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Research Article

Double Discontinuous Inverse Problems for Sturm-Liouville Operator with Parameter-Dependent Conditions

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The purpose of this paper is to solve the inverse spectral problems for Sturm-Liouville operator with boundary conditions depending on spectral parameter and double discontinuities inside the interval. It is proven that the coefficients of the problem can be uniquely determined by either Weyl function or given two different spectral sequences.

1. Introduction

Spectral problems of differential operators are studied in two main branches, namely, direct spectral problems and inverse spectral problems. Direct problems of spectral analysis consist in investigating the spectral properties of an operator. On the other hand, inverse problems aim at recovering operators from their spectral characteristics. Such problems often appear in mathematics, physics, mechanics, electronics, geophysics, and other branches of natural sciences.

First and most important results for inverse problem of a regular Sturm-Liouville operator were given by Ambartsumyan in 1929 [1] and Borg in 1946 [2]. Physical applications of inverse spectral problems can be found in several works (see, e.g., [3–9] and references therein).

Eigenvalue-dependent boundary conditions were studied extensively. The references [10, 11] are well-known examples for problems with boundary conditions that depend linearly on the eigenvalue parameter. In [10, 12], an operator-theoretic formulation of the problems with the spectral parameter contained in only one of the boundary conditions has been given. Inverse problems according to various spectral data for eigenparameter linearly dependent Sturm-Liouville operator were investigated in [13–17]. Boundary conditions that depend nonlinearly on the spectral parameter were also considered in [18–23].

Boundary value problems with discontinuity condition appear in the various problems of the applied sciences. These kinds of problems are well studied (see, e.g., [24–31]).

In this study, we consider a boundary value problem generated by the Sturm-Liouville equation:

$$\ell y := -y'' + q(x)\, y = \lambda y, \quad x \in I = \bigcup_{i=0}^{2} (d_i, d_{i+1}) \tag{1}$$

subject to the boundary conditions

$$U(y) := \lambda\left(y'(0) + h_0 y(0)\right) - h_1 y'(0) - h_2 y(0) = 0, \tag{2}$$

$$V(y) := \lambda\left(y'(1) + H_0 y(1)\right) - H_1 y'(1) - H_2 y(1) = 0 \tag{3}$$

and double discontinuity conditions

$$\begin{aligned} y(d_i + 0) &= \alpha_i y(d_i - 0), \\ y'(d_i + 0) &= \alpha_i^{-1} y'(d_i - 0) - (\gamma_i \lambda + \beta_i)\, y(d_i - 0), \end{aligned} \tag{4}$$

where $q(x)$ is real valued function in $L_2(0,1)$; h_j and H_j, $j = 0, 1, 2$, are real numbers; $\alpha_i, \gamma_i \in \mathbb{R}^+$, $\beta_i \in \mathbb{R}$, $i = 1, 2$; $d_0 = 0, d_1, d_2 \in (0,1)$, $d_3 = 1$; $\rho_1 := h_2 - h_0 h_1 > 0$, $\rho_2 := H_0 H_1 - H_2 > 0$; and λ is a spectral parameter. We denote the problem (1)–(4) by $L = L(q, \mathbf{h}, \mathbf{H}, \mathbf{s}_1, \mathbf{s}_2)$, where $\mathbf{h} = (h_0, h_1, h_2)$, $\mathbf{H} = (H_0, H_1, H_2)$, and $\mathbf{s}_i = (d_i, \alpha_i, \gamma_i, \beta_i)$, $i = 1, 2$.

It is proven that the coefficients of the problem can be uniquely determined by either Weyl function or given two different spectral sequences. The obtained results are generalizations of the similar results for the classical Sturm-Liouville operator on a finite interval.

2. Preliminaries

Let the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1) under the following initial conditions and the jump conditions (4):

$$\begin{pmatrix}\varphi\\ \varphi'\end{pmatrix}(0,\lambda)=\begin{pmatrix}-\lambda+h_1\\ \lambda h_0-h_2\end{pmatrix},\qquad \begin{pmatrix}\psi\\ \psi'\end{pmatrix}(1,\lambda)=\begin{pmatrix}-\lambda+H_1\\ \lambda H_0-H_2\end{pmatrix}. \tag{5}$$

These solutions are the entire functions of λ and satisfy the relation $\psi(x,\lambda_n)=\boldsymbol{\beta}_n\varphi(x,\lambda_n)$ for each eigenvalue λ_n, where $\boldsymbol{\beta}_n=-(\psi'(0,\lambda_n)+h_0\psi(0,\lambda_n))/\rho_1$.

The following asymptotics can be obtained from the integral equations given in the appendix:

$$\varphi(x,\lambda)=\begin{cases}-\lambda\cos\sqrt{\lambda}x+O\left(\sqrt{\lambda}\exp|\tau|x\right), & x<d_1,\\ \gamma_1\lambda^2\left[\dfrac{\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}-\dfrac{\sin\sqrt{\lambda}(2d_1-x)}{2\sqrt{\lambda}}\right]+O\left(\lambda\exp|\tau|x\right), & d_1<x<d_2,\\ \dfrac{\gamma_1\gamma_2}{4}\lambda^2\left[\cos\sqrt{\lambda}x+\cos\sqrt{\lambda}(2d_1-x)-\cos\sqrt{\lambda}(2d_2-x)-\cos\sqrt{\lambda}(2d_2-2d_1-x)\right] & \\ \quad+O\left(\lambda^{3/2}\exp|\tau|x\right), & x>d_2,\end{cases} \tag{6}$$

$$\varphi'(x,\lambda)=\begin{cases}\lambda^{3/2}\sin\sqrt{\lambda}x+O\left(\lambda\exp|\tau|x\right), & x<d_1,\\ \dfrac{\gamma_1\lambda^2}{2}\left[\cos\sqrt{\lambda}x+\cos\sqrt{\lambda}(2d_1-x)\right]+O\left(\lambda^{3/2}\exp|\tau|x\right), & d_1<x<d_2,\\ -\dfrac{\gamma_1\gamma_2}{4}\lambda^{5/2}\left\{\sin\sqrt{\lambda}x-\sin\sqrt{\lambda}(2d_1-x)+\sin\sqrt{\lambda}(2d_2-x)+\sin\sqrt{\lambda}(2d_2-2d_1-x)\right\} & \\ \quad+O\left(\lambda^2\exp|\tau|x\right), & x>d_2,\end{cases} \tag{7}$$

where $\tau=\operatorname{Im}\sqrt{\lambda}$.

The values of the parameter λ for which the problem L has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions.

The characteristic function $\Delta(\lambda)$ and norming constants α_n of the problem L are defined as follows:

$$\begin{aligned}\Delta(\lambda)&=W[\varphi,\psi]\\ &=\lambda\left(\varphi'(1,\lambda)+H_0\varphi(1,\lambda)\right)-H_1\varphi'(1,\lambda)-H_2\varphi(1,\lambda)\\ &=-\lambda\left(\psi'(0,\lambda)+h_0\psi(0,\lambda)\right)+h_1\psi'(0,\lambda)+h_2\psi(0,\lambda),\\ \boldsymbol{\alpha}_n&:=\int_0^1\varphi^2(x,\lambda_n)\,dx+\frac{1}{\rho_1}\left(\varphi'(0,\lambda_n)+h_0\varphi(0,\lambda_n)\right)^2\\ &\quad+\frac{1}{\rho_2}\left(\varphi'(1,\lambda_n)+H_0\varphi(1,\lambda_n)\right)^2\\ &\quad+\alpha_1\gamma_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\gamma_2\varphi^2(d_2-0,\lambda_n).\end{aligned} \tag{8}$$

It is obvious that $\Delta(\lambda)$ is an entire function in λ and the zeros, namely, $\{\lambda_n\}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the problem L. Now, from (6) and (8), we can write

$$\begin{aligned}\Delta(\lambda)&=-\frac{\gamma_1\gamma_2}{4}\lambda^4\\ &\quad\times\left[\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}-\frac{\sin\sqrt{\lambda}(2d_1-1)}{\sqrt{\lambda}}+\frac{\sin\sqrt{\lambda}(2d_2-1)}{\sqrt{\lambda}}\right.\\ &\quad\left.+\frac{\sin\sqrt{\lambda}(2d_2-2d_1-1)}{\sqrt{\lambda}}\right]+O\left(\lambda^3\exp|\tau|\right).\end{aligned} \tag{9}$$

Lemma 1. *See the following.*

(i) *All eigenvalues of the problem L are real and algebraically simple; that is, $\Delta'(\lambda_n)\neq 0$.*

(ii) *Two eigenfunctions $\varphi(x,\lambda_1)$ and $\varphi(x,\lambda_2)$, corresponding to different eigenvalues λ_1 and λ_2, are orthogonal in the sense of*

$$\begin{aligned}&\int_0^1\varphi(x,\lambda_1)\varphi(x,\lambda_2)\,dx\\ &\quad+\frac{1}{\rho_1}\left(\varphi'(0,\lambda_1)+h_0\varphi(0,\lambda_1)\right)\left(\varphi'(0,\lambda_2)+h_0\varphi(0,\lambda_2)\right)\\ &\quad+\frac{1}{\rho_2}\left(\varphi'(1,\lambda_1)+H_0\varphi(1,\lambda_1)\right)\left(\varphi'(1,\lambda_2)+H_0\varphi(1,\lambda_2)\right)\\ &\quad+\alpha_1\gamma_1\varphi(d_1-0,\lambda_1)\varphi(d_1-0,\lambda_2)\\ &\quad+\alpha_2\gamma_2\varphi(d_2-0,\lambda_1)\varphi(d_2-0,\lambda_2)=0.\end{aligned} \tag{10}$$

Proof. Consider a Hilbert Space $H = L_2(0,1) \oplus \mathbb{C}^4$, equipped with the inner product

$$\langle Y, Z\rangle := \int_0^1 y(x)\overline{z(x)}dx + \frac{1}{\rho_1}Y_1\overline{Z_1} + \frac{1}{\rho_2}Y_2\overline{Z_2} + \alpha_1\gamma_1 Y_3\overline{Z_3} + \alpha_2\gamma_2 Y_4\overline{Z_4} \tag{11}$$

for $Y = (y(x), Y_1, Y_2, Y_3, Y_4)^T$, $Z = (z(x), Z_1, Z_2, Z_3, Z_4)^T \in H$.

Define an operator T with the domain $D(T) = \{Y \in H : y(x)$, and $y'(x)$ are absolutely continuous in I, $\ell Y \in L_2(0,1)$, $y(d_i+0) = \alpha_i y(d_i-0)$, $Y_1 = y'(0)+h_0y(0)$, $Y_2 = y_1'(1)+H_0y(1)$, $Y_3 = \gamma_1 y(d_1-0)$, and $Y_4 = \gamma_2 y(d_2-0)\}$ such that

$$T(Y) := \begin{pmatrix} -y''(x) + q(x)y(x) \\ h_1y'(0) + h_2y(0) \\ H_1y'(1) + H_2y(1) \\ -y'(d_1+0) + \alpha_1^{-1}y'(d_1-0) - \beta_1 y(d_1-0) \\ -y'(d_2+0) + \alpha_2^{-1}y'(d_2-0) - \beta_2 y(d_2-0) \end{pmatrix}. \tag{12}$$

It is easily proven, using classical methods in the similar works (see, e.g., [28]), that the operator T is symmetric in H; the eigenvalue problem for the operator T and the problem L coincide. Therefore, all eigenvalues are real, and two different eigenfunctions are orthogonal.

Let us show the simplicity of the eigenvalues λ_n by writting the following equations:

$$\begin{aligned} -\psi''(x,\lambda) + q(x)\psi(x,\lambda) &= \lambda\psi(x,\lambda), \\ -\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n) &= \lambda_n\varphi(x,\lambda_n). \end{aligned} \tag{13}$$

If these equations are multiplied by $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$, respectively, subtracting them side by side and finally integrating over the interval $[0,1]$, the equality

$$\begin{aligned} &\left[\varphi'(x,\lambda_n)\psi(x,\lambda) - \psi'(x,\lambda)\varphi(x,\lambda_n)\right] \\ &\quad\times\left(\Big|_0^{d_1-0} + \Big|_{d_1+0}^{d_2-0} + \Big|_{d_2+0}^{1}\right) \\ &= (\lambda-\lambda_n)\int_0^1 \psi(x,\lambda)\varphi(x,\lambda_n)\,dx \end{aligned} \tag{14}$$

is obtained. Add and subtract $\Delta(\lambda)$ in the left-hand side of the last equality, and use initial conditions (5) to get

$$\begin{aligned} &\Delta(\lambda) + (\lambda-\lambda_n)\left(\psi'(0,\lambda) + h_0\psi(0,\lambda)\right) \\ &\quad - (\lambda-\lambda_n)\left(\varphi'(1,\lambda_n) + H_0\varphi(1,\lambda_n)\right) \\ &\quad - (\lambda-\lambda_n)\alpha_1\gamma_1\psi(d_1-0,\lambda)\varphi(d_1-0,\lambda_n) \\ &\quad - (\lambda-\lambda_n)\alpha_2\gamma_2\psi(d_2-0,\lambda)\varphi(d_2-0,\lambda_n) \\ &= (\lambda-\lambda_n)\int_0^1 \psi(x,\lambda)\varphi(x,\lambda_n)\,dx. \end{aligned} \tag{15}$$

Rewrite this equality as

$$\begin{aligned} \frac{\Delta(\lambda)}{\lambda-\lambda_n} &= \int_0^1 \psi(x,\lambda)\varphi(x,\lambda_n)\,dx \\ &\quad + \left(\varphi'(1,\lambda_n) + H_0\varphi(1,\lambda_n)\right) \\ &\quad - \left(\psi'(0,\lambda) + h_0\psi(0,\lambda)\right) \\ &\quad + \alpha_1\gamma_1\psi(d_1-0,\lambda)\varphi(d_1-0,\lambda_n) \\ &\quad + \alpha_2\gamma_2\psi(d_2-0,\lambda)\varphi(d_2-0,\lambda_n) \end{aligned} \tag{16}$$

$$\begin{aligned} &= \int_0^1 \psi(x,\lambda)\varphi(x,\lambda_n)\,dx \\ &\quad - \frac{\left(\psi'(0,\lambda)+h_0\psi(0,\lambda)\right)\left(\varphi'(0,\lambda_n)+h_0\varphi(0,\lambda_n)\right)}{h_0h_1 - h_2} \\ &\quad + \frac{\left(\varphi'(1,\lambda_n)+H_0\varphi(1,\lambda_n)\right)\left(\psi'(1,\lambda)+H_0\psi(1,\lambda)\right)}{H_0H_1 - H_2} \\ &\quad + \alpha_1\gamma_1\psi(d_1-0,\lambda)\varphi(d_1-0,\lambda_n) \\ &\quad + \alpha_2\gamma_2\psi(d_2-0,\lambda)\varphi(d_2-0,\lambda_n) \end{aligned} \tag{17}$$

$$\begin{aligned} &= \int_0^1 \psi(x,\lambda)\varphi(x,\lambda_n)\,dx \\ &\quad + \frac{\left(\psi'(0,\lambda)+h_0\psi(0,\lambda)\right)\left(\varphi'(0,\lambda_n)+h_0\varphi(0,\lambda_n)\right)}{\rho_1} \\ &\quad + \frac{\left(\varphi'(1,\lambda_n)+H_0\varphi(1,\lambda_n)\right)\left(\psi'(1,\lambda)+H_0\psi(1,\lambda)\right)}{\rho_2} \\ &\quad + \alpha_1\gamma_1\psi(d_1-0,\lambda)\varphi(d_1-0,\lambda_n) \\ &\quad + \alpha_2\gamma_2\psi(d_2-0,\lambda)\varphi(d_2-0,\lambda_n). \end{aligned} \tag{18}$$

As $\lambda \to \lambda_n$,

$$\Delta'(\lambda_n) = \beta_n\boldsymbol{\alpha}_n \tag{19}$$

is obtained by using the equality $\psi(x,\lambda_n) = \boldsymbol{\beta}_n\varphi(x,\lambda_n)$. Thus, $\Delta'(\lambda_n) \neq 0$. □

3. Main Results

We consider three statements of the inverse problem for the boundary value problem L; from the Weyl function, from the spectral data $\{\lambda_n, \boldsymbol{\alpha}_n\}_{n\geq 0}$, and from two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$. For studying the inverse problem, we consider a boundary value problem $\widetilde{L}$, together with L, of the same form but with different coefficients $\widetilde{q}(x)$, $\widetilde{\mathbf{h}}$, $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{s}}_i$, $i = 1,2$.

Let the function $\varkappa(x,\lambda)$ denote the solution of (1) under the initial conditions $\varkappa(0,\lambda) = \rho_1^{-1}$, $\varkappa'(0,\lambda) = -\rho_1^{-1}h_0$

and the jump conditions (4). It is clear that the function $\psi(x, \lambda)$ can be represented by

$$\psi(x,\lambda) = \Delta(\lambda)\varkappa(x,\lambda) - \frac{\psi'(0,\lambda) + h_0\psi(0,\lambda)}{\rho_1}\varphi(x,\lambda). \tag{20}$$

Denote

$$m(\lambda) := \frac{\psi'(0,\lambda) + h_0\psi(0,\lambda)}{\rho_1\Delta(\lambda)}. \tag{21}$$

Then, we have

$$\frac{\psi(x,\lambda)}{\Delta(\lambda)} = \varkappa(x,\lambda) - m(\lambda)\varphi(x,\lambda). \tag{22}$$

The function $m(\lambda)$ is called Weyl function [32].

Theorem 2. *If $m(\lambda) = \widetilde{m}(\lambda)$, then $L = \widetilde{L}$; that is, $q(x) = \widetilde{q}(x)$, always everywhere in I; $\mathbf{h} = \widetilde{\mathbf{h}}$, $\mathbf{H} = \widetilde{\mathbf{H}}$, and $\mathbf{s}_i = \widetilde{\mathbf{s}}_i$, $i = 1, 2$.*

Proof. Let us define the functions $P_1(x,\lambda)$ and $P_2(x,\lambda)$ as follows:

$$\begin{aligned} P_1(x,\lambda) &= \varphi(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ P_2(x,\lambda) &= \Phi(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\widetilde{\Phi}(x,\lambda), \end{aligned} \tag{23}$$

where $\Phi(x,\lambda) = \psi(x,\lambda)/\Delta(\lambda)$. If $m(\lambda) = \widetilde{m}(\lambda)$, then from (22)-(23), $P_1(x,\lambda)$ and $P_2(x,\lambda)$ are entire functions in λ. Denote $G_\delta = \{\lambda : \lambda = k^2, |k - k_n| > \delta, n = 1, 2, \ldots\}$ and $\widetilde{G}_\delta = \{\lambda : \lambda = k^2, |k - \widetilde{k}_n| > \delta, n = 1, 2, \ldots\}$, where δ is sufficiently small number and k_n and $\widetilde{k}_n$ are square roots of the eigenvalues of the problem L and $\widetilde{L}$, respectively. One can easily show that the asymptotics

$$\begin{aligned} \Phi(x,\lambda) &= O\left(\lambda^{-(i+3)/2}\exp(-|\tau|x)\right), \\ \Phi'(x,\lambda) &= O\left(\lambda^{-(i+2)/2}\exp(-|\tau|x)\right) \end{aligned} \tag{24}$$

are valid for $d_i < x < d_{i+1}$, $i = 0, 1, 2$, and sufficiently large $|\lambda|$ in $G_\delta \cap \widetilde{G}_\delta$. Thus, the following inequalities are obtained from (6) and (24):

$$\left|P_1(x,\lambda)\right| \le C_\delta, \qquad \left|P_2(x,\lambda)\right| \le C_\delta|\lambda|^{-1/2}, \qquad \lambda \in G_\delta \cap \widetilde{G}_\delta. \tag{25}$$

According to the last inequalities and Liouville's theorem, $P_1(x,\lambda) = A(x)$ and $P_2(x,\lambda) = 0$. Use (23) again to take

$$\begin{aligned} \varphi(x,\lambda) &= A(x)\widetilde{\varphi}(x,\lambda), \\ \Phi(x,\lambda) &= A(x)\widetilde{\Phi}(x,\lambda). \end{aligned} \tag{26}$$

Since $W[\Phi(x,\lambda), \varphi(x,\lambda)] = 1$ and similarly $W[\widetilde{\Phi}(x,\lambda), \widetilde{\varphi}(x,\lambda)] = 1$, then $A^2(x) = 1$.

On the other hand, the asymptotic expressions

$$\begin{aligned} \varphi(x,\lambda) &= C(\lambda)\exp\left(-i\sqrt{\lambda}x\right)(1 + o(1)), \\ \varphi(x,\lambda) &= \widetilde{C}(\lambda)\exp\left(-i\sqrt{\lambda}x\right)(1 + o(1)) \end{aligned} \tag{27}$$

are valid for $\sqrt{\lambda} \to \infty$ on the imaginary axis, where

$$\begin{aligned} C(\lambda) &= \begin{cases} -\dfrac{1}{2}\lambda, & 0 < x < d_1, \\ \dfrac{\gamma_1}{4}\lambda^{3/2}, & d_1 < x < d_2, \\ \dfrac{\gamma_1\gamma_2}{8}\lambda^2, & d_2 < x < 1, \end{cases} \\ \widetilde{C}(\lambda) &= \begin{cases} -\dfrac{1}{2}\lambda, & 0 < x < \widetilde{d}_1, \\ \dfrac{\widetilde{\gamma}_1}{4}\lambda^{3/2}, & \widetilde{d}_1 < x < \widetilde{d}_2, \\ \dfrac{\widetilde{\gamma}_1\widetilde{\gamma}_2}{8}\lambda^2, & \widetilde{d}_2 < x < 1. \end{cases} \end{aligned} \tag{28}$$

Assume that $d_1 \ne \widetilde{d}_1$ and $d_2 \ne \widetilde{d}_2$. There are six different cases for the permutation of the numbers d_i and $\widetilde{d}_i$. Without loss of generality, let $0 < d_1 < \widetilde{d}_1 < d_2 < \widetilde{d}_2 < 1$.

From (26)-(27), we get $\gamma_1 = \widetilde{\gamma}_1$, $\gamma_2 = \widetilde{\gamma}_2$, and $A(x) \equiv 1$, while $x \in [0, d_1) \cup (\widetilde{d}_1, d_2) \cup (\widetilde{d}_2, 1]$.

Moreover, we get

$$2\lambda^{-1/2}(1 + o(1))A(x) + \gamma_1 = o(1), \tag{29}$$

while $x \in (d_1, \widetilde{d}_1)$. By taking limit in (29) as $|\lambda| \to \infty$, we condradict $\gamma_1 > 0$. Thus, $d_1 = \widetilde{d}_1$. Similarly, $d_2 = \widetilde{d}_2$, and $A(x) = 1$ in I. Hence,

$$\varphi(x,\lambda) = \widetilde{\varphi}(x,\lambda), \qquad \frac{\psi'(x,\lambda)}{\psi(x,\lambda)} = \frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\psi}(x,\lambda)}. \tag{30}$$

It can be obtained from (1), (4), and (5) that $q(x) = \widetilde{q}(x)$, a.e. in I; $\mathbf{s}_i = \widetilde{\mathbf{s}}_i$, $i = 1, 2$, and $\mathbf{h} = \widetilde{\mathbf{h}}$, $\mathbf{H} = \widetilde{\mathbf{H}}$. Consequently, $L = \widetilde{L}$. □

Theorem 3. *If $\{\lambda_n, \boldsymbol{\alpha}_n\}_{n\ge0} = \{\widetilde{\lambda}_n, \widetilde{\boldsymbol{\alpha}}_n\}_{n\ge0}$, then $L = \widetilde{L}$.*

Proof. The meromorphic function $m(\lambda)$ has simple poles at λ_n, and its residues at these poles are

$$\begin{aligned} \operatorname{Res}\{m(\lambda), \lambda_n\} &= \frac{\psi'(0,\lambda_n) + h_0\psi(0,\lambda_n)}{\rho_1\Delta'(\lambda_n)} \\ &= -\frac{\boldsymbol{\beta}_n}{\Delta'(\lambda_n)} = -\frac{1}{\boldsymbol{\alpha}_n}. \end{aligned} \tag{31}$$

Denote $\Gamma_n = \{\lambda : |\lambda| = (\sqrt{\lambda_n} + \varepsilon)^2\}$, where ε is sufficiently small number. Consider the contour integral

$$F_n(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{m(\eta)}{(\eta - \lambda)}d\eta, \quad \lambda \in \operatorname{int}\Gamma_n. \tag{32}$$

There exists a constant $C_\delta > 0$ such that $\Delta(\lambda) \ge |\lambda|^{7/2}C_\delta\exp|\tau|$ holds for $\lambda \in G_\delta$. Use this inequality and (21) to get $|m(\lambda)| \le C_\delta/|\lambda|^{3/2}$, for $\lambda \in G_\delta$. Hence, $\lim_{n\to\infty}F_n(\lambda) = 0$, and so

$$m(\lambda) = \sum_{n=0}^{\infty}\frac{1}{\alpha_n(\lambda_n - \lambda)} \tag{33}$$

is obtained from residue theorem. Consequently, if $\lambda_n = \tilde{\lambda}_n$ and $\boldsymbol{\alpha}_n = \tilde{\boldsymbol{\alpha}}_n$ for all n, then from (33), $m(\lambda) = \tilde{m}(\lambda)$. Hence, Theorem 2 yields $L = \tilde{L}$. □

We consider the boundary value problem L_1 with the condition

$$y'(0,\lambda) + h_0 y(0,\lambda) = 0 \tag{34}$$

instead of (2) in L. Let $\{\eta_n^2\}_{n\geq 0}$ be the eigenvalues of the problem L_1. It is obvious that η_n are zeros of $\Delta_1(\eta) := \psi'(0,\eta) + h_0\psi(0,\eta)$.

Theorem 4. *If* $\{\lambda_n, \eta_n\}_{n\geq 0} = \{\tilde{\lambda}_n, \tilde{\eta}_n\}_{n\geq 0}$ *and* $\mathbf{h} = \tilde{\mathbf{h}}$, *then* $L = \tilde{L}$.

Proof. The functions $\Delta(\lambda)$ and $\Delta_1(\eta)$ which are entire of order 1/2 can be represented by Hadamard's factorization theorem as follows:

$$\begin{aligned} \Delta(\lambda) &= C\prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \\ \Delta_1(\eta) &= C_1\prod_{n=0}^{\infty}\left(1 - \frac{\eta}{\eta_n}\right), \end{aligned} \tag{35}$$

where C and C_1 are constants which depend only on $\{\lambda_n\}$ and $\{\eta_n\}$, respectively. Therefore, $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$ and $\Delta_1(\eta) \equiv \tilde{\Delta}_1(\eta)$, when $\lambda_n = \tilde{\lambda}_n$ and $\eta_n = \tilde{\eta}_n$ for all n. Thus, $\psi'(0,\eta) + h_0\psi(0,\eta) = \tilde{\psi}'(0,\eta) + h_0\tilde{\psi}(0,\eta)$. Moreover, $\rho_1 = \tilde{\rho}_1$ since $\mathbf{h} = \tilde{\mathbf{h}}$. Consequently, the equality (21) yields $m(\lambda) \equiv \tilde{m}(\lambda)$. Hence, the proof is completed by Theorem 2. □

Appendix

The solution $\varphi(x,\lambda)$ satisfies the following integral equations.

If $x < d_1$,

$$\begin{aligned} \varphi(x,\lambda) = {} & (\lambda h_0 - h_2)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + (h_1 - \lambda)\cos\sqrt{\lambda}x \\ & + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\sqrt{\lambda}(x-t)\, q(t)\,\varphi(t,\lambda)\,dt; \end{aligned} \tag{A.1}$$

if $d_1 < x < d_2$,

$$\begin{aligned} \varphi(x,\lambda) = {} & \frac{\lambda h_0 - h_2}{\sqrt{\lambda}} \\ & \times\left(\alpha_1^+ \sin\sqrt{\lambda}x + \alpha_1^- \sin\sqrt{\lambda}(2d_1 - x)\right) \\ & + (h_1 - \lambda) \\ & \times\left(\alpha_1^+ \cos\sqrt{\lambda}x + \alpha_1^- \cos\sqrt{\lambda}(2d_1 - x)\right) \\ & + \frac{(\gamma_1\lambda + \beta_1)(h_1 - \lambda)}{2\sqrt{\lambda}} \\ & \times\left(\sin\sqrt{\lambda}(2d_1 - x) - \sin\sqrt{\lambda}x\right) \\ & + \frac{(\gamma_1\lambda + \beta_1)(\lambda h_0 - h_2)}{2\lambda} \\ & \times\left(\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}(2d_1 - x)\right) \\ & + \frac{(\gamma_1\lambda + \beta_1)}{2\lambda} \\ & \times\int_0^{d_1}\left(\cos\sqrt{\lambda}(x-t) - \cos\sqrt{\lambda}(2d_1 - x - t)\right) \\ & \qquad \times q(t)\,\varphi(t,\lambda)\,dt \\ & + \frac{1}{\sqrt{\lambda}}\int_0^{d_1}\left(\alpha_1^+ \sin\sqrt{\lambda}(x-t)\right. \\ & \qquad \left. + \alpha_1^- \sin\sqrt{\lambda}(2d_1 - x - t)\right) \\ & \qquad \times q(t)\,\varphi(t,\lambda)\,dt \\ & + \frac{1}{\sqrt{\lambda}}\int_{d_1}^x \sin\sqrt{\lambda}(x-t)\, q(t)\,\varphi(t,\lambda)\,dt; \end{aligned} \tag{A.2}$$

if $x > d_2$,

$$\begin{aligned} \varphi(x,\lambda) = {} & -\frac{(h_1 - \lambda)(\gamma_1\lambda + \beta_1)}{2\sqrt{\lambda}} \\ & \times\left[\alpha_2^+\left(\sin\sqrt{\lambda}x - \sin\sqrt{\lambda}(2d_1 - x)\right)\right. \\ & \quad + \alpha_2^-\left(\sin\sqrt{\lambda}(2d_2 - x)\right. \\ & \qquad \left.\left. + \sin\sqrt{\lambda}(2d_2 - 2d_1 - x)\right)\right] \\ & + \frac{(\lambda h_0 - h_2)(\gamma_1\lambda + \beta_1)}{2\lambda} \\ & \times\left[\alpha_2^+\left(\cos\sqrt{\lambda}x - \cos\sqrt{\lambda}(2d_1 - x)\right)\right. \\ & \quad + \alpha_2^-\left(\cos\sqrt{\lambda}(2d_1 - x)\right. \\ & \qquad \left.\left. - \cos\sqrt{\lambda}(2d_2 - 2d_1 - x)\right)\right] \\ & + (h_1 - \lambda) \\ & \times\left[\alpha_1^+\left(\alpha_2^+ \cos\sqrt{\lambda}x + \alpha_2^- \cos\sqrt{\lambda}(2d_2 - x)\right)\right. \\ & \quad + \alpha_1^-\left(\alpha_2^+ \cos\sqrt{\lambda}(2d_1 - x)\right. \\ & \qquad \left.\left. + \alpha_2^- \cos\sqrt{\lambda}(2d_2 - 2d_1 - x)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\lambda h_0-h_2)}{\sqrt{\lambda}}\\
&\times\Big[\alpha_1^+\Big(\alpha_2^+\sin\sqrt{\lambda}x+\alpha_2^-\sin\sqrt{\lambda}(2d_2-x)\Big)\\
&\quad+\alpha_1^-\Big(\alpha_2^+\sin\sqrt{\lambda}(2d_1-x)\\
&\qquad-\alpha_2^-\sin\sqrt{\lambda}(2d_2-2d_1-x)\Big)\Big]\\
&-\frac{(h_1-\lambda)(\gamma_1\lambda+\beta_1)(\gamma_2\lambda+\beta_2)}{4\lambda}\\
&\times\Big[\cos\sqrt{\lambda}x+\cos\sqrt{\lambda}(2d_1-x)\\
&\quad-\cos\sqrt{\lambda}(2d_2-x)\\
&\quad-\cos\sqrt{\lambda}(2d_2-2d_1-x)\Big]\\
&-\frac{(\lambda h_0-h_2)(\gamma_1\lambda+\beta_1)(\gamma_2\lambda+\beta_2)}{4\lambda^{3/2}}\\
&\times\Big[\sin\sqrt{\lambda}x-\sin\sqrt{\lambda}(2d_1-x)\\
&\quad-\sin\sqrt{\lambda}(2d_2-x)\\
&\quad+\sin\sqrt{\lambda}(2d_2-2d_1-x)\Big]\\
&+\frac{(h_1-\lambda)(\gamma_2\lambda+\beta_2)}{2\sqrt{\lambda}}\\
&\times\Big[\alpha_1^+\Big(\sin\sqrt{\lambda}(2d_2-x)-\sin\sqrt{\lambda}x\Big)\\
&\quad+\alpha_1^-\Big(\sin\sqrt{\lambda}(2d_1-x)\\
&\qquad+\sin\sqrt{\lambda}(2d_2-2d_1-x)\Big)\Big]\\
&+\frac{(\lambda h_0-h_2)(\gamma_2\lambda+\beta_2)}{2\lambda}\\
&\times\Big[\alpha_1^+\Big(\cos\sqrt{\lambda}x-\cos\sqrt{\lambda}(2d_2-x)\Big)\\
&\quad-\alpha_1^-\Big(\cos\sqrt{\lambda}(2d_1-x)\\
&\qquad-\cos\sqrt{\lambda}(2d_2-2d_1-x)\Big)\Big]\\
&+\frac{\alpha_2}{2\sqrt{\lambda}}\int_0^{d_1}\Big[\alpha_1^+\Big(\sin\sqrt{\lambda}(2d_2-x-t)\\
&\qquad+\sin\sqrt{\lambda}(x-t)\Big)\\
&\quad+\alpha_1^-\Big(\sin\sqrt{\lambda}(2d_1-x-t)\\
&\qquad-\sin\sqrt{\lambda}(2d_2-2d_1-x+t)\Big)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt+\frac{\alpha_2(\gamma_1\lambda+\beta_1)}{4\lambda}\\
&\times\int_0^{d_1}\Big[\cos\sqrt{\lambda}(x-t)-\cos\sqrt{\lambda}(2d_1-x-t)\\
&\quad+\cos\sqrt{\lambda}(2d_2-x-t)\\
&\quad+\cos\sqrt{\lambda}(2d_2-2d_1-x+t)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt-\frac{1}{2\alpha_2\sqrt{\lambda}}\\
&\times\int_0^{d_2}\Big[\alpha_1^+\Big(\sin\sqrt{\lambda}(2d_2-x-t)-\sin\sqrt{\lambda}(x-t)\Big)\\
&\quad-\alpha_1^-\Big(\sin\sqrt{\lambda}(2d_1-x-t)\\
&\qquad+\sin\sqrt{\lambda}(2d_2-2d_1-x+t)\Big)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt\\
&+\frac{(\gamma_1\lambda+\beta_1)}{4\alpha_2\lambda}\\
&\times\int_0^{d_2}\Big[\cos\sqrt{\lambda}(x-t)-\cos\sqrt{\lambda}(2d_1-x-t)\\
&\quad-\cos\sqrt{\lambda}(2d_2-x-t)\\
&\quad+\cos\sqrt{\lambda}(2d_2-2d_1-x+t)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt\\
&-\frac{1}{\sqrt{\lambda}}\int_{d_1}^{d_2}\Big[\alpha_2^+\sin\sqrt{\lambda}(x-t)\\
&\qquad+\alpha_2^-\sin\sqrt{\lambda}(2d_2-x-t)\Big]\\
&\quad\times q(t)\varphi(t,\lambda)\,dt\\
&+\frac{(\gamma_2\lambda+\beta_2)}{2\lambda}\\
&\times\int_0^{d_1}\Big[\alpha_1^+\Big(\cos\sqrt{\lambda}(x-t)-\cos\sqrt{\lambda}(2d_2-x-t)\Big)\\
&\quad-\alpha_1^-\Big(\cos\sqrt{\lambda}(2d_1-x-t)\\
&\qquad-\cos\sqrt{\lambda}(2d_2-2d_1-x+t)\Big)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt\\
&-\frac{(\gamma_1\lambda+\beta_1)(\gamma_2\lambda+\beta_2)}{4\lambda^{3/2}}\\
&\times\int_0^{d_1}\Big[\sin\sqrt{\lambda}(x-t)-\sin\sqrt{\lambda}(2d_1-x-t)\\
&\quad-\sin\sqrt{\lambda}(2d_2-x-t)\\
&\quad+\sin\sqrt{\lambda}(2d_2-2d_1-x+t)\Big]\\
&\times q(t)\varphi(t,\lambda)\,dt\\
&-\frac{(\gamma_2\lambda+\beta_2)}{2\lambda}
\end{aligned}
$$

$$\times \int_{d_1}^{d_2} \left[\cos\sqrt{\lambda}(x-t) - \cos\sqrt{\lambda}(2d_2 - x - t)\right] \times q(t)\varphi(t,\lambda)\,dt + \frac{1}{\sqrt{\lambda}}\int_{d_2}^{x} \sin\sqrt{\lambda}(x-t)\,q(t)\varphi(t,\lambda)\,dt, \tag{A.3}$$

where $\alpha_i^{\pm} = (1/2)(\alpha_i \pm 1/\alpha_i)$, $i = 1, 2$.

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