+1})\,d(fx_{2n}, gx_{2n+1}) \\ &\quad \leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \\ &\qquad\quad d(x_{2n}, gx_{2n+1}), d(x_{2n+1}, fx_{2n})) \\ &\quad = \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad\quad d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})) \\ &\quad \leq \psi(0, 0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 0). \end{aligned} \tag{17}$$

If $d(x_{2n+1}, x_{2n+2}) > 0$, then $\psi(0, 0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 0) > 0$. By Remark 9, we get a contradiction. So we conclude that $d(x_{2n+1}, x_{2n+2}) = 0$; that is, $x_{2n+1} = x_{2n+2}$. Similarly, we may show that $x_{2n+2} = x_{2n+3}$. Hence $\{x_n\}$ is a constant sequence, and so $\{x_n\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$.

Case 2. Suppose that $x_{2n} \neq x_{2n+1}$ for all $n \in \mathbb{N}$ in the inequality (16).

Substep 1. We show that the sequence $\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is decreasing.

Subcase 1. If n is even, then we let $n = 2m$ for some $m \in \mathbb{N}$. Since $x_{2m} \in A$, $x_{2m+1} \in B$, and x_{2m}, x_{2m+1} are comparable in X, we have

$$\begin{aligned} &d(x_{n+1}, x_{n+2}) \\ &\quad = d(x_{2m+1}, x_{2m+2}) = d(fx_{2m}, gx_{2m+1}) \\ &\quad \leq \alpha(fx_{2m}, fx_{2m})\,\alpha(gx_{2m+1}, gx_{2m+1})\,d(fx_{2m}, gx_{2m+1}) \\ &\quad < \psi(d(x_{2m}, x_{2m+1}), d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1}), \\ &\qquad\quad d(x_{2m}, gx_{2m+1}), d(x_{2m+1}, fx_{2m})) \\ &\quad = \psi(d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), \\ &\qquad\quad d(x_{2m}, x_{2m+2}), d(x_{2m+1}, x_{2m+1})) \\ &\quad \leq \psi(d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), \\ &\qquad\quad d(x_{2m}, x_{2m+1}) + d(x_{2m+1}, x_{2m+2}), 0) \\ &\quad = \psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\qquad\quad d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0). \end{aligned} \tag{18}$$

If $d(x_{2m}, x_{2m+1}) < d(x_{2m+1}, x_{2m+2})$, then the above inequality becomes

$$\begin{aligned} & d(x_{2m+1}, x_{2m+2}) \\ & \quad < \psi(d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}), \\ & \qquad d(x_{2m+1}, x_{2m+2}), 2d(x_{2m+1}, x_{2m+2}), 0) \\ & \quad \leq d(x_{2m+1}, x_{2m+2}), \end{aligned} \tag{19}$$

which is a contradiction. So we have that

$$d(x_{2m+1}, x_{2m+2}) \leq d(x_{2m}, x_{2m+1}). \tag{20}$$

Subcase 2. If n is odd, then we let $n = 2m+1$ for some $m \in \mathbb{N}$. Since $x_{2m+2} \in A$, $x_{2m+3} \in B$ and x_{2m+2}, x_{2m+3} are comparable in X, we have

$$\begin{aligned} & d(x_{n+2}, x_{n+1}) \\ & \quad = d(x_{2m+3}, x_{2m+2}) = d(fx_{2m+2}, gx_{2m+1}) \\ & \quad \leq \alpha(fx_{2m+2}, fx_{2m+2})\,\alpha(gx_{2m+1}, gx_{2m+1}) \\ & \qquad \times d(fx_{2m+2}, gx_{2m+1}) \\ & \quad < \psi(d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, fx_{2m+2}), \\ & \qquad d(x_{2m+1}, gx_{2m+1}), d(x_{2m+2}, gx_{2m+1}), \\ & \qquad d(x_{2m+1}, fx_{2m+2})) \\ & \quad = \psi(d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, x_{2m+3}), d(x_{2m+1}, x_{2m+2}), \\ & \qquad d(x_{2m+2}, x_{2m+2}), d(x_{2m+1}, x_{2m+3})) \\ & \quad \leq \psi(d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, x_{2m+3}), d(x_{2m+1}, x_{2m+2}), \\ & \qquad 0, d(x_{2m+1}, x_{2m+2}) + d(x_{2m+2}, x_{2m+3})) \\ & \quad = \psi(d(x_{n+1}, x_n), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}), \\ & \qquad 0, d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})). \end{aligned} \tag{21}$$

If $d(x_{2m+1}, x_{2m+2}) < d(x_{2m+2}, x_{2m+3})$, then the above inequality becomes

$$\begin{aligned} & d(x_{2m+2}, x_{2m+3}) \\ & \quad < \psi(d(x_{2m+2}, x_{2m+3}), d(x_{2m+2}, x_{2m+3}), \\ & \qquad d(x_{2m+3}, x_{2m+3}), 0, 2d(x_{2m+2}, x_{2m+3})) \\ & \quad \leq d(x_{2m+2}, x_{2m+3}), \end{aligned} \tag{22}$$

which is a contradiction. So we have that

$$d(x_{2m+2}, x_{2m+3}) < d(x_{2m+1}, x_{2m+2}). \tag{23}$$

From (20) and (23), we conclude that

$$d(x_{+1}, x_{n+2}) < d(x_{n+1}, x_{n+1}). \tag{24}$$

From the above argument, we have that the sequence $\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is decreasing, and it must converge to some $\eta \geq 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \eta. \tag{25}$$

Substep 2. We next claim that

$$\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{26}$$

Notice that $\eta = \inf\{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\eta = 0$. Suppose, to the contrary, that $\eta > 0$.

If n is even, by the argument of Subcase 1 and the inequality (25), we have

$$\begin{aligned} \lim_{n \to \infty} & \psi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 0) = \eta. \end{aligned} \tag{27}$$

Since (f, g) is a generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction, corresponding to η use and taking into account the above (27), there exist $\delta > 0$ and a natural number k such that

$$\begin{aligned} & \eta \leq \psi(d(x_k, x_{k+1}), d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}), d(x_k, x_{k+1}) \\ & \qquad + d(x_{k+1}, x_{k+2}), 0) < \eta + \delta \\ & \Longrightarrow \alpha(fx_k, fx_k)\,\alpha(gx_{k+1}, gx_{k+1})\,d(fx_k, gx_{k+1}) < \eta, \end{aligned} \tag{28}$$

which implies

$$\begin{aligned} & d(fx_k, gx_{k+1}) \\ & \quad \leq \alpha(fx_k, fx_k)\,\alpha(gx_{k+1}, gx_{k+1})\,d(fx_k, gx_{k+1}) < \eta. \end{aligned} \tag{29}$$

So we get a contradiction, since $\eta = \inf\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}$. Thus we have that

$$\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{30}$$

If n is odd, by the argument of Subcase 2 and the inequality (25), we have

$$\begin{aligned} \lim_{n \to \infty} & \psi(d(x_{n+1}, x_n), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}), \\ & 0, d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) = \eta. \end{aligned} \tag{31}$$

Similarly, we can prove that

$$\lim_{n \to \infty} d(x_n, fx_{n+1}) = 0. \tag{32}$$

Substep 3. We show that $\{x_n\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$.

Suppose, to the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence in $(X, \sqsubseteq, d)$. Then there exist $\epsilon > 0$ and two

subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest integer for which $n(k) > m(k) > k$,

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon, \qquad d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon, \tag{33}$$

and we get

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) \\ &\quad + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \tag{34}$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon. \tag{35}$$

On the other hand, we also obtain that

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)+1}, x_{2n(k)-1}) + d(x_{2m(k)}, x_{2m(k)+1}) \\ &\quad + d(x_{2n(k)-1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-1}) + 2d(x_{2m(k)}, x_{2m(k)+1}) \\ &\quad + d(x_{2n(k)-1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)}) + 2d(x_{2m(k)}, x_{2m(k)+1}) \\ &\quad + 2d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \tag{36}$$

Letting $k \to \infty$ in the above inequality, we get

$$\begin{aligned} &\lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)}) \\ &\quad = \lim_{n\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) \\ &\quad = \lim_{n\to\infty} d(x_{2m(k)+1}, x_{2n(k)-1}) \\ &\quad = \epsilon. \end{aligned} \tag{37}$$

Since

$$\begin{aligned} &d(x_{2m(k)+1}, x_{2n(k)-1}) \\ &\quad \leq d(x_{2m(k)+1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)-1}) \\ &\quad \leq d(x_{2m(k)+1}, x_{2n(k)-1}) + 2d(x_{2n(k)}, x_{2n(k)-1}), \end{aligned} \tag{38}$$

letting $k \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) = \epsilon. \tag{39}$$

Since $x_{2m(k)} \in A$, $x_{2n(k)-1} \in B$, and $x_{2m(k)}$, $x_{2n(k)-1}$ are comparable in X, we have

$$\begin{aligned} &d(x_{2m(k)+1}, x_{2n(k)}) \\ &\quad = d(fx_{2m(k)}, gx_{2n(k)-1}) \\ &\quad \leq \alpha(fx_{2m(k)}, fx_{2m(k)})\,\alpha(gx_{2n(k)-1}, gx_{2n(k)-1}) \\ &\qquad \times d(fx_{2m(k)}, gx_{2n(k)-1}) \\ &\quad < \psi(d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2m(k)}, fx_{2m(k)}), \\ &\qquad\quad d(x_{2n(k)-1}, gx_{2n(k)-1}), d(x_{2m(k)}, gx_{2n(k)-1}), \\ &\qquad\quad d(x_{2n(k)-1}, fx_{2m(k)})) \\ &\quad = \psi(d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2m(k)}, x_{2m(k)+1}), \\ &\qquad\quad d(x_{2n(k)-1}, x_{2n(k)}), d(x_{2m(k)}, x_{2n(k)}), \\ &\qquad\quad d(x_{2n(k)-1}, x_{2m(k)+1})). \end{aligned} \tag{40}$$

Letting $k \to \infty$ in the above inequality and using (37) and (39), we get

$$\epsilon = \lim_{n\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) < \psi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \epsilon, \tag{41}$$

which implies a contradiction. So we get that $\{x_n\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$.

Step 2. Finally, we prove the existence of common fixed point of f and g.

Since $(X, \sqsubseteq, d)$ is complete and $\{x_n\}$ is a Cauchy sequence in $(X, \sqsubseteq, d)$, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n-1} = \nu. \tag{42}$$

From (42) and since $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(f\nu, f\nu) \geq 1$ and $\alpha(g\nu, g\nu) \geq 1$.

Since $\{x_{2n}\}$ is a sequence in A and A is closed, by (42), we have that $\nu \in A$. Similarly, since $\{x_{2n+1}\}$ is a sequence in B and B is closed, by (42), we have that $\nu \in B$. We now claim that ν is a common fixed point of f and g. Without loss of generality, we assume that f is continuous, and by (42), we have

$$x_{2n+1} = fx_{2n} \longrightarrow \nu, \quad \text{as } n \to \infty. \tag{43}$$

By the uniqueness of the limit, we have that $\nu = f\nu$.

Since $\nu \sqsubseteq \nu$ with $\nu \in A$ and $\nu \in B$, we have

$$\begin{aligned} d(\nu, g\nu) &= d(f\nu, g\nu) \\ &\leq \alpha(f\nu, f\nu)\,\alpha(g\nu, g\nu)\,d(f\nu, g\nu) \\ &< \psi(d(\nu, \nu), d(\nu, f\nu), d(\nu, g\nu), \\ &\qquad d(\nu, g\nu), d(\nu, f\nu)) \\ &= \psi(0, 0, d(\nu, g\nu), d(\nu, g\nu), 0) \\ &\leq d(\nu, g\nu). \end{aligned} \tag{44}$$

This implies that $\nu = g\nu$. So we complete the proof. □

Applying Theorem 10 and if we let $\alpha(x, y) = 1$, then we immediately get the following theorem.

Theorem 11. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let A, B be nonempty closed subsets of X with $X = A \cup B$, and let $f, g : X \to X$ be two mappings such that the pair (f, g) is a generalized cyclic Meir-Keeler-type (ψ, A, B)-contraction and (A, B)-weakly increasing. If f or g is continuous, then f and g have a common fixed point in X.*

We next state our second main result for the generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction as follows.

Theorem 12. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let A, B be nonempty closed subsets of X with $X = A \cup B$, let $\alpha : X \times X \to \mathbb{R}^+$, and let $f, g : X \to X$ be two mappings such that the pair (f, g) is a generalized cyclic Meir-Keeler-type (α, ψ, A, B)-contraction and (A, B)-weakly increasing. Suppose that the following conditions hold:*

(i) *if $\{x_n\}$ is a nondecreasing sequence in X and $\lim_{n\to\infty} x_n = \nu$, then $x_n \sqsubseteq \nu$;*

(ii) *there exists $x_0 \in A$ with $\alpha(fx_0, fx_0) \geq 1$;*

(iii) *if $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \nu$, then $\alpha(f\nu, f\nu) \geq 1$ and $\alpha(g\nu, g\nu) \geq 1$.*

Then f and g have a common fixed point in X.

Proof. From the same proof's process of Theorem 10, we can construct a nondecreasing sequence $\{x_n\}$ in X with $x_{2n} \in A$, $x_{2n+1} \in B$, and $x_n \to \nu$ for some $\nu \in X$. Since $x_n \to \nu$ and A, B are nonempty closed subsets of X, we have $x_{2n} \to \nu$, $x_{2n+1} \to \nu$, and $\nu \in A \cap B$. By the condition (i), we get $x_n \sqsubseteq \nu$ for all $n \in \mathbb{N}$.

Since $x_{2n} \in A$ and $\nu \in B$, we have

$$\begin{aligned} d&(x_{2n+1}, g\nu) \\ &= d(fx_{2n}, g\nu) \\ &\leq \alpha(fx_{2n}, fx_{2n})\,\alpha(g\nu, g\nu)\, d(fx_{2n}, g\nu) \\ &< \psi(d(x_{2n}, \nu), d(x_{2n}, fx_{2n}), d(\nu, g\nu), \\ &\qquad d(x_{2n}, g\nu), d(\nu, fx_{2n})) \\ &= \psi(d(x_{2n}, \nu), d(x_{2n}, x_{2n+1}), d(\nu, g\nu), \\ &\qquad d(x_{2n}, g\nu), d(\nu, x_{2n+1})). \end{aligned} \tag{45}$$

Letting $n \to \infty$ in the above inequality, we get

$$d(\nu, g\nu) < \psi(0, 0, d(\nu, g\nu), d(\nu, g\nu), 0) \leq d(\nu, g\nu). \tag{46}$$

This implies that $d(\nu, g\nu) = 0$; that is, $\nu = g\nu$. Similarly, we may show that $\nu = f\nu$. So ν is a common fixed point of f and g. □

Applying Theorem 12, it is easy to get the following theorem.

Theorem 13. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let A, B be nonempty closed subsets of X with $X = A \cup B$, and let $f, g : X \to X$ be two mappings such that the pair (f, g) is a generalized cyclic Meir-Keeler-type (ψ, A, B)-contraction and (A, B)-weakly increasing. Suppose that the following condition holds:*

if $\{x_n\}$ is a nondecreasing sequence in X and $\lim_{n\to\infty} x_n = \nu$, then $x_n \sqsubseteq \nu$.

Then f and g have a common fixed point in X.

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