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Research Article

Hardy-Type Space Associated with an Infinite-Dimensional Unitary Matrix Group

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We investigate an orthogonal system of the homogenous Hilbert-Schmidt polynomials with respect to a probability measure which is invariant under the right action of an infinite-dimensional unitary matrix group. With the help of this system, a corresponding Hardy-type space of square-integrable complex functions is described. An antilinear isomorphism between the Hardy-type space and an associated symmetric Fock space is established.

1. Introduction

We investigate an orthogonal system of the Hilbert-Schmidt polynomials in the space L^2_χ of square-integrable complex functions on the projective limit $\mathfrak{U} = \varprojlim U(m)$ of unitary $(m \times m)$-dimensional matrix groups $U(m)$ $(m \in \mathbb{N})$, called the space of virtual unitary matrices and endowed with the projective limit measure $\chi = \varprojlim \chi_m$ of the probability Haar measures χ_m on $U(m)$. The measure χ on the space $\mathfrak{U}$ is invariant under the right action of the infinite-dimensional unitary group $U(\infty) \times U(\infty)$, where $U(\infty) = \bigcup_m U(m)$.

The space of virtual unitary matrices $\mathfrak{U}$ was studied by Neretin [1] and Olshanski [2]. This notion relates to D. Pickrell's space of virtual Grassmannian [3] and to Kerov, Olshanski, and Vershik's space of virtual permutations [4]. Various spaces of integrable functions with respect to measures that are invariant under infinite-dimensional groups have been widely applied in stochastic processes [5], infinite-dimensional probability [6, 7], complex analysis [8], and so forth.

The main results of the present paper are Theorems 6-7 that describe a Hardy-type subspace $\mathscr{H}^2_\chi \subset L^2_\chi$ spanned by the finite type homogenous Hilbert-Schmidt polynomials that are generated by an associated symmetric Fock space.

2. Preliminaries

We consider the following infinite-dimensional unitary matrix groups:

$$U(\infty) = \bigcup \{U(m) : m \in \mathbb{N}\}, \qquad U^2(\infty) := U(\infty) U(\infty), \tag{1}$$

where $U(m)$ is the group of unitary $(m \times m)$-matrices which is identified with the subgroup in $U(m+1)$ fixing the $(m+1)$th basis vector. In other words, $U(\infty)$ is the group of infinite unitary matrices $u = [u_{ij}]_{i,j\in\mathbb{N}}$ with finitely many matrix entries u_{ij} distinct from δ_{ij}. We equip every group $U(m)$ with the probability Haar measure χ_m.

Following [1, 2], every matrix $u_m \in U(m)$ with $m > 1$, we write in the following block matrix form:

$$u_m = \begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix}, \tag{2}$$

corresponding to the partition $m = (m-1) + 1$ so that $z_{m-1} \in U(m-1)$ and $t \in \mathbb{C}$. Over the group $U(\infty)$ (resp., $U(m)$) the right action is well defined:

$$u \cdot g = w^{-1} u v, \tag{3}$$

where u belongs to $U(\infty)$ (resp., to $U(m)$) and $g = (v, w)$ belongs to $U^2(\infty)$ (resp., to $U^2(m) := U(m) \times U(m)$). In [1, Proposition 0.1], [2, Lemma 3.1], it was proven that the following Livšic-type mapping:

$$\pi_{m-1}^m : U(m) \ni u_m \longrightarrow u_{m-1} \in U(m-1), \tag{4}$$

such that

$$\begin{bmatrix} z_{m-1} & a \\ b & t \end{bmatrix} \longmapsto \begin{cases} z_{m-1} - a(1+t)^{-1}b: & t \neq -1, \\ z_{m-1}: & t = -1, \end{cases} \tag{5}$$

(which is not a group homomorphism) is Borel and surjective onto $U(m-1)$ and commutes with the right action of $U^2(m-1)$.

As is known [1, Theorem 1.6], the pullback of the probability Haar measure χ_{m-1} on $U(m-1)$ under the mapping π_{m-1}^m is the probability Haar measure χ_m on $U(m)$, that is,

$$\chi_{m-1} \circ \pi_{m-1}^m = \chi_m. \tag{6}$$

Let $U'(m) \subset U(m)$ be the subset of unitary matrices which do not have $\{-1\}$, as an eigenvalue. Then, $U'(m)$ is open in $U(m)$, and the complement $U(m) \setminus U'(m)$ is a χ_m-negligible set. Moreover (see [2, Lemma 3.11]), the mapping

$$\pi_{m-1}^m : U'(m) \longrightarrow U'(m-1) \tag{7}$$

is continuous and surjective.

Consider the projective limits, taken with respect to the surjective Borel projections π_{m-1}^m and their continuous restrictions $\pi_{m-1}^m|_{U'(m)}$, respectively,

$$\mathfrak{U} = \varprojlim U(m), \qquad \mathfrak{U}' = \varprojlim U'(m), \tag{8}$$

called the spaces of virtual unitary matrices. Notice that $\mathfrak{U}$ is a Borel subset in the Cartesian product $\bigtimes_{m\in\mathbb{N}} U(m) = \{u = (u_m) : u_m \in U(m)\}$ endowed with the product topology, because all mapping π_{m-1}^m are Borel. Moreover, the canonical projections

$$\pi_m : \mathfrak{U} \longrightarrow U(m), \qquad \pi_m : \mathfrak{U}' \longrightarrow U'(m), \tag{9}$$

such that $\pi_{m-1} = \pi_{m-1}^m \circ \pi_m$, are surjective by surjectivity of π_{m-1}^m and $\pi_{m-1}^m|_{U'(m)}$.

Following [2, Lemma 4.8], [1, Section 3.1], with the help of the Kolmogorov consistent theorem, we uniquely define a probability measure χ on $\mathfrak{U}'$ as the projective limit under the mapping (6),

$$\chi = \varprojlim \chi_m, \tag{10}$$

which satisfies the equality $\chi = \chi_m \circ \pi_m$ for all $m \in \mathbb{N}$. On $\mathfrak{U} \setminus \mathfrak{U}'$, the measure χ is zero, because χ_m is zero on $U(m) \setminus U'(m)$ for all $m \in \mathbb{N}$.

Using (3), right action of the group $U^2(\infty)$ on the space of virtual unitary matrices $\mathfrak{U}$ can be defined (see [2, Definition 4.5]) as follows:

$$\pi_m(u \cdot g) = w^{-1}\pi_m(u)v, \quad u \in \mathfrak{U}, \tag{11}$$

where m is so large that $g = (v, w) \in U^2(m)$.

The canonical dense embedding $\imath : U(\infty) \hookrightarrow \mathfrak{U}$ to any element $u_m \in U(m)$ assigns the unique sequence $u = (u_l)_{l\in\mathbb{N}}$, such that

$$\imath : U(m) \ni u_m \longmapsto (u_l) \in \mathfrak{U},$$
$$u_l = \begin{cases} \pi_l^{l+1} \circ \cdots \circ \pi_{m-1}^m(u_m): & l < m, \\ u_m: & l = m, \\ \begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1}_{l-m} \end{bmatrix}: & l > m, \end{cases} \tag{12}$$

where $\mathbb{1}_{l-m}$ is the unit in $U(l-m)$. So, the image $\imath \circ U(\infty)$ consists of stabilizing sequences in $\mathfrak{U}$ (see [2, Section 4]).

3. Invariant Probability Measure

In what follows, we will endow the space of virtual unitary matrices $\mathfrak{U}$ with the measure $\chi = \varprojlim \chi_m$. A complex function on $\mathfrak{U}$ is called cylindrical [2, Definition 4.5] if it has the following form:

$$f(u) = (f_m \circ \pi_m)(u), \quad u \in \mathfrak{U}, \tag{13}$$

for a certain $m \in \mathbb{N}$ and a certain complex function f_m on $U(m)$.

Any continuous bounded function f on $\mathfrak{U}'$ has a unique χ-essentially bounded extension on $\mathfrak{U}$, because the set $\mathfrak{U} \setminus \mathfrak{U}'$ is χ-negligible. Therefore, if the function $U'(m) \ni \pi_m(u) \mapsto f_m[\pi_m(u)]$ in the definition (13) is continuous and bounded, then the corresponding cylindrical function f is χ essentially bounded.

By $\mathscr{L}_\chi^\infty$, we denote closure of the algebraic hull of all cylindrical χ-essentially bounded functions (13) with respect to the following norm:

$$\|f\|_{\mathscr{L}_\chi^\infty} = \operatorname*{ess\,sup}_{u\in\mathfrak{U}} |f(u)|. \tag{14}$$

Lemma 1. *The measure $\chi = \varprojlim \chi_m$ on $\mathfrak{U}$ is a Radon probability measure such that*

$$\int_{\mathfrak{U}} f(u \cdot g)\, d\chi(u) = \int_{\mathfrak{U}} f(u)\, d\chi(u), \tag{15}$$

for all $g \in U^2(\infty)$ and $f \in \mathscr{L}_\chi^\infty$. For any compact set $K \subset U(m)$ the following equality holds:

$$(\chi \circ \imath)(K) = \chi_m(K). \tag{16}$$

Proof. Recall the Prohorov criterion, which is adapted to our notation (see [9, Chapter IX.4.2, Theorem 1] or [6, Theorem 6]): there exists a Radon probability measure χ' on $\mathfrak{U}'$ such that

$$\chi' = \chi_m \circ \pi_m|_{\mathfrak{U}'} \quad \forall m \in \mathbb{N}, \tag{17}$$

if and only if for every $\varepsilon > 0$ there exists a compact set $\mathscr{K}$ in $\mathfrak{U}'$ such that the following inequality

$$(\chi_m \circ \pi_m)(\mathscr{K}) \geq 1 - \varepsilon \quad \forall m \in \mathbb{N} \tag{18}$$

holds; in this case, χ' is uniquely determined by means of the formula $\chi'(\mathscr{K}) = \inf_{m\in\mathbb{N}}(\chi_m \circ \pi_m)(\mathscr{K})$, where $\mathscr{K}$ is a compact set in $\mathfrak{U}'$.

Let $K_n \subset U'(n)$ be a compact set with a fixed n. Putting $K_{n-1} = \pi^n_{n-1}(K_n)$, we have

$$\chi_{n-1}(K_{n-1}) = (\chi_{n-1} \circ \pi^n_{n-1})(K_n) = \chi_n(K_n). \tag{19}$$

On the other hand, if we put $K_{n+1} = \left[\begin{smallmatrix} K_n & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then via (6),

$$\begin{aligned} \chi_{n+1}(K_{n+1}) &= \left(\chi_n \circ \pi^{n+1}_n\right)(K_{n+1}) \\ &= \left(\chi_n \circ \pi^{n+1}_n\right)\begin{bmatrix} K_n & 0 \\ 0 & 1 \end{bmatrix} = \chi_n(K_n). \end{aligned} \tag{20}$$

As a consequence, the compact set $\mathscr{K} = (K_m)$ in $\mathfrak{U}'$, generated by a compact set $K_n \subset U'(n)$ with the help of mappings π^n_{n-1}, satisfies the following condition:

$$\chi_n(K_n) = \chi_m(K_m) \quad \forall m \in \mathbb{N}. \tag{21}$$

The probability Haar measure χ_n is regular on $U(n)$, and the complement $U(n) \setminus U'(n)$ is a negligible set. Hence, if K_n runs over all compact sets in $U'(n)$, then

$$\sup_{K_n \subset U'(n)} \chi_n(K_n) = 1. \tag{22}$$

Therefore, for every $\varepsilon > 0$ there exists a compact set $K_n \subset U'(n)$ such that $\chi_n(K_n) \geq 1 - \varepsilon$. From (21), it follows that for every $\varepsilon > 0$ the compact set $\mathscr{K}$ satisfies the hypothesis of Prohorov's criterion:

$$(\chi_m \circ \pi_m)(\mathscr{K}) = \chi_m(K_m) \geq 1 - \varepsilon \quad \forall m \in \mathbb{N}. \tag{23}$$

So, in view of this criterion, there exists a unique Radon probability measure χ' on $\mathfrak{U}'$ which satisfies the condition (17). However, on the projective limits $\mathfrak{U}' = \varprojlim U'(m)$, there exists a unique $U^2(\infty)$-invariant Radon measure χ, determined by the equality (15). Using the uniqueness property of projective limits, we obtain $\chi' = \chi$. The measure χ on $\mathfrak{U}\setminus\mathfrak{U}'$ is defined to be zero, because χ_m is zero on $U(m) \setminus U'(m)$.

As a consequence of (21), we obtain (16), because

$$\chi(\mathscr{K}) = \inf_{m\in\mathbb{N}} \chi_m(K_m) = \chi_n(K_n). \tag{24}$$

As is known [1, Proposition 3.2], the measure χ is $U^2(\infty)$-invariant under the right actions (11) on the space $\mathfrak{U}$. Hence, for every $f \in \mathscr{L}^\infty_\chi$, the equality (15) holds. □

4. Shift Groups

Consider that in the space $\mathscr{L}^\infty_\chi$, the group of shifts

$$Q_g f(u) = f(u \cdot g), \quad g \in U^2(\infty) \ \ u \in \mathfrak{U}, \tag{25}$$

is generated by the right action of $U^2(\infty)$ over $\mathfrak{U}$. Choosing instead of $U(\infty)$ a compact subgroup $U(m)$ or the compact subgroups

$$\begin{aligned} U_0 &= \{g_0(\vartheta) = \exp(\mathrm{i}\vartheta) : \vartheta \in (-\pi, \pi]\}, \\ U_j(m) &= \{g_{mj}(\vartheta) = \mathbb{1}_{j-1} \otimes \exp(\mathrm{i}\vartheta) \otimes \mathbb{1}_{m-j} : \vartheta \in (-\pi, \pi]\} \\ & \qquad j = 1, \ldots, m, \end{aligned} \tag{26}$$

we obtain the corresponding subgroups of shifts Q_g with elements $g \in U^2(m)$ or with elements $g_0(\vartheta) \in U_0^2$ and $g_{mj}(\vartheta) \in U_j^2(m)$, respectively. Here, $\mathbb{1}_m$ means the unit element in $U(m)$.

Lemma 2. *For any $f \in \mathscr{L}^\infty_\chi$ the following equalities:*

$$\int_{\mathfrak{U}} f d\chi = \int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} Q_g f(u)\, d(\chi_m \otimes \chi_m)(g), \tag{27}$$

$$\int_{\mathfrak{U}} f \ d\chi = \frac{1}{2\pi} \int_{\mathfrak{U}} d\chi(u) \int_{-\pi}^{\pi} Q_{g(\vartheta)} f(u)\, d\vartheta, \tag{28}$$

with $g(\vartheta) \in U_0^2$ or $U_j^2(m)$ hold.

Proof. For any $f \in \mathscr{L}^\infty_\chi$, the function $(u, g) \mapsto Q_g f(u) = f(u \cdot g)$ is integrable on the Cartesian product $\mathfrak{U} \times U^2(m)$. By the Fubini theorem, we obtain

$$\begin{aligned} &\int_{\mathfrak{U}} d\chi(u) \int_{U^2(m)} Q_g f(u)\, d(\chi_m \otimes \chi_m)(g) \\ &\quad = \int_{U^2(m)} d(\chi_m \otimes \chi_m)(g) \int_{\mathfrak{U}} Q_g f(u)\, d\chi(u). \end{aligned} \tag{29}$$

This equality yields the required formula (27), because the internal integral on the right-hand side is independent of g and $\int_{U^2(m)} d(\chi_m \otimes \chi_m) = 1$. In turn, putting instead of $U(m)$ the subgroups U_0 and $U_j(m)$, we obtain equalities (28). □

5. The Homogeneous Hilbert-Schmidt Polynomials

Consider the countable orthogonal Hilbertian sum

$$\mathsf{E} := \bigoplus_{m\in\mathbb{N}} \mathbb{C}^m = \{x = (x_m) : x_m \in \mathbb{C}^m, \|x\|_{\mathsf{E}} < \infty\}, \tag{30}$$

with the scalar product $\langle x \mid y\rangle_{\mathsf{E}} = \sum_m \langle x_m \mid y_m\rangle_{\mathbb{C}^m}$, where every coordinate $x_m \in \mathbb{C}^m$ is identified with its image $(0, \ldots, 0, x_m, 0, \ldots) \in \mathsf{E}$ under the embedding $\mathbb{C}^m \hookrightarrow \mathsf{E}$.

Let $\otimes^n_{\mathfrak{h}}\mathsf{E}$ stand for the complete nth tensor power of the Hilbert subspace E, endowed with the Hilbertian scalar product and norm, respectively,

$$\begin{aligned} \langle x_1 \otimes \cdots \otimes x_n \mid \psi_n\rangle_{\otimes^n_{\mathfrak{h}}\mathsf{E}} &= \sum_j \langle x_1 \mid y_{1j}\rangle_{\mathsf{E}} \ldots \langle x_n \mid y_{nj}\rangle_{\mathsf{E}}, \\ \|\psi_n\|_{\otimes^n_{\mathfrak{h}}\mathsf{E}} &= \langle \psi_n \mid \psi_n\rangle^{1/2}_{\otimes^n_{\mathfrak{h}}\mathsf{E}}, \end{aligned} \tag{31}$$

where $x_1 \otimes \cdots \otimes x_n$, $y_{1j} \otimes \cdots \otimes y_{nj} \in \otimes_{\mathfrak{h}}^n \mathsf{E}$ with $x_{tj}, y_{tj} \in \mathsf{E}$ for all $t = 1, \ldots, n$ and $\psi_n = \sum_j y_{1j} \otimes \cdots \otimes y_{nj}$ denotes a finite sum. Put $\otimes_{\mathfrak{h}}^0 \mathsf{E} = \mathbb{C}$. We use the following short denotation:

$$x^{\otimes n} = x \otimes \cdots \otimes x, \quad x \in \mathsf{E}. \tag{32}$$

Replacing the space E by the subspace $\mathbb{C}^m$, we similarly define the tensor product $\otimes_{\mathfrak{h}}^n \mathbb{C}^m$. There is the unitary embedding $\otimes_{\mathfrak{h}}^n \mathbb{C}^m \hookrightarrow \otimes_{\mathfrak{h}}^n \mathsf{E}$. If $m = 1$, then $\otimes_{\mathfrak{h}}^n \mathbb{C} = \mathbb{C}$.

For any finite sum $\psi_n = \sum_j y_{1j} \otimes \cdots \otimes y_{nj}$ from the space $\otimes_{\mathfrak{h}}^n \mathbb{C}^m$ (or $\otimes_{\mathfrak{h}}^n \mathsf{E}$), we can to define the finite type n-homogeneous Hilbert-Schmidt polynomials:

$$\mathbb{C}^m \ni x \longmapsto \langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = \sum_j \prod_{t=1}^n \langle x \mid y_{tj} \rangle_{\mathbb{C}^m}. \tag{33}$$

Consider the canonical orthonormal bases:

$$\begin{gathered} \mathscr{E}(\mathbb{C}^m) = \{\mathfrak{e}_{m1}, \ldots, \mathfrak{e}_{mm}\} \quad \text{in } \mathbb{C}^m, \\ \mathscr{E}(\mathsf{E}) = \bigcup \{\mathscr{E}(\mathbb{C}^m) : m \in \mathbb{N}\} \quad \text{in } \mathsf{E}, \end{gathered} \tag{34}$$

where $\mathfrak{e}_{ml} = \underbrace{(\overbrace{0, \ldots, 0, 1}^{l}, 0, \ldots, 0)}_{m}$.

If $\mathfrak{s} : \{1, \ldots, n\} \mapsto \{\mathfrak{s}(1), \ldots, \mathfrak{s}(n)\}$ runs over all n-elements permutations $\mathfrak{S}(n)$, then the symmetric nth tensor power $\odot_{\mathfrak{h}}^n \mathbb{C}^m$ is defined to be a codomain of the symmetrization mapping:

$$\begin{gathered} \otimes_{\mathfrak{h}}^n \mathbb{C}^m \ni x_1 \otimes \cdots \otimes x_n \longmapsto x_1 \odot \cdots \odot x_n, \\ x_1 \odot \cdots \odot x_n := \frac{1}{n!} \sum_{\mathfrak{s} \in \mathfrak{S}(n)} x_{\mathfrak{s}(1)} \otimes \cdots \otimes x_{\mathfrak{s}(n)}, \end{gathered} \tag{35}$$

which is an orthogonal projector. Similarly, the symmetric nth tensor power $\odot_{\mathfrak{h}}^n \mathsf{E}$ can be defined. Clearly, $\odot_{\mathfrak{h}}^n \mathbb{C}^m$ is a closed subspace in $\odot_{\mathfrak{h}}^n \mathsf{E}$.

Given a pair of numbers $(m, n) \in \mathbb{N} \times \mathbb{Z}_+$, we consider the n-fold tensor power of the canonical mapping $\pi_m : \mathfrak{U} \ni u \mapsto \pi_m(u) \in U(m)$,

$$\mathfrak{U} \ni u \longmapsto \pi_m^{\otimes n}(u) \in \mathscr{L}(\odot_{\mathfrak{h}}^n \mathbb{C}^m), \tag{36}$$

where $\pi_m^{\otimes n}(u) := \underbrace{\pi_m(u) \otimes \cdots \otimes \pi_m(u)}_{n}$. If $n = 0$, we put $\pi_m^{\otimes 0}(u) = 1$ for all $u \in \mathfrak{U}$ and $m \in \mathbb{N}$. The mapping (36) is Borel and has a continuous restriction to $\mathfrak{U}'$, because π_m has the same property (see Section 2).

Let $\mathfrak{a}_m \in \mathbb{C}^m$ be an arbitrary fixed element such that $\|\mathfrak{a}_m\|_{\mathbb{C}^m} = 1$. Then, $\mathfrak{a}_m^{\otimes n} \in \odot_{\mathfrak{h}}^n \mathbb{C}^m$. Using the mapping (36), we can write

$$[\pi_m^{\otimes n}(u)](\mathfrak{a}_m^{\otimes n}) = \underbrace{[\pi_m(u)](\mathfrak{a}_m) \otimes \cdots \otimes [\pi_m(u)](\mathfrak{a}_m)}_{n}. \tag{37}$$

To any n-homogeneous Hilbert-Schmidt polynomial (33), there corresponds the function

$$\begin{aligned} \psi_n^*(u) &:= \langle [\pi_m^{\otimes n}(u)](\mathfrak{a}_m^{\otimes n}) \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} \\ &= \sum_j \prod_{t=1}^n \langle [\pi_m(u)](\mathfrak{a}_m) \mid y_{tj} \rangle_{\mathbb{C}^m} \end{aligned} \tag{38}$$

of the variable $u \in \mathfrak{U}$. Any cylindrical function of the form $\mathfrak{U} \ni u \mapsto \langle [\pi_m(u)](\mathfrak{a}_m) \mid y_{tj} \rangle_{\mathbb{C}^m}$ has a continuous bounded restriction to $\mathfrak{U}'$. Therefore, it is χ-essentially bounded on $\mathfrak{U}$, because $\mathfrak{U} \setminus \mathfrak{U}'$ is a χ-negligible set. Consequently, $\psi_n^* \in L_\chi^\infty$ and $\psi_n^*|_{\mathfrak{U}'}$ is continuous and bounded.

Definition 3. We define $\mathscr{P}_{\mathfrak{h}}^n(\mathbb{C}^m)$ to be the space of all functions ψ_n^* of the variable $u \in \mathfrak{U}$, determined by the finite type n-homogeneous Hilbert-Schmidt polynomials (33).

Lemma 4. *For any element $\mathfrak{a}_m \in \mathbb{C}^m$ such that $\|\mathfrak{a}_m\|_{\mathbb{C}^m} = 1$ the set*

$$\mathsf{S}^m = \{x = [\pi_m(u)](\mathfrak{a}_m) : u \in \mathfrak{U}\} \tag{39}$$

coincides with the unit sphere in $\mathbb{C}^m$. As a consequence, the one-to-one antilinear corresponding

$$\odot_{\mathfrak{h}}^n \mathbb{C}^m \ni \psi_n \rightleftarrows \psi_n^* \in \mathscr{P}_{\mathfrak{h}}^n(\mathbb{C}^m). \tag{40}$$

Holds, and any function ψ_n^* is independent of the choice of an element $\mathfrak{a}_m \in \mathsf{S}^m$.

Proof. Suppose, on the contrary, that there is an element $\psi_n \in \odot_{\mathfrak{h}}^n \mathbb{C}^m$ such that $\langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = 0$ for all $x = [\pi_m(u)](\mathfrak{a}_m) \in \mathsf{S}^m$ with $u \in \mathfrak{U}$. The mapping

$$\pi_m : \mathfrak{U} \ni u \longmapsto \pi_m(u) \in U(m) \tag{41}$$

is surjective by surjectivity of the mapping π_m (see [2, Lemma 3.1]). Hence, the set S^m coincides with the unit sphere in $\mathbb{C}^m$ and is independent on the choice of an element $\mathfrak{a}_m$. By n-homogeneity, we have $\langle x^{\otimes n} \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = 0$ for all $x \in \mathbb{C}^m$.

Apply the following polarization formula for symmetric tensor products (see, e.g., [10, Section 1.5]):

$$z_1 \odot \cdots \odot z_n = \frac{1}{2^n n!} \sum_{1 \le t \le n} \sum_{\delta_t = \pm 1} \delta_1 \cdots \delta_n \, x^{\otimes n}, \tag{42}$$

with $x = \sum_{t=1}^n \delta_t z_t \in \mathbb{C}^m$, which is valid for all $z_1, \ldots, z_n \in \mathbb{C}^m$. It follows that $\langle z_1 \odot \cdots \odot z_n \mid \psi_n \rangle_{\otimes_{\mathfrak{h}}^n \mathbb{C}^m} = 0$ for all elements $z_1, \ldots, z_n \in \mathbb{C}^m$. Hence, $\psi_n = 0$, because the subset of all elements $z_1 \odot \cdots \odot z_n$ is total in $\odot_{\mathfrak{h}}^n \mathbb{C}^m$. As a consequence, the subset

$$\{x^{\otimes n} = [\pi_m^{\otimes n}(u)](\mathfrak{a}_m^{\otimes n}) : u \in \mathfrak{U}\} \tag{43}$$

is also total in $\odot_{\mathfrak{h}}^n \mathbb{C}^m$. It immediately yields the correspondence (40). □

Consider the symmetric Fock space F and its closed subspace F_m, where

$$\begin{gathered} \mathsf{F} := \mathbb{C} \oplus \mathsf{E} \oplus (\odot_{\mathfrak{h}}^2 \mathsf{E}) \oplus (\odot_{\mathfrak{h}}^3 \mathsf{E}) \oplus \cdots, \\ \mathsf{F}_m := \mathbb{C} \oplus \mathbb{C}^m \oplus (\odot_{\mathfrak{h}}^2 \mathbb{C}^m) \oplus (\odot_{\mathfrak{h}}^3 \mathbb{C}^m) \oplus \cdots. \end{gathered} \tag{44}$$

We will use the following notations:

$$\begin{aligned} (m) &:= (m1,\dots,mm),\\ k_{(m)} &:= (k_{m1},\dots,k_{mm})\in\mathbb{Z}_+^m,\\ \left|k_{(m)}\right| &:= k_{m1}+\cdots+k_{mm},\\ k_{(m)}! &:= k_{m1}!\cdot\ldots\cdot k_{mm}!. \end{aligned}\tag{45}$$

As is well known (see, e.g., [11]), the system of symmetric tensor elements, indexed by the set $k_{(m)}$,

$$\begin{aligned}\mathscr{E}\left(\odot_{\mathfrak{h}}^n\mathbb{C}^m\right)=\Big\{&\mathbf{e}_{(m)}^{\otimes k_{(m)}}=\mathbf{e}_{m1}^{\otimes k_{m1}}\odot\cdots\odot\mathbf{e}_{mm}^{\otimes k_{mm}}:\\ &k_{(m)}\in\mathbb{Z}_+^m;\ \left|k_{(m)}\right|=n\Big\}\end{aligned}\tag{46}$$

forms an orthogonal basis in the subspace

$$\odot_{\mathfrak{h}}^n\mathbb{C}^m\subset\mathsf{F}_m.\tag{47}$$

We will also use the following notations:

$$\begin{aligned} [m] &:= \{(11),(21,22),\dots,(m1,\dots,mm)\},\\ \{k\} &:= \{k_{(1)},\dots,k_{(m)}\}\in\bigtimes_{r=1}^{m}\mathbb{Z}_+^r,\\ |\{k\}| &:= \left|k_{(1)}\right|+\cdots+\left|k_{(m)}\right|,\\ \{k\}! &:= k_{(1)}!\cdot\ldots\cdot k_{(m)}!. \end{aligned}\tag{48}$$

Then, the system of symmetric tensor elements with a fixed n, indexed by the sets $[m]$ and $\{k\}$,

$$\begin{aligned}\mathscr{E}_n=\bigcup_{m\in\mathbb{N}}\Big\{&\mathbf{e}_{[m]}^{\otimes\{k\}}=\mathbf{e}_{(1)}^{\otimes k_{(1)}}\odot\cdots\odot\mathbf{e}_{(m)}^{\otimes k_{(m)}}:\\ &\mathbf{e}_{(1)}^{\otimes k_{(1)}}\in\mathscr{E}\left(\odot_{\mathfrak{h}}^{|k_{(1)}|}\mathbb{C}\right),\dots,\mathbf{e}_{(m)}^{\otimes k_{(m)}}\in\mathscr{E}\left(\odot_{\mathfrak{h}}^{|k_{(m)}|}\mathbb{C}^m\right)\\ &\text{with fixed } |\{k\}|=n\Big\},\end{aligned}\tag{49}$$

forms an orthogonal basis in the subspace $\odot_{\mathfrak{h}}^n\mathsf{E}\subset\mathsf{F}$. Thus, the system

$$\mathscr{E}=\{\mathscr{E}_n:n\in\mathbb{Z}_+\}\tag{50}$$

forms an orthogonal basis in the symmetric Fock space F.

By virtue of the one-to-one mapping (40), the system of symmetric tensor elements $\mathscr{E}(\odot_{\mathfrak{h}}^n\mathbb{C}^m)$ uniquely defines the following corresponding system:

$$\mathscr{E}_{m,n}^*\subset\mathscr{P}_{\mathfrak{h}}^n(\mathbb{C}^m),\tag{51}$$

of the following χ_m-integrable cylindrical functions:

$$\begin{aligned}\mathbf{e}_{(m)}^{*k_{(m)}}(u)&:=\left\langle\left[\pi_m^{\otimes n}(u)\right]\left(\mathbf{e}_{m1}^{\otimes n}\right)\mid\mathbf{e}_{(m)}^{\otimes k_{(m)}}\right\rangle_{\otimes_{\mathfrak{h}}^n\mathbb{C}^m}\\ &=\prod_{r=1}^{m}\left\langle(\pi_m\circ u)(\mathbf{e}_{m1})\mid\mathbf{e}_{mr}\right\rangle_{\mathbb{C}^m}^{k_{mr}},\end{aligned}\tag{52}$$

of the variable $u\in\mathfrak{U}$, where we take $\mathfrak{a}_m=\mathbf{e}_{m1}$. Consider the system of functions of the variable $u\in\mathfrak{U}$,

$$\begin{aligned}\mathscr{E}_n^*=\bigcup_{m\in\mathbb{N}}\Big\{&\mathbf{e}_{[m]}^{*\{k\}}=\mathbf{e}_{(1)}^{*k_{(1)}}\cdot\ldots\cdot\mathbf{e}_{(m)}^{*k_{(m)}}:\\ &\mathbf{e}_{(1)}^{*k_{(1)}}\in\mathscr{E}_{1,|k_{(1)}|}^*,\dots,\mathbf{e}_{(m)}^{*k_{(m)}}\in\mathscr{E}_{m,|k_{(m)}|}^*\\ &\text{with fixed } |\{k\}|=n\Big\},\end{aligned}\tag{53}$$

generated by the system of symmetric tensor elements $\mathscr{E}_n$. All these functions belong to the space $\mathscr{L}_\chi^\infty$ by their definition. Denote

$$\mathscr{E}^*=\{\mathscr{E}_n^*:n\in\mathbb{Z}_+\},\qquad\mathscr{E}_m^*=\{\mathscr{E}_{m,n}^*:n\in\mathbb{Z}_+\}.\tag{54}$$

6. The Hardy-Type Space

Let L_χ^2 be the space of square χ-integrable complex functions , f on the space of virtual matrices $\mathfrak{U}$. Since χ is a probability measure, the embedding $\mathscr{L}_\chi^\infty\subset L_\chi^2$ holds and

$$\|f\|_{L_\chi^2}\le\operatorname*{ess\,sup}_{u\in\mathfrak{U}}|f(u)|,\quad f\in\mathscr{L}_\chi^\infty.\tag{55}$$

Denote by $\mathscr{H}_{\chi_m}^2$ the L_χ^2-closure of complex linear spans of the subsystem $\mathscr{E}_m^*$. As is well known (see, e.g., [12, Theorem 5.6.8]), the space $\mathscr{H}_{\chi_m}^2$ is isomorphic to the classic Hardy space $\mathscr{H}_{\chi_m}^2(\mathsf{B}^m)$ of analytic complex functions on the open unit ball $\mathsf{B}^m=\{x_m\in\mathbb{C}^m:\|x_m\|_{\mathbb{C}^m}<1\}$. Therefore, the following more general definition seems natural (see, also [8]).

Definition 5. The Hardy-type space $\mathscr{H}_\chi^2$ on the space of virtual unitary matrices $\mathfrak{U}$ is defined to be the L_χ^2-closure of the complex linear span of the system $\mathscr{E}^*$.

Theorem 6. *The system $\mathscr{E}^*$ of all functions $\mathbf{e}_{[m]}^{*\{k\}}=\mathbf{e}_{(1)}^{*k_{(1)}}\cdot\ldots\cdot\mathbf{e}_{(m)}^{*k_{(m)}}$ with $m\in\mathbb{N}$, such that $\mathbf{e}_{(r)}^{*k_{(r)}}\in\mathscr{E}_{r,|k_{(r)}|}^*$ as $r=1,\dots,m$, forms an orthogonal basis in the Hardy-type spaces $\mathscr{H}_\chi^2$ with norms*

$$\left\|\mathbf{e}_{[m]}^{*\{k\}}\right\|_{L_\chi^2}=\left(\prod_{r=1}^{m}\frac{(r-1)!(k)_r!}{(r-1+|(k)_r|)!}\right)^{1/2}.\tag{56}$$

Proof. If $|\{k\}|\ne|\{q\}|$, then from (28), it follows that

$$\begin{aligned}&\int_{\mathfrak{U}}\mathbf{e}_{[m]}^{*\{k\}}\cdot\overline{\mathbf{e}}_{[n]}^{*\{q\}}d\chi\\ &\quad=\int_{\mathfrak{U}}\mathbf{e}_{[m]}^{*\{k\}}(\exp(\mathrm{i}\vartheta)u)\cdot\overline{\mathbf{e}}_{[n]}^{*\{q\}}(\exp(\mathrm{i}\vartheta)u)\,d\chi(u)\\ &\quad=\frac{1}{2\pi}\int_{\mathfrak{U}}\mathbf{e}_{[m]}^{*\{k\}}\overline{\mathbf{e}}_{[n]}^{*\{q\}}d\chi\int_{-\pi}^{\pi}\exp\left(\mathrm{i}\left(|\{k\}|-|\{q\}|\right)\vartheta\right)d\vartheta\\ &\quad=0.\end{aligned}\tag{57}$$

So, $\mathbf{e}_{[m]}^{*\{k\}}\perp\mathbf{e}_{[n]}^{*\{q\}}$ in the space L_χ^2 if $|\{k\}|\ne|\{q\}|$ for all indices $[m],[n]$.

Let $|\{k\}| = |\{q\}|$ and $m > n$ for definiteness. If the elements $\mathbf{e}_{[m]}^{*\{k\}}$ and $\mathbf{e}_{[n]}^{*\{q\}}$ are different, then there exists a subindex $ms \in \{11, 21, 22, \ldots, m1, \ldots, mm\}$ in the block-index $[m] = [(11), (21, 22), \ldots, (m1, \ldots, mm)]$ such that $ms \notin \{11, 21, 22, \ldots, n1, \ldots, nn\}$, where $[n] = [(11), (21, 22), \ldots, (n1, \ldots, nn)]$. The formula (28) implies that for the group of shifts $Q_{g_{ms}(\vartheta)}$ generated by elements $g_{ms}(\vartheta) \in U_s^2(m)$ with the subindex ms,

$$\begin{aligned}\int_{\mathfrak{U}} \mathbf{e}_{[m]}^{*\{k\}} \cdot \overline{\mathbf{e}}_{[n]}^{*\{q\}} d\chi &= \int_{\mathfrak{U}} Q_{g_{ms}(\vartheta)} \mathbf{e}_{[m]}^{*\{k\}} \cdot Q_{g_{ms}(\vartheta)} \overline{\mathbf{e}}_{[n]}^{*\{q\}} d\chi \\ &= \frac{1}{2\pi} \int_{\mathfrak{U}} \mathbf{e}_{[m]}^{*\{k\}} \cdot \overline{\mathbf{e}}_{[n]}^{*\{q\}} d\chi \int_{-\pi}^{\pi} \exp\left(\mathrm{i} k_{ms} \vartheta\right) d\vartheta = 0.\end{aligned} \tag{58}$$

Hence, $\mathbf{e}_{[m]}^{*\{k\}} \perp \mathbf{e}_{[n]}^{*\{q\}}$ in L_χ^2.

Let now $|\{k\}| = |\{q\}|$ and $m = n$. If $\mathbf{e}_{[m]}^{*\{k\}} \neq \mathbf{e}_{[n]}^{*\{q\}}$, then $\{k\} \neq \{q\}$. Hence, there exists a sub-index rs in the block-index $[m] = [n]$ such that $k_{rs} \neq q_{rs}$. Similarly as previous mentioned, applying the formula (28) to the group of shifts $Q_{g_{rs}(\vartheta)}$ generated by elements $g_{rs}(\vartheta) \in U_s^2(r)$ with the sub-index rs, we get

$$\begin{aligned}\int_{\mathfrak{U}} \mathbf{e}_{[m]}^{*\{k\}} \cdot \overline{\mathbf{e}}_{[n]}^{*\{q\}} d\chi &= \frac{1}{2\pi} \int_{\mathfrak{U}} \mathbf{e}_{[m]}^{*\{k\}} \overline{\mathbf{e}}_{[n]}^{*\{q\}} d\chi \int_{-\pi}^{\pi} \exp\left(\mathrm{i}\left(k_{rs} - q_{rs}\right) \vartheta\right) d\vartheta \\ &= 0.\end{aligned} \tag{59}$$

Hence, in this case also $\mathbf{e}_{[m]}^{*\{k\}} \perp \mathbf{e}_{[n]}^{*\{q\}}$ under the measure χ.

Let $g_r = (\mathbb{1}_r, w_r) \in U^2(r)$ and $u \in \mathfrak{U}$. Using (11) and (52), we have

$$\begin{aligned}&\int_{U^2(r)} Q_{g_r} \left|\mathbf{e}_{(r)}^{*(k)_r}\right|^2 (u)\, d\left(\chi_r \otimes \chi_r\right)\left(g_r\right) \\ &\quad = \int_{U(r)} \prod_{l=1}^{r} \left|\left\langle \left[w_r^{-1} \pi_r(u)\right]\left(\mathrm{e}_{r1}\right) \mid \mathrm{e}_{rl}\right\rangle_{\mathbb{C}^r}^{k_{rl}}\right|^2 d\chi_r\left(w_r\right).\end{aligned} \tag{60}$$

However, the previous integral with the Haar measure χ_r is independent of $\pi_r(u) \in U(r)$. It follows that

$$\begin{aligned}&\int_{U^2(r)} Q_{g_r} \left|\mathbf{e}_{(r)}^{*(k)_r}\right|^2 (u)\, d\left(\chi_r \otimes \chi_r\right)\left(g_r\right) \\ &\quad = \int_{U(r)} \prod_{l=1}^{r} \left|\left\langle w_r^{-1}\left(\mathrm{e}_{r1}\right) \mid \mathrm{e}_{rl}\right\rangle_{\mathbb{C}^r}^{k_{rl}}\right|^2 d\chi_r\left(w_r\right) \\ &\quad = \frac{(r-1)!(k)_r!}{\left(r-1+\left|(k)_r\right|\right)!} = \left\|\mathbf{e}_{(r)}^{*(k)_r}\right\|_{L_{\chi_r}^2}^2\end{aligned} \tag{61}$$

by the well-known formula [12, Section 1.4.9]. Using the formula (27) m-times for $r = 1, \ldots, m$, we get

$$\begin{aligned}&\int_{\mathfrak{U}} \left|\mathbf{e}_{[m]}^{*\{k\}}\right|^2 d\chi \\ &\quad = \int_{\mathfrak{U}} d\chi(u) \prod_{r=1}^{m} \int_{U^2(r)} Q_{g_r} \left|\mathbf{e}_{(r)}^{*(k)_r}\right|^2 (u)\, d\left(\chi_r \otimes \chi_r\right)\left(g_r\right) \\ &\quad = \prod_{r=1}^{m} \left\|\mathbf{e}_{(r)}^{*k_{(r)}}\right\|_{L_{\chi_r}^2}^2,\end{aligned} \tag{62}$$

because $\int_{\mathfrak{U}} d\chi = 1$. It follows that

$$\left\|\mathbf{e}_{[m]}^{*\{k\}}\right\|_{L_\chi^2}^2 = \prod_{r=1}^{m} \left\|\mathbf{e}_{(r)}^{*k_{(r)}}\right\|_{L_{\chi_r}^2}^2 = \prod_{r=1}^{m} \frac{(r-1)!(k)_r!}{\left(r-1+\left|(k)_r\right|\right)!}, \tag{63}$$

for all $\mathbf{e}_{[m]}^{*\{k\}} = \mathbf{e}_{(1)}^{*k_{(m)}} \cdot \cdots \cdot \mathbf{e}_{(m)}^{*k_{(m)}}$. □

As is known (see, e.g., [11]), the system $\mathscr{E}_m$ of symmetric tensors $\mathbf{e}_{(m)}^{\otimes(k)_m}$ with a fixed m forms an orthogonal basis in the symmetric Fock space F_m with norms $\|\mathbf{e}_{(m)}^{\otimes(k)_m}\|_{\mathsf{F}_m} = \sqrt{(k)_m!/|(k)_m|!}$. Similarly, the system $\mathscr{E}$ of symmetric tensors $\mathbf{e}_{[m]}^{\otimes\{k\}} = \mathbf{e}_{(1)}^{\otimes(k)_1} \odot \cdots \odot \mathbf{e}_{(m)}^{\otimes k_{(m)}}$ with all $m \in \mathbb{N}$, such that $\mathbf{e}_{(r)}^{\otimes(k)_r} \in \mathscr{E}_{r,|(k)_r|}$ as $r = 1, \ldots, m$, forms an orthogonal basis in the symmetric Fock space F with norms $\|\mathbf{e}_{[m]}^{\otimes\{k\}}\|_{\mathsf{F}} = \sqrt{\{k\}!/|\{k\}|!}$.

Combining Lemma 4, Theorem 6, and [12, Theorem 5.6.8], we obtain the following.

Theorem 7. *Antilinear extensions of the one-to-one mappings between the orthonormal bases*

$$\begin{aligned}\frac{\mathbf{e}_{(m)}^{\otimes(k)_m}}{\left\|\mathbf{e}_{(m)}^{\otimes(k)_m}\right\|_{\mathsf{F}_m}} &\rightleftarrows \frac{\mathbf{e}_{(m)}^{*(k)_m}}{\left\|\mathbf{e}_{(m)}^{*(k)_m}\right\|_{L_{\chi_m}^2}}, \\ \frac{\mathbf{e}_{[m]}^{\otimes\{k\}}}{\left\|\mathbf{e}_{[m]}^{\otimes\{k\}}\right\|_{\mathsf{F}}} &\rightleftarrows \frac{\mathbf{e}_{[m]}^{*\{k\}}}{\left\|\mathbf{e}_{[m]}^{*\{k\}}\right\|_{L_\chi^2}},\end{aligned} \tag{64}$$

uniquely define the corresponding anti-linear isometric isomorphisms

$$\mathsf{F}_m \simeq \mathscr{H}_{\chi_m}^2\left(\mathsf{B}^m\right), \qquad \mathsf{F} \simeq \mathscr{H}_\chi^2. \tag{65}$$

Reasoning by analogy with [8, Proposition 6.1 and Theorem 7.1], it is easy to show that the Hardy space $\mathscr{H}_\chi^2$ possesses the reproducing kernel of a Cauchy type

$$\begin{aligned}\mathfrak{C}(v, u) &= \sum_{n \in \mathbb{Z}_+} \sum_{|\{k\}|=n} \frac{\mathbf{e}_{[m]}^{*\{k\}}(v)\, \overline{\mathbf{e}}_{[m]}^{*\{k\}}(u)}{\left\|\mathbf{e}_{[m]}^{*\{k\}}\right\|_{L_\chi^2}^2} \\ &= \prod_{m=1}^{\infty} \left(1 - \left\langle \left(\pi_m \circ v\right)\left(\mathrm{e}_{m1}\right) \mid \left(\pi_m \circ u\right)\left(\mathrm{e}_{m1}\right)\right\rangle_{\mathsf{E}}\right)^{-m},\end{aligned} \tag{66}$$

with $u, v \in \mathfrak{U}$, where the sum $\sum_{|\{k\}|=n}$ is over all indices $\{k\} \in \{\bigtimes_{r=1}^{m} \mathbb{Z}_{+}^{r} : m \in \mathbb{N}\}$ such that $|\{k\}| = n$. As a consequence, the integral representation of any function $f \in \mathscr{H}_{\chi}^{2}$,

$$f(\lambda v) = \int_{\mathfrak{U}} f(u)\, \mathfrak{C}(\lambda v, u)\, d\chi(u) \tag{67}$$

gives a unique analytic extension in the complex variable $\lambda \in \mathsf{B}^1$ for all elements $v \in \mathfrak{U}$ such that

$$\sum_{m \in \mathbb{N}} m \left\| (\pi_m \circ v)(\mathfrak{e}_{m1}) \right\|_{\mathbb{C}^m}^{2} < \infty. \tag{68}$$

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