Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 792431, 9 pages
http://dx.doi.org/10.1155/2013/792431

Research Article

Relaxation Problems Involving Second-Order Differential Inclusions

Adel Mahmoud Gomaa[1,2]

[1] *Taibah University, Faculty of Applied Science, Department of Applied Mathematics, Al-Madinah, Saudi Arabia*
[2] *Mathematics Department, Faculty of Science, Helwan University, Cairo, Egypt*

Correspondence should be addressed to Adel Mahmoud Gomaa; gomaa_5@hotmail.com

Received 12 November 2012; Accepted 12 March 2013

Academic Editor: Malisa R. Zizovic

Copyright © 2013 Adel Mahmoud Gomaa. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present relaxation problems in control theory for the second-order differential inclusions, with four boundary conditions, $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $[0, 1]$; $u(0) = 0$, $u(\eta) = u(\theta) = u(1)$ and, with $m \geq 3$ boundary conditions, $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $[0, 1]$; $\dot{u}(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, where $0 < \eta < \theta < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and F is a multifunction from $[0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$ to the nonempty compact convex subsets of $\mathbb{R}^n$. We have results that improve earlier theorems.

1. Introduction

Second-order differential inclusions of three boundary conditions were studied by many authors [1–6], using Hartman-type functions. Such a function was first introduced by [7] for two boundary conditions. Moreover, in [8] we consider second-order differential inclusions with four boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, T], \\ u(0) = x_0, \quad u(\eta) = u(\theta) = u(T), \end{gathered} \tag{1}$$

where $0 < \eta < \theta < T$ and F is a multifunction from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ to the nonempty compact subsets of $\mathbb{R}^n$, while in [9] we study four-point boundary value problems for differential inclusions and differential equations with and without multivalued moving constraints.

In the present paper, we study relaxation results for the second-order differential inclusions, with four boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \\ u(0) = 0, \quad u(\eta) = u(\theta) = u(1) \end{gathered} \tag{P}$$

and, with $m \geq 3$ boundary conditions,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \\ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{gathered} \tag{Q}$$

where $0 < \eta < \theta < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and F is a multifunction from $[0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$ to the non-empty compact subsets of $\mathbb{R}^n$.

In conjunction with Problem (P) and Problem (Q) we also consider the following problems:

$$\begin{gathered} \ddot{u}(t) \in \operatorname{ext} F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \\ u(0) = 0, \quad u(\eta) = u(\theta) = u(1), \end{gathered} \tag{P_e}$$

$$\begin{gathered} \ddot{u}(t) \in \operatorname{ext} F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \\ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \end{gathered} \tag{Q_e}$$

By $\operatorname{ext} F(t, u(t), \dot{u}(t))$, we denote the set of extreme points of $F(t, u(t), \dot{u}(t))$.

2. Notations and Preliminaries

Throughout this paper we let $I = [0,1]$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. We will use the following definitions, notations, and summarize some results.

(i) A multifunction F from a metric space (X,d) to the set $P_f(Y)$ of all closed subsets of another metric space Y is lower semicontinuous *(l. s. c.)* at $x_0 \in X$ if for every open subset V in Y with $F(x_0) \cap V \neq \emptyset$ there exists an open subset U in X such that $x_0 \in U$ and $F(x) \cap V \neq \emptyset$ for all $x \in U$. F is *l. s. c.* if it is *l. s. c.* at each $x_0 \in X$.

(ii) F is upper semicontinuous *(u. s. c.)* at $x_0 \in X$ if for every open subset V in Y and containing $F(x_0)$ there exists an open subset U in X such that $x_0 \in U$ and $F(x) \subseteq V$, for all $x \in U$. F is *u. s. c.* if it is *u. s. c.* at each $x_0 \in X$.

(iii) A multifunction F from I into the set $P_f(X)$ of all closed subsets of X is measurable if for all $x \in X$ the function $t \to d(x, F(t)) = \inf\{\| x - y \| : y \in F(t)\}$ is measurable [10–13].

(iv) Let (Ω, Σ) be a measurable space and X a separable Banach space. We say that $F : \Omega \to P_f(X)$ is graph measurable if

$$gr(F) = \{(z,x) \in \Omega \times X : x \in F(z)\} \in \Sigma \times \mathscr{B}(X), \quad (2)$$

where $\mathscr{B}(X)$ is the Borel σ-field of X. For further details we refer to [14–16].

(v) F is continuous if it is lower and upper semicontinuous.

(vi) For each $A, B \in P_f(X)$, the Hausdorff metric is defined by

$$d_H(A,B) = \max\left[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right]. \quad (3)$$

It is known that the space $(P_f(X), d_H)$ is a generalized metric space, if the sets are not bounded (see, for instance, [14, 15]).

(vii) A multifunction F is Hausdorff continuous (d_H-continuous) if it is continuous from X into the metric space $(P_f(Y), d_H)$.

(viii) If F has compact values in Y, then F is d_H-continuous if and only if it is continuous [14, 17].

(ix) We denote by $P_{kc}(\mathbb{R}^n)$ the nonempty compact convex subsets of $\mathbb{R}^n$.

(x) The Banach spaces $C(I,\mathbb{R}^n)$, $C^1(I,\mathbb{R}^n)$, and $C^2(I,\mathbb{R}^n)$ endowed with the norms

$$\|u\|_C = \max_{t\in I} |u(t)|, \qquad \|u\|_{C^1} = \max\{\|u\|_C, \|\dot{u}\|_C\}, \qquad \|u\|_{C^2} = \max\{\|u\|_C, \|\dot{u}\|_C, \|\ddot{u}\|_C\}, \quad (4)$$

respectively.

(xi) $L^1_w(I,\mathbb{R}^n)$ denotes the space $L^1(I,\mathbb{R}^n)$ equipped with weak norm $\|\cdot\|_w$ which is defined by

$$\|h\|_w = \sup\left\{\left\|\int_a^b h(t)\,dt\right\| : 0 \le a \le b \le 1\right\}. \quad (5)$$

(xii) $W^{2,1}(I,\mathbb{R}^n)$ is the Sobolev space of functions $u : I \to \mathbb{R}^n$, u and $\dot{u}$ are both absolutely continuous functions so $\ddot{u}(t) \in L^1(I,\mathbb{R}^n)$ and it is equipped with the norm $\|u\|_{W^{2,1}(I,\mathbb{R}^n)} = \|u\|_{L^1(I,\mathbb{R}^n)} + \|\dot{u}\|_{L^1(I,\mathbb{R}^n)} + \|\ddot{u}\|_{L^1(I,\mathbb{R}^n)}$.

(xiii) Let $R : I \to 2^{\mathbb{R}^n}$ be a multifunction and $\delta^1_R = \{h \in L^1(I,\mathbb{R}^n) : h(t) \in R(t)\}$.

(xiv) By a solution of (P) (resp., of (P_e)) we mean a function $u \in W^{2,1}(I,\mathbb{R}^n)$ such that $\ddot{u}(t) = h(t)$ a.e. on I with $h \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ (resp., $h \in \delta^1_{\text{ext}F(\cdot,u(\cdot),\dot{u}(\cdot))}$) and $u(0) = 0$, $u(\eta) = u(\theta) = u(1)$.

(xv) By a solution of (Q) (resp., of (Q_e)) we mean a function $u \in W^{2,1}(I,\mathbb{R}^n)$ such that $\ddot{u}(t) = h(t)$ a.e. on I with $h \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ (resp., $h \in \delta^1_{\text{ext}F(\cdot,u(\cdot),\dot{u}(\cdot))}$) and $\dot{u}(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$.

(xvi) In the sequel by Δ_P (resp., Δ_{P_e}) we denote the solution set of Problem (P) (resp., of Problem (P_e)). Moreover, by Δ_Q (resp., Δ_{Q_e}) we denote the solution set of Problem (Q) (resp., of Problem (Q_e)).

Definition 1. Let E be a Banach space and let Y be a metric space. A multifunction $G : I \times Y \to P_{ck}(E)$ has the Scorza-Dragoni property (the SD-property) if for every $\varepsilon > 0$ there exists a closed set $A \subset I$ such that the Lebesgue measure $\mu(I \setminus A)$ is less than ε and $G|_{A\times Y}$ is continuous. The multifunction G is called integrably bounded on compacta in Y if, for any compact subset $Q \subset Y$, we can find an integrable function $\mu_Q : I \to \mathbb{R}^+$ such that $\sup\{\| y \| : y \in G(t,z)\} \le \mu_Q(t)$, for almost every $z \in Q$.

Theorem 2 (see [18]). *Let Y be a complete metric space, E a separable Banach space, E_σ the Banach space E endowed with the weak topology, $M : I \times Y \to P_{ck}(E_\sigma)$, and K a compact subset of $C(I,Y)$. Furthermore, let $R : K \to 2^{L^1(I,E)}$ be a multifunction defined by*

$$R(y) = \left\{g \in L^1(I,E) : g(t) \in M(t, y(t)) \text{ a.e. on } I\right\}. \quad (6)$$

If M has the SD-property and is integrably bounded on compacta in Y, then the set

$$A_K = \left\{f \in C\left(K, L^1_w(I,E)\right) : f(y) \in R(y) \ \forall y \in K\right\} \quad (7)$$

is nonempty complete subset of the space $C(K, L^1_w(I,E))$. Moreover, $A_K = \overline{A_{\text{ext}\,K}}$ where $L^1_w(I,E)$ is the space of equivalence classes of Bochner-integrable functions $v : I \to E$ with the norm $\| v\|_w = \sup_{t\in T} \| \int_0^t v(s)ds \|$ and

$$A_{\text{ext}\,K} = \left\{f \in C\left(K, L^1_w(I,E)\right) : f(y) \in \text{ext}R(y) \ \forall y \in K\right\}. \quad (8)$$

Lemma 3 (see [19]). *For p such that $1 < p < \infty$ let $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^p(I,\mathbb{R}^n)$, $\sup_{n\in\mathbb{N}} \|u_n\|_p < \infty$ and $u_n \to u$ with respect to the weak norm $\|\cdot\|_w$. Then $u_n \to u$ weakly in $L^p(I,\mathbb{R}^n)$.*

Next we state a preliminary lemma, for $0 < \eta < \theta < 1$, which is useful in the study of four boundary problems for the differential equations and the differential inclusions, and moreover we summarize some properties of a Hartman-type function.

Lemma 4 (see [8]). *Let $G : I\times I \to \mathbb{R}$ be the function defined as follows:*
as $0 \le t < \eta$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\le\tau\le t\\ -t & \text{if } t<\tau\le\eta\\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } \eta<\tau\le\theta\\ \dfrac{1-\tau}{1-\theta} & \text{if } \theta<\tau\le 1,\end{cases}\tag{9}$$

when $\eta \le t < \theta$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\le\tau\le\eta\\ \dfrac{\tau(t-\theta+1)+\eta(\tau-t-1)}{\theta-\eta} & \text{if } \eta<\tau\le t\\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } t<\tau\le\theta\\ \dfrac{1-\tau}{1-\theta} & \text{if } \theta<\tau\le 1,\end{cases}\tag{10}$$

lastly if $\theta \le t \le 1$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\le\tau\le\eta\\ \dfrac{\eta(\tau-t-1)+\tau(t-\theta+1)}{\theta-\eta} & \text{if } \eta<\tau\le\theta\\ \dfrac{1-\tau}{1-\theta}+(t-\tau) & \text{if } \theta<\tau\le t\\ \dfrac{1-\tau}{1-\theta} & \text{if } t<\tau\le 1.\end{cases}\tag{11}$$

Then the following hold.

(i) *If $u \in W^{2,1}(I,\mathbb{R}^n)$ with $u(0)=x_0, u(1)=u(\theta)=u(\eta)$, then*

$$u(t)=x_0+\int_0^1 G(t,\tau)\ddot{u}(\tau)\,d\tau,\quad \forall t\in I;\tag{12}$$

(ii) *if $w \in L^1(I,\mathbb{R}^n)$, then for all $t\in I$,*

$$\begin{aligned}\int_0^1 G(t,\tau)w(\tau)\,d\tau &= \int_0^t (t-\tau)w(\tau)\,d\tau\\ &\quad-\int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}w(\tau)\,d\tau\\ &\quad+\int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}w(\tau)\,d\tau\\ &\quad+\int_\theta^1 \frac{1-\tau}{1-\theta}w(\tau)\,d\tau;\end{aligned}\tag{13}$$

(iii) $\sup_{t,\tau\in I}|G(t,\tau)|\le 2$, $\sup_{t,\tau\in I}|\partial G(t,\tau)/\partial t|\le 1$.

Let $c_1, c_2, a \in L^p(I,\mathbb{R}^+)$, $1<p<\infty$, and let L be a linear operator from $C(I,\mathbb{R})\times C(I,\mathbb{R})$ to $C(I,\mathbb{R})\times C(I,\mathbb{R})$ defined by $L(f,g)=(\underline{f},\underline{g})$ such that, for all $t\in I$,

$$\begin{aligned}\underline{f}(t)&=\int_0^T|G(t,\tau)|\left(c_1(\tau)f(\tau)+c_2(\tau)g(\tau)\right)d\tau,\\ \underline{g}(t)&=\int_0^T\left|\frac{\partial G(t,\tau)}{\partial t}\right|\left(c_1(\tau)f(\tau)+c_2(\tau)g(\tau)\right)d\tau.\end{aligned}\tag{14}$$

If $c_1=c_2=0$, then clearly $L=0$. We note that if $\mathscr{K}=\{(h_1,h_2)\in C(I,\mathbb{R})\times C(I,\mathbb{R}) : h_1(t),h_2(t)\ge 0,\ \forall t\in I\}$, then $L(\mathscr{K})\subseteq\mathscr{K}$. Moreover, the spectral radius $r(L)=\lim\|L^n\|^{1/n}$ is an eigenvalue of L with an eigenvector in $\mathscr{K}$ [20].

3. Relaxation Theorems

In this section, both Theorems 5 and 7 improve [19, Theorem 4.1] with [21, Theorem 6]. Indeed in [19] Papageorgiou considered (P) and (P_e) with the two boundary conditions $u(0)=u(1)=0$ and in [21] Ibrahim and Gomaa study the same problems with three boundary conditions $u(0)=x_0,\ u(\eta)=u(1)$.

Theorem 5. *Let $F : I\times\mathbb{R}^n\times\mathbb{R}^n \to P_{kc}(\mathbb{R}^n)$ be a multifunction such that*

(i) *for each $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, the multifunction $F(\cdot,x,y)$ is measurable,*

(ii) $d_H(F(t,x,y),F(t,x',y'))\le\alpha_1(t)\|x-x'\|+\alpha_2(t)\|y-y'\|$ *a.e. with $\alpha_1, \alpha_2\in L^1(I,\mathbb{R}^+)$ and $\|\alpha_1+\alpha_2\|<1/2$,*

(iii) *for each $(t,x,y)\in I\times\mathbb{R}^n\times\mathbb{R}^n$,*

$$\begin{aligned}\|F(t,x,y)\|&=\sup\{\|v\| : v\in F(t,x,y)\}\\ &\le a(t)+c_1(t)\|x\|+c_2(t)\|y\|\end{aligned}\tag{15}$$

with $a,c_1,c_2\in L^p(I,\mathbb{R}^+)$ $1<p<\infty$,

(iv) *the spectral radius, $r(L)$, is less than* 1.

Then for each solution $u\in\Delta_{P_e}$, there is a sequence $(u_m(\cdot))_{m\in\mathbb{N}}\subset\Delta_P$ converging to $u(\cdot)$ in $(C^1(I,\mathbb{R}^n),\|\cdot\|_{C^1})$.

Proof. From [9, Theorem 2.1], we obtain $\Delta_{P_e}\ne\emptyset$. Moreover, we can say that $\|F(t,x,y)\|\le a_1(t)$ a.e. on I for some $a_1\in L^p(I,\mathbb{R}^+)$. Let $u\in\Delta_P$. Then

$$\begin{aligned}&\ddot{u}(t)=h(t),\quad \text{a.e. on } I,\\ &u(0)=0,\ u(\eta)=u(\theta)=u(1),\end{aligned}\tag{16}$$

where $h(t)\in F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $f : L^1(I,\mathbb{R}^n)\to C^1(I,\mathbb{R}^n)$ is a function such that, for each $h\in L^1(I,\mathbb{R}^n)$, $f(h)\in W^{1,2}(I,\mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$\begin{aligned}&\ddot{u}(t)=h(t),\quad \text{a.e. on } I,\\ &u(0)=0,\ u(\eta)=u(\theta)=u(1).\end{aligned}\tag{P_h}$$

Let $\mathcal{S} = \{u \in L^1(I,\mathbb{R}^n) : \| u(t) \| \le a_1(t)$ a.e. on $I\}$. It is easy to see that $f(\mathcal{S})$ is convex. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $f(\mathcal{S})$. Hence, $u_n \in W^{2,1}(I,\mathbb{R}^n)$ with $u_n(0) = x_0$, $u_n(\eta) = u_n(\theta) = u_n$ (17) and

$$u_n(t) = x_0 + \int_0^t (t-\tau)\ddot{u}_n(\tau)\,d\tau - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}\ddot{u}_n(\tau)\,d\tau + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}\ddot{u}_n(\tau)\,d\tau + \int_\theta^1 \frac{1-\tau}{1-\theta}\ddot{u}_n(\tau)\,d\tau. \tag{17}$$

Then,

$$\lim_{n\to\infty} u_n(t) = \int_0^1 G(t,\tau)\ddot{u}(\tau)\,d\tau = u(t), \tag{18}$$

which means that $f(\mathcal{S})$ is a compact subset of $C^1(I,\mathbb{R}^n)$. Set

$$\mathcal{P}_\varepsilon(t) = \{x \in F(t,v(t),\dot{v}(t)) : \|h(t)-x\| < \varepsilon + d(h(t), F(t,v(t),\dot{v}(t)))\}, \tag{19}$$

where $\varepsilon > 0$ and $v \in f(\mathcal{S})$. Hence, for each $t \in I$, $\mathcal{P}_\varepsilon(t) \neq \emptyset$. Assume that $\mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R}^n)$ are the Borel σ-fields of I and $\mathbb{R}^n$, respectively. From condition, (i) the function $t \to F(t,v(t),\dot{v}(t))$ is measurable. Hence, $grF(\cdot,v(\cdot),\dot{v}(\cdot)) \in \mathcal{B}(I)\times\mathcal{B}(\mathbb{R}^n)$ and $(t,x) \to \varepsilon d(h(t),F(t,v(t),\dot{v}(t))) - \| h(t)-x \|$ is measurable in t and continuous in x that is jointly measurable. Thus, by Aumann's selection theorem, there exists a measurable selection s_ε of $\mathcal{P}_\varepsilon$ such that $s_\varepsilon(t) \in \mathcal{P}_\varepsilon(t)$ for each $t \in I$. Now we define a multifunction $\mathcal{Q}_\varepsilon : f(\mathcal{S}) \to 2^{L^1(I,\mathbb{R}^n)}$ by the following:

$$\mathcal{Q}_\varepsilon(v) = \Big\{x \in \delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} : \|h(t)-x\| < \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \text{ a.e. on } I\Big\}, \tag{20}$$

with $\mathcal{Q}_\varepsilon(v)(t) \neq \emptyset$ for each $v \in f(\mathcal{S})$. From [22, Proposition 4], $\mathcal{Q}_\varepsilon$ is *l. s. c.* and clearly has decomposable values. Applying [22, Theorem 3], we have a continuous selection S_ε of $\overline{\mathcal{Q}_\varepsilon}$. Therefore,

$$\begin{aligned}\|h(t)-S_\varepsilon(v)(t)\| &\le \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \\ &\le \varepsilon + \alpha_1(t)\|u(t)-v(t)\| \\ &\quad + \alpha_2(t)\|\dot{u}(t)-\dot{v}(t)\| \quad \text{a.e. on } I.\end{aligned} \tag{21}$$

From Theorem 2, we find a continuous function $\xi_\varepsilon : f(\mathcal{S}) \to L^1_w(I,\mathbb{R}^n)$ such that $\xi_\varepsilon(v) \in \text{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))}$ and $\| S_\varepsilon(v) - \xi_\varepsilon(v) \| < \varepsilon$ for each $v \in f(\mathcal{S})$. Define a multifunction $R : f(\mathcal{S}) \to 2^{L^1(I,\mathbb{R}^n)}$ by

$$R(u) = \Big\{g \in L^1(I,\mathbb{R}^n) : g(t) \in F(t,u(t),\dot{u}(t)) \text{ a.e. on } I\Big\}. \tag{22}$$

Assume that $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t,(x,y)) = F(t,x,y)$. From Theorem 3.1 in [23], M has SD-property. R has nonempty convex values. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $R(u)$ for some $u \in f(\mathcal{S})$. So, for each $t \in I$,

$$\lim_{n\to\infty} g_n(t) = g(t) \in F(t,u(t),\dot{u}(t)) \tag{23}$$

because F has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ which implies that $R(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $\theta : f(\mathcal{S}) \to L^1_w(I,\mathbb{R}^n)$ such that $\theta(u) \in \text{ext}(R(u))$, for all $u \in f(\mathcal{S})$. We see that $\theta(u)(t) \in \text{ext}(M(t,(u(t),\dot{u}(t))))$ [24], hence $\theta(u)(t) \in \text{ext}F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $\eta : f(\mathcal{S}) \to W^{1,2}(I,\mathbb{R}^n)$ is the function which for each $u \in f(\mathcal{S})$, $\eta(u) = g(\theta(u))$. For each $u \in f(\mathcal{S})$, we have $\| \theta(u)(t) \| \le a_1$ and so $\theta(u) \in \mathcal{S}$. Then, η is a function from $f(\mathcal{S})$ into $f(\mathcal{S})$ and also we see that η is continuous [19]. Now let $\varepsilon_n \to 0$, $S_{\varepsilon_n} = S_n$ and $\xi_n = \xi_{\varepsilon_n}$. Then, for each $n \in \mathbb{N}$, the function $fo\xi_n$ is a continuous function from the compact set $f(\mathcal{S})$ into itself. From Schauder's fixed point theorem, $fo\xi_n$ has a fixed point u_n, but $\text{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} = \delta^1_{\text{ext}F(\cdot,v(\cdot),\dot{v}(\cdot))}$ [24] so $u_n \in \Delta_{P_e}$. By passing to a subsequence if necessary, we may assume that $u_n \to \widehat{u}$ in $C^1(I,\mathbb{R}^n)$. Then, we obtain

$$\begin{aligned}\|u_n(t)-u(t)\| &\le \int_0^1 \Bigg\| \Bigg[\int_0^t (t-\tau)(\xi_n(\tau)-h(\tau))\,d\tau - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}(\xi_n(\tau)-h(\tau))\,d\tau \\ &\quad + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}(\xi_n(\tau)-h(\tau))\,d\tau + \int_\theta^1 \frac{1-\tau}{1-\theta}(\xi_n(\tau)-h(\tau))\,d\tau\Bigg]\Bigg\|\,ds \\ &\le \int_0^1 \Bigg[\int_0^t (t-\tau)\|\xi_n(\tau)-S_n(\tau)\|\,d\tau + \int_0^t (t-\tau)\|h(\tau)-S_n(\tau)\|\,d\tau \\ &\quad + \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}(\xi_n(\tau)-S_n(\tau))\,d\tau + \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}\|h(\tau)-S_n(\tau)\|\,d\tau\end{aligned}$$

$$+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau$$
$$\left.+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|h(\tau)-S_n(\tau)\right\|d\tau\right]ds. \tag{24}$$

But $\xi_n-S_n \to 0$ with respect to the norm $\|\cdot\|_w$ from Lemma 3 we get $\xi_n-S_n \to 0$ weakly in $L^1(I,\mathbb{R}^n)$. So we have

$$\begin{aligned}&\int_0^t (t-\tau)\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau\\&+\int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau\\&+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|\xi_n(\tau)-S_n(\tau)\right\|d\tau\longrightarrow 0.\end{aligned} \tag{25}$$

Moreover,

$$\begin{aligned}&\int_0^1\left[\int_0^t (t-\tau)\left\|h(\tau)-S_n(\tau)\right\|d\tau\right.\\&\quad+\int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\xi_n(\tau)-S_n(\tau)\right)d\tau\\&\quad\left.+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left\|h(\tau)-S_n(\tau)\right\|d\tau\right]ds\\&\le\int_0^1\left[\int_0^t (t-\tau)\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\|\right.\right.\\&\qquad\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot u_n(\tau)\right\|\right)\\&\quad+\int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\|\right.\\&\qquad\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot u_n(\tau)\right\|\right)\\&\quad+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\varepsilon_n+\alpha_1(\tau)\left\|u(\tau)-u_n(\tau)\right\|\right.\\&\qquad\left.\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot u_n(\tau)\right\|\right)\right]ds.\end{aligned} \tag{26}$$

As $n\to\infty$, we have

$$\begin{aligned}&\left\|\hat u(t)-u(t)\right\|\\&\le\int_0^1\left[\int_0^t (t-\tau)\left(\alpha_1(\tau)\left\|u(\tau)-\hat u(\tau)\right\|\right.\right.\\&\qquad\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot{\hat u}(\tau)\right\|\right)d\tau\\&\quad+\int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\alpha_1(\tau)\left\|u(\tau)-\hat u(\tau)\right\|\right.\\&\qquad\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot{\hat u}(\tau)\right\|\right)d\tau\\&\quad+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\alpha_1(\tau)\left\|u(\tau)-\hat u(\tau)\right\|\right.\\&\qquad\left.\left.+\alpha_2(\tau)\left\|\dot u(\tau)-\dot{\hat u}(\tau)\right\|\right)d\tau\right]ds\\&\le\left\|u-\hat u\right\|_{C^1(I,\mathbb{R}^n)}\left(\int_0^t(t-\tau)\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\right.\\&\qquad+\int_0^{\eta}\frac{t(\tau-\eta)(t+1)}{\theta-\eta}\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\\&\qquad\left.+\int_{\theta}^{1}\frac{1-\tau}{1-\theta}\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\right)\\&=\left\|u-\hat u\right\|_{C^1(I,\mathbb{R}^n)}\int_0^1\left|G(t,\tau)\right|\left(\alpha_1(\tau)+\alpha_2(\tau)\right)d\tau\\&\le 2\left\|u-\hat u\right\|_{C^1(I,\mathbb{R}^n)}\left\|\alpha_1(\tau)+\alpha_2(\tau)\right\|.\end{aligned} \tag{27}$$

Since by assumption (ii), $\|\alpha_1+\alpha_2\|<1/2$ we get $u=\hat u$. So $u_n\to u$ in $C^1(I,\mathbb{R}^n)$ and $u\in\overline{\Delta}_{P_e}$ where the closure is taken in $C^1(I,\mathbb{R}^n)$ which means that $\Delta_P\subseteq\overline{\Delta}_{P_e}$. Therefore, the proof is complete if we show that Δ_P is closed. Indeed if $v_n\in\Delta_P$ and $v_n\to v$ in $C^1(I,\mathbb{R}^n)$, then $v_n=f(y_n)$ for $y_n\in\delta^1_{F(\cdot,v(\cdot),\dot v(\cdot))}$. From assumption (iii) and the Dunford-Pettis theorem, $\{y_n\}_{n\in\mathbb{N}}$ is weakly sequentially compact in $L^1(I,\mathbb{R}^n)$. So we can say that $\{y_n\}_{n\in\mathbb{N}}$ in $L^1(I,\mathbb{R}^n)$. By [25, Theorem 3.1], we get

$$\begin{aligned}y(t)&\in\overline{\mathrm{conv}}\,\overline{\lim}\{y_n(t)\}_{n\in\mathbb{N}}\subseteq\overline{\mathrm{conv}}\,\overline{\lim}\,F\left(t,v_n(t),\dot v_n(t)\right)\\&=F\left(t,v(t),\dot v(t)\right)\quad\text{a.e. on } I.\end{aligned} \tag{28}$$

Moreover, $f(y_n)\to f(y)$ in $L^1(I,\mathbb{R}^n)$ for $y\in L^1(I,\mathbb{R}^n)$ and $y(t)\in F(t,v(t),\dot v(t))$ a.e. on I. Hence, $v\in\Delta_P$; that is Δ_P is closed in $C^1(I,\mathbb{R}^n)$. □

Now we consider the following assumptions:

(A_1) $\beta\in(0,\pi/2)$, $a_i>0$ and $\sum_{i=1}^{m-2}a_i<1$;

(A_2) $\sum_{i=1}^{m-2}a_i\cos\beta\xi_i-\cos\beta>0$ and $K_m=1/\sum_{i=1}^{m-2}a_i\cos\beta\xi_i-\cos\beta$;

(A_3) $C_0=(\sin\beta/\beta)(1+K_m)$ and $C_1=\min\{K_m+1, K_m\sin^2\beta\}$;

(A_4) $S = \{u \in C^2(I, \mathbb{R}^n) : \dot{u}(0) = 0,\ u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)\}$;

(A_5) $\mathcal{G} : I \times I \to \mathbb{R}$ is defined by

$$\mathcal{G}(t,s) = \begin{cases} \dfrac{1}{\beta}\sin\beta(t-s) & \text{if } 0 \le s \le t \le 1 \\ 0 & \text{if } 0 \le t \le s \le 1 \end{cases} + \frac{K_m}{\beta}\cos\beta t \begin{cases} \sin\beta(1-s) - \sum_{i=1}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } 0 \le s \le \xi_1, \\ \sin\beta(1-s) - \sum_{i=2}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_1 < s \le \xi_2, \\ \sin\beta(1-s) - \sum_{i=3}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_2 < s \le \xi_3, \\ \vdots \\ \sin\beta(1-s) - \sum_{i=k}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_{k-1} < s \le \xi_k, \\ \vdots \\ \sin\beta(1-s), \\ \quad \text{if } \xi_{m-2} < s \le 1. \end{cases} \tag{29}$$

Lemma 6 (see [26]). *If the assumptions (A_1)–(A_5) hold, then*

(i) $0 \le \mathcal{G}(t,s) \le C_0$ *for all* $(t,s) \in I \times I$,

(ii) $\sup_{t,s\in I} |\partial\mathcal{G}(t,s)/\partial t| \le C_1$,

(iii) *for each* $x \in C^1(I, \mathbb{R}^n)$ *there exists a unique function* $u_x \in S$ *such that*

$$u_x(t) = \int_0^1 \mathcal{G}(t,s)\, x(s)\, ds, \tag{30}$$

(iv) $(\int_0^1 |\mathcal{G}(t,s)|^k ds)^{1/k} \le C_0$ *and* $(\int_0^1 |(\partial\mathcal{G}/\partial t)(t,s)|^k ds)^{1/k} \le C_1$.

Proof. (ii) Since

$$\frac{\partial\mathcal{G}(t,s)}{\partial t} = \begin{cases} \cos\beta(t-s) & \text{if } 0 \le s \le t \le 1 \\ 0 & \text{if } 0 \le t \le s \le 1 \end{cases} - K_m \sin\beta t \begin{cases} \sin\beta(1-s) - \sum_{i=1}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } 0 \le s \le \xi_1, \\ \sin\beta(1-s) - \sum_{i=2}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_1 < s \le \xi_2, \\ \sin\beta(1-s) - \sum_{i=3}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_2 < s \le \xi_3, \\ \vdots \\ \sin\beta(1-s) - \sum_{i=k}^{m-2} a_i \sin\beta(\xi_i - s), \\ \quad \text{if } \xi_{k-1} < s \le \xi_k, \\ \vdots \\ \sin\beta(1-s), \\ \quad \text{if } \xi_{m-2} < s \le 1, \end{cases} \tag{31}$$

then $\sup_{t,s\in I} \partial\mathcal{G}(t,s)/\partial t \le 1 + K_m$. Furthermore,

$$\frac{\partial\mathcal{G}(t,s)}{\partial t} \ge K_m \sin\beta t \left[\sum_{i=1}^{m-2} a_i \sin(\xi_i - s) - \sin\beta(1-\beta)\right] \ge -K_m \sin^2\beta \tag{32}$$

and thus $\sup_{t,s\in I} |\partial\mathcal{G}(t,s)/\partial t| \le C_1$. □

Theorem 7. *Assume that the assumptions (A_1) and (A_2) hold. Let F be a multifunction from $I \times \mathbb{R}^n \times \mathbb{R}^n$ to $P_{kc}(\mathbb{R}^n)$ satisfying the following conditions:*

(a) *for each $(x, y) \in \mathbb{R} \times \mathbb{R}$, the multifunction $F(\cdot, x, y)$ is measurable;*

(b) *for each $t \in I$, the function $(x, y) \to F(t, x, y)$ is continuous with respect to the Hausdorff metric d_H;*

(c) *for each $(t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$*

$$\|F(t,x,y)\| \le \sup\{\|v\| : v \in F(t,x,y)\} \le a(t) + c_1(t)\|x\| + c_2(t)\|y\|; \tag{33}$$

(d) *the spectral radius $r(L)$ of L is less than one.*

Then Problem (Q_e) admits a solution in S.

Proof. We can say that $\| F(t,x,y) \| \le a_1(t)$ a.e. on I for some $a_1 \in L^p(I,\mathbb{R}^+)$ [9]. Let $x \in C^1(I,\mathbb{R}^n)$ and let $u \in C^2(I,\mathbb{R}^n)$ be the unique solution of the problem

$$\ddot{u}(t) = x(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{$*$}$$

From Lemma 6, we have $u(t) = \int_0^1 \mathcal{G}(t,s)x(s)ds$, $\forall t \in I$. Thus, we define a function $f : C^1(I,\mathbb{R}^n) \to C^2(I,\mathbb{R}^n)$ such that $f(x)$ is the unique solution of $(*)$. Let

$$\mathcal{V} = \left\{x \in C^1(I,\mathbb{R}^n) : \|x(t)\| \le a_1(t) \text{ a.e. on } I\right\}. \tag{34}$$

From the Dunford-Pettis theorem, $\mathcal{V}$ is weakly compact and then $f(\mathcal{V})$ is convex and compact subset of $C^2(I,\mathbb{R}^n)$. Let $\mathcal{Y} = \mathbb{R}^n \times \mathbb{R}^n$. If $\mathcal{K} = f(\mathcal{V})$, $\mathcal{R} : \mathcal{K} \to 2^{L^1(I,\mathbb{R}^n)}$ and $\mathcal{M} : I \times \mathcal{Y} \to 2^{\mathbb{R}^n}$, where $\mathcal{R}(u) = \{g \in L^1(I,\mathbb{R}^n) : g(t) \in F(t,u(t),\dot{u}(t))$ a.e. on $I\}$ and $\mathcal{M}(t,(x,y)) = F(t,x,y)$, then $\mathcal{M}$ has SD-property [23]. It is easy to show that $\mathcal{R}$ is nonempty and convex subset of $L^1(I,\mathbb{R}^n)$. If f_n is a sequence in $\mathcal{R}(u)$ for some $u \in \mathcal{K}$, then $\lim_{n\to\infty} f_n(t) = f(t) \in F(t,u(t),\dot{u}(t))$, where the values of F are closed. Therefore, the values of $\mathcal{R}$ are weakly compact. According to Theorem 5 there exists a continuous function $r : \mathcal{K} \to L^1_w(I,\mathbb{R}^n)$ with $r(u) \in \operatorname{ext}(\mathcal{R}(u))$, for all $u \in \mathcal{K}$. Thus, $r(u)(t) \in \operatorname{ext}(\mathcal{M}(t,u(t),\dot{u}(t)))$ a.e. on I [24] which implies $r(u)(t) \in \operatorname{ext}(F(t,u(t),\dot{u}(t)))$ a.e. on I. If $u \in f(\mathcal{V})$, then $\| r(u)(t) \| \le a_1$ and so $r(u) \in \mathcal{V}$. Put $\theta : f(\mathcal{V}) \to W^{2,1}(I,\mathbb{R}^n)$ such that $\theta(u) = f(r(u))$, thus θ is a continuous function from $f(\mathcal{V})$ into $f(\mathcal{V})$ [19]. From Schauder's fixed point theorem, there exists $x \in f(\mathcal{V})$ such that $x = \theta(x) = f(r(x))$ which means that there is $x \in S \subseteq C^2(I,\mathbb{R}^n)$ such that $\ddot{x}(t) \in \operatorname{ext}(F(t,x(t),\dot{x}(t)))$. □

Theorem 8. *In the setting of Theorem 7, if one replaces condition (b) by the following condition:*

(b)′ $d_H(F(t,x,y), F(t,x',y')) \le k_1 \| x - x' \| + k_2 \| y - y' \|$ a.e. with $k_1 \ge 0$, $k_2 \ge 0$ and $|k_1 + k_2| < 1/2C_0$.

Then Δ_{Q_e} is nonempty and $\overline{\Delta_{Q_e}} = \Delta_Q$ where the closure taken in $C^2(I,\mathbb{R}^n)$.

Proof. From Theorem 7, we have $\Delta_{Q_e} \neq \emptyset$. Moreover, $\| F(t,x,y) \| \le b_1(t)$ a.e. on I for some $b_1 \in L^p(I,\mathbb{R}^+)$. Let $u \in \Delta_Q$. Then

$$\ddot{u}(t) = h(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{35}$$

where $h(t) \in F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $f' : C^1(I,\mathbb{R}^n) \to C^2(I,\mathbb{R}^n)$ is a function such that, for each $h \in C^1(I,\mathbb{R}^n)$, $f'(h) \in C^2(I,\mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$\ddot{u}(t) = h(t), \quad \text{a.e. on } I,$$
$$\dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{Q_h}$$

Let $S = \{u \in C^1(I,\mathbb{R}^n) : \| u(t) \| \le b_1(t)$ a.e. on $I\}$. So $f'(S)$ is convex. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $f'(S)$. Hence, $u_n \in C^2(I,\mathbb{R}^n)$ with $u_n(0) = 0$, $\dot{u}_n(0) = 0$, $u_n(1) = \sum_{i=1}^{m-2} a_i u_n(\xi_i)$. Then from Lemma 6,

$$\lim_{n\to\infty} u_n(t) = \int_0^1 \mathcal{G}(t,\tau)\ddot{u}(\tau)d\tau = u(t), \tag{36}$$

hence, $f'(S)$ is a compact subset of $C^2(I,\mathbb{R}^n)$. Set

$$\mathcal{Q}_\varepsilon(t) = \{x \in F(t,v(t),\dot{v}(t)) : \|h(t) - x\| < \varepsilon + d(h(t),F(t,v(t),\dot{v}(t)))\}, \tag{37}$$

where $\varepsilon > 0$ and $v \in f'(S)$. Hence, for each $t \in I$, $\mathcal{Q}_\varepsilon(t) \neq \emptyset$. Assume that $\mathcal{B}(I)$ and $\mathcal{B}(\mathbb{R}^n)$ are the Borel σ-fields of I and $\mathbb{R}^n$, respectively. From condition (i), the function $t \to F(t,v(t),\dot{v}(t))$ is measurable. Hence, $grF(\cdot,v(\cdot),\dot{v}(\cdot)) \in \mathcal{B}(I) \times \mathcal{B}(\mathbb{R}^n)$ and $(t,x) \to \varepsilon d(h(t),F(t,v(t),\dot{v}(t))) - \| h(t) - x \|$ is measurable in t and continuous in x that is jointly measurable. Thus, by Aumann's selection theorem, there exists a measurable selection s_ε of $\mathcal{Q}_\varepsilon$ such that $s_\varepsilon(t) \in \mathcal{Q}_\varepsilon(t)$ for each $t \in I$. Now we define a multifunction $\mathcal{Q}_\varepsilon : f'(S) \to 2^{C^1(I,\mathbb{R}^n)}$ by the following:

$$\mathcal{Q}_\varepsilon(v) = \left\{x \in \delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} : \|h(t) - x\| < \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \text{ a.e. on } I\right\}, \tag{38}$$

with $\mathcal{Q}_\varepsilon(v)(t) \neq \emptyset$ for each $v \in f'(S)$. From [22, Proposition 4], $\mathcal{Q}_\varepsilon$ is *l. s. c.* and clearly has decomposable values. Applying [22, Theorem 3], we have a continuous selection S_ε of $\overline{\mathcal{Q}_\varepsilon}$. Therefore,

$$\begin{aligned}\|h(t) - S_\varepsilon(v)(t)\| &\le \varepsilon + d(h(t),F(t,v(t),\dot{v}(t))) \\ &\le \varepsilon + k_1(t)\|u(t) - v(t)\| \\ &\quad + k_2(t)\|\dot{u}(t) - \dot{v}(t)\| \quad \text{a.e. on } I.\end{aligned} \tag{39}$$

From Theorem 2, we find a continuous function $\xi'_\varepsilon : f'(S) \to L^1_w(I,\mathbb{R}^n)$ such that $\xi'_\varepsilon(v) \in \operatorname{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))}$ and $\| S_\varepsilon(v) - \xi'_\varepsilon(v) \| < \varepsilon$ for each $v \in f'(S)$. Define a multifunction $R' : f'(S) \to 2^{C^1(I,\mathbb{R}^n)}$ by

$$R'(u) = \left\{g \in C^1(I,\mathbb{R}^n) : g(t) \in F(t,u(t),\dot{u}(t)) \text{ a.e. on } I\right\}. \tag{40}$$

As in Theorem 5, let $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t,(x,y)) = F(t,x,y)$. From [23, Theorem 3.1], M has SD-property. R' has nonempty convex values. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $R'(u)$ for some $u \in f'(S)$. So, for each $t \in I$,

$$\lim_{n\to\infty} g_n(t) = g(t) \in F(t, u(t), \dot{u}(t)) \tag{41}$$

because F has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F(\cdot,u(\cdot),\dot{u}(\cdot))}$ which implies $R'(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $\theta' : f'(S) \to L^1_w(I,\mathbb{R}^n)$ such that $\theta'(u) \in \operatorname{ext}(R'(u))$, for all $u \in f'(S)$. We see that $\theta'(u)(t) \in \operatorname{ext}(M(t,(u(t),\dot{u}(t))))$ [24], hence $\theta'(u)(t) \in \operatorname{ext}F(t,u(t),\dot{u}(t))$ a.e. on I. Assume that $\eta' : f'(S) \to C^2(I,\mathbb{R}^n)$ is the function which for each $u \in f'(S)$, $\eta'(u) = g(\theta'(u))$. For each $u \in f'(S)$, we have $\| \theta'(u)(t) \| \le b_1$ and so $\theta'(u) \in S$. Then, η' is a function from $f'(S)$ into $f'(S)$ and also we see that η' is continuous [19]. Now let $\varepsilon_n \to 0$, $S_{\varepsilon_n} = S_n$ and $\xi'_n = \xi'_{\varepsilon_n}$. Then, for each $n \in \mathbb{N}$, the function $f' o \xi'_n$ is a continuous function from the compact set $f'(S)$ into itself. From Schauder's fixed point theorem, $fo\xi'_n$ has a fixed point u_n, but $\operatorname{ext}\delta^1_{F(\cdot,v(\cdot),\dot{v}(\cdot))} = \delta^1_{\operatorname{ext}F(\cdot,v(\cdot),\dot{v}(\cdot))}$ [24] so $u_n \in \Delta_{P_e}$. Assume that $u_n \to \hat{u}$ in $C^2(I,\mathbb{R}^n)$. From Lemma 6, we obtain

$$\begin{aligned} \left\|u_n(t) - u(t)\right\| \le \int_0^1 \Bigg[\int_0^1 |\mathscr{G}(t,\tau)| \left\|\xi'{}_n(\tau) - S_n(\tau)\right\| d\tau \\ + \int_0^1 |\mathscr{G}(t,\tau)| \left\|(S_n(\tau) - h(\tau))\right\| d\tau \Bigg] ds. \end{aligned} \tag{42}$$

But $\xi'_n - S_n \to 0$ with respect to the norm $\| \cdot \|_w$ and from Lemma 3 we get $\xi'_n - S_n \to 0$ weakly in $C^1(I,\mathbb{R}^n)$. So we have

$$\int_0^1 |\mathscr{G}(t,\tau)| \left\|\xi'_n(\tau) - S_n(\tau)\right\| \, d\tau \longrightarrow 0. \tag{43}$$

Moreover, as $n \to \infty$ we have

$$\begin{aligned} \|\hat{u}(t) - u(t)\| &\le \|u - \hat{u}\|_{C^1(I,\mathbb{R}^n)} \int_0^1 |\mathscr{G}(t,\tau)| \left(k_1(\tau) + k_2(\tau)\right) d\tau \\ &\le \|u - \hat{u}\|_{C^1(I,\mathbb{R}^n)} \left\|k_1(\tau) + k_2(\tau)\right\| C_0. \end{aligned} \tag{44}$$

Since by assumption (ii), $\| k_1 + k_2 \| < 1/2C_0$, thus from Lemma 6, we get $u = \hat{u}$. So $u_n \to u$ in $C^2(I,\mathbb{R}^n)$ and $u \in \overline{\Delta}_{Q_e}$ where the closure is taken in $C^2(I,\mathbb{R}^n)$ which means that $\Delta_P \subseteq \overline{\Delta}_{P_e}$. If $v_n \in \Delta_Q$ and $v_n \to v$ in $C^2(I,\mathbb{R}^n)$, then $v_n = f'(y_n)$ for $y_n \in \delta^1_{F'(\cdot,v(\cdot),\dot{v}(\cdot))}$. From assumption (iii) and the Dunford-Pettis theorem, $\{y_n\}_{n\in\mathbb{N}}$ is weakly sequentially compact in $C^2(I,\mathbb{R}^n)$. By [25, Theorem 3.1], we get

$$\begin{aligned} y(t) \in \overline{\operatorname{conv}}\, \overline{\lim}\{y_n(t)\}_{n\in\mathbb{N}} &\subseteq \overline{\operatorname{conv}}\, \overline{\lim}\, F(t, v_n(t), \dot{v}_n(t)) \\ &= F(t, v(t), \dot{v}(t)) \quad \text{a.e. on } I. \end{aligned} \tag{45}$$

Moreover, $f'(y_n) \to f'(y)$ in $C^2(I,\mathbb{R}^n)$ for $y \in C^2(I,\mathbb{R}^n)$ and $y(t) \in F(t,v(t),\dot{v}(t))$ a.e. on I. Hence, $v \in \Delta_Q$; that is, Δ_Q is closed in $C^2(I,\mathbb{R}^n)$. □

Acknowledgments

The author is deeply indebted and thankful to the deanship of the scientific research and his helpful and distinct team of employees at Taibah University, Al-Madinah Al-Munawarah, Saudia Arabia. This research work was supported by a Grant no. 3029/1434.

References

[1] A. M. Gomaa, *Set-valued functions and set-valued differential equations [Ph.D. thesis]*, Faculty of Science, Cairo University, 1999.

[2] A. G. Ibrahim and A. M. Gomaa, "Existence theorems for a functional multivalued three-point boundary value problem of second order," *Journal of the Egyptian Mathematical Society*, vol. 8, no. 2, pp. 155–168, 2000.

[3] A. G. Ibrahim and A. G. Gomaa, "Extremal solutions of classes of multivalued differential equations," *Applied Mathematics and Computation*, vol. 136, no. 2-3, pp. 297–314, 2003.

[4] A. M. Gomaa, "On the solution sets of three-point boundary value problems for nonconvex differential inclusions," *Journal of the Egyptian Mathematical Society*, vol. 12, no. 2, pp. 97–107, 2004.

[5] D. L. Azzam, C. Castaing, and L. Thibault, "Three boundary value problems for second order differential inclusions in Banach spaces," *Control and Cybernetics*, vol. 31, no. 3, pp. 659–693, 2002.

[6] B. Satco, "Second order three boundary value problem in Banach spaces via Henstock and Henstock-Kurzweil-Pettis integral," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 919–933, 2007.

[7] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, NY, USA, 1964.

[8] A. M. Gomaa, "On the solution sets of four-point boundary value problems for nonconvex differential inclusions," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 1, pp. 23–37, 2011.

[9] A. M. Gomaa, "On four-point boundary value problems for differential inclusions and differential equations with and without multivalued moving constraints," *Czechoslovak Mathematical Journal*, vol. 62, no. 137, pp. 139–154, 2012.

[10] C. J. Himmelberg and F. S. Van Vleck, "Selection and implicit function theorems for multifunctions with Souslin graph," *Bulletin de l'Académie Polonaise des Sciences*, vol. 19, pp. 911–916, 1971.

[11] C. J. Himmelberg and F. S. Van Vleck, "Some selection theorems for measurable functions," *Canadian Journal of Mathematics*, vol. 21, pp. 394–399, 1969.

[12] C. J. Himmelberg and F. S. Van Vleck, "Extreme points of multifunctions," *Indiana University Mathematics Journal*, vol. 22, pp. 719–729, 1973.

[13] K. Kuratowski and C. Ryll-Nardzewski, "A general theorem on selectors," *Bulletin de l'Académie Polonaise des Sciences*, vol. 13, pp. 397–403, 1965.

[14] E. Klein and A. C. Thompson, *Theory of Correspondences*, John Wiley and Sons, New York, NY, USA, 1984.

[15] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, vol. 580 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.

[16] J.-P. Aubin and A. Cellina, *Differential Inclusions Set-Valued Maps and Viability Theory*, vol. 264, Springer, Berlin, Germany, 1984.

[17] F. S. De Blasi and J. Myjak, "On continuous approximations for multifunctions," *Pacific Journal of Mathematics*, vol. 123, no. 1, pp. 9–31, 1986.

[18] A. A. Tolstonogov, "Extremal selectors of multivalued mappings and the "bang-bang" principle for evolution inclusions," *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques*, vol. 317, no. 3, pp. 589–593, 1991.

[19] D. Kravvaritis and N. S. Papageorgiou, "Boundary value problems for nonconvex differential inclusions," *Journal of Mathematical Analysis and Applications*, vol. 185, no. 1, pp. 146–160, 1994.

[20] S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value*, Academic Press, New York, NY, USA, 1974.

[21] A. G. Ibrahim and A. M. Gomaa, "Topological properties of the solution sets of some differential inclusions," *Pure Mathematics and Applications*, vol. 10, no. 2, pp. 197–223, 1999.

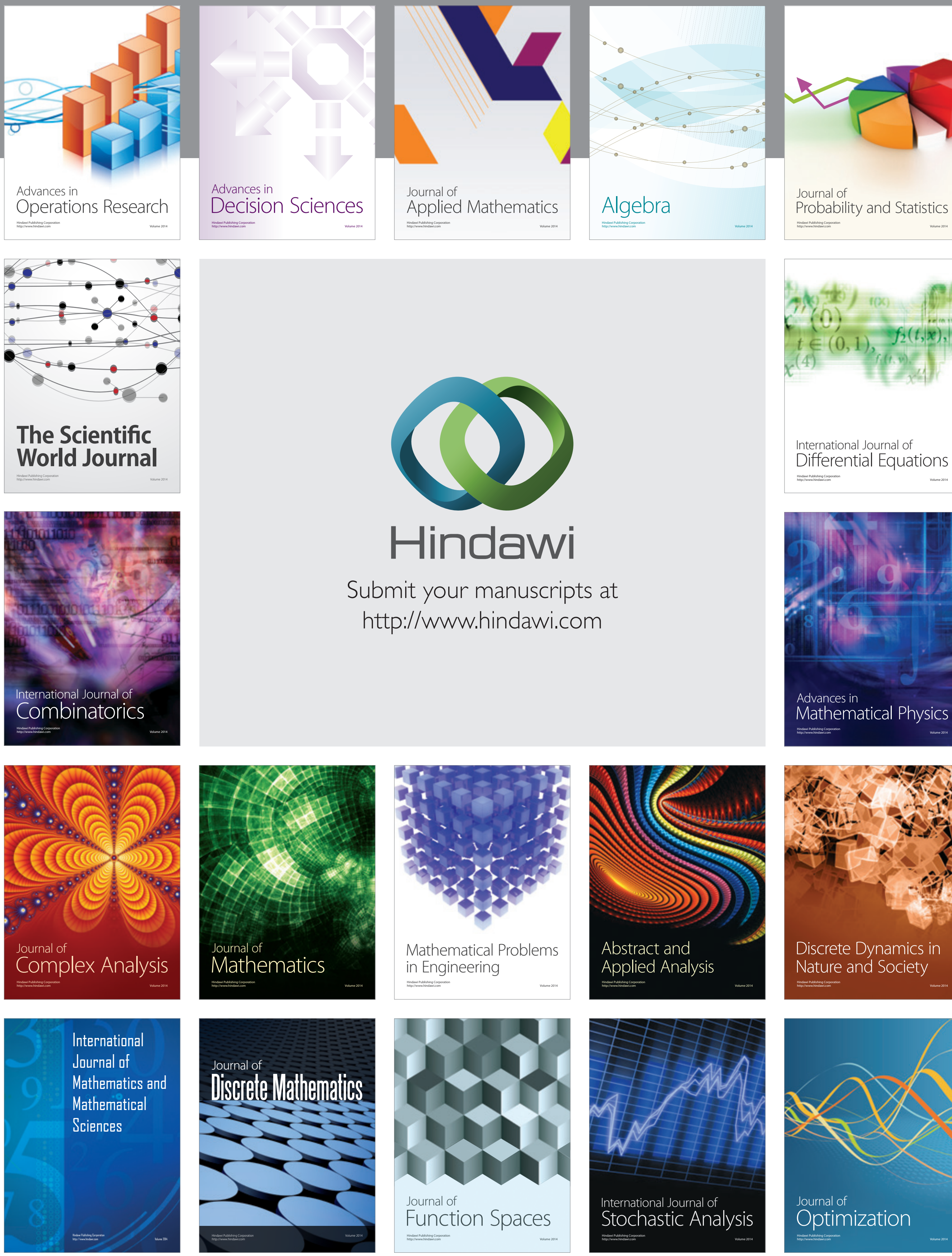
[22] A. Bressan and G. Colombo, "Extensions and selections of maps with decomposable values," *Studia Mathematica*, vol. 90, no. 1, pp. 69–86, 1988.

[23] N. S. Papageorgiou, "On measurable multifunctions with applications to random multivalued equations," *Mathematica Japonica*, vol. 32, no. 3, pp. 437–464, 1987.

[24] M. Benamara, *Point Extrémaux, Multi-applications et Fonctionelles Intégrales [M.S. thesis]*, Université de Grenoble, 1975.

[25] N. S. Papageorgiou, "Convergence theorems for Banach space valued integrable multifunctions," *International Journal of Mathematics and Mathematical Sciences*, vol. 10, no. 3, pp. 433–442, 1987.

[26] L. X. Truong, L. T. P. Ngoc, and N. T. Long, "Positive solutions for an m-point boundary-value problem," *Electronic Journal of Differential Equations*, vol. 2008, no. 111, 10 pages, 2008.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization