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Research Article

Fixed Point and Common Fixed Point Theorems on Ordered Cone b-Metric Spaces

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The concept of a cone b-metric space has been introduced recently as a generalization of a b-metric space and a cone metric space in 2011. The aim of this paper is to establish some fixed point and common fixed point theorems on ordered cone b-metric spaces. The proposed theorems expand and generalize several well-known comparable results in the literature to ordered cone b-metric spaces. Some supporting examples are given.

1. Introduction

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem. This theorem supplies a method for solving a variety of applied dilemma in mathematical sciences and engineering. A large literature on this subject exists, and this is a very active area of research at present. Banach contraction principle has been generalized in dissimilar directions in different spaces by mathematicians over the years; for more details on this and related topics, we refer to [1–6] and references therein.

In contemporary time, fixed point theory has evolved speedily in partially ordered cone metric spaces; that is, cone metric spaces equipped with a partial ordering, for some new results in ordered metric spaces see [7]. A coming early result in this bearing was constituted by Altun and Durmaz [8] under the condition of normality for cones. Then, Altun et al. [9] generalized the results of Altun and Durmaz [8] by omitting the assumption of normality condition for cones. Afterward, several authors have studied fixed point and common fixed point problems in ordered cone metric spaces; for more details see [10–17].

In 2011, Hussain and Shah [18] presented cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces; for some new results in b-metric spaces see [19]. They not only constructed some topological properties in such spaces but also ameliorated some current results about KKM mappings in the setting of a cone b-metric space. After some time, many authors have been motivated to demonstrate fixed point theorems as well as common fixed point theorems for two or more mappings on cone b-metric spaces by the incipient work of Hussain and Shah [18] (see [20–23] and the references therein).

In [8], Altun and Durmaz proved the following results under the condition of normality for cones.

Theorem 1 (see [8]). *Let $(X, \sqsubseteq)$ be a partially ordered set, suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete, and let P be a normal cone with normal constant K. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exists $k \in [0, 1)$ such that $d(fx, fy) \preceq kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point in X.

In [9], Altun et al. generalized the above theorem and proved it without normality condition for cones.

Theorem 2 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that*

$$\begin{aligned} d(fx, fy) \preceq\, & kd(x, y) + l(d(fx, x) + d(fy, y)) \\ & + r(d(fx, y) + d(fy, x)) \end{aligned} \tag{1}$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point in X.

Theorem 3 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P. Let $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that*

$$\begin{aligned} d(fx, fy) \preceq\, & kd(x, y) + l(d(fx, x) + d(fy, y)) \\ & + r(d(fx, y) + d(fy, x)) \end{aligned} \tag{2}$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*

(iii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f has a fixed point in X.

In the same paper, they also presented the following two common fixed point results in ordered cone metric spaces.

Theorem 4 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that*

$$\begin{aligned} d(fx, gy) \preceq\, & kd(x, y) + l(d(x, fx) + d(y, gy)) \\ & + r(d(y, fx) + d(x, gy)) \end{aligned} \tag{3}$$

for all comparative $x, y \in X$;

(ii) *f or g is continuous.*

Then f and g have a common fixed point $x^* \in X$.

Theorem 5 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete over a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist $k, l, r \in [0, 1)$ with $k + 2l + 2r < 1$ such that*

$$\begin{aligned} d(fx, gy) \preceq\, & kd(x, y) + l(d(x, fx) + d(y, gy)) \\ & + r(d(y, fx) + d(x, gy)) \end{aligned} \tag{4}$$

for all comparative $x, y \in X$;

(ii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f and g have a common fixed point $x^* \in X$.

In this paper, we prove some fixed point and common fixed point theorems on ordered cone b-metric spaces. Our results extend and generalize several well-known comparable results in the literature to ordered cone b-metric spaces. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is, $\operatorname{int} P \neq \emptyset$.

The following definitions and results shall be needed in the sequel.

Let E be a real Banach space and θ denotes the zero element in E. A cone P is a subset of E such that

(1) P is nonempty closed set and $P \neq \{\theta\}$;

(2) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;

(3) $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called normal if there exists the number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K \|y\|, \tag{5}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

Definition 6 (see [18]). Let X be a nonempty set and E a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. A vector-valued function $d : X \times X \to E$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then pair (X, d) is called a cone b-metric space (or a cone metric type space); we shall use the first mentioned term.

Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied; however, it does not hold true when $s > 1$. Thus the class of cone b-metric spaces is

effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. The following examples show the above remarks.

Example 7. Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \to P$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = \theta$, $x \in X$, and $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, d) is a complete cone b-metric space but the triangle inequality is not satisfied. Indeed, we have that $d(-1, 1) + d(1, 0) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = d(-1, 0)$. It is not hard to verify that $s = 3/2$.

Example 8. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, and $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$. Define $d : X \times X \to E$ by $d(x, y) = (|x - y|^2, |x - y|^2)$. Then, it is easy to see that (X, d) is a cone b-metric space with the coefficient $s = 2$. But it is not a cone metric spaces since the triangle inequality is not satisfied.

Definition 9 (see [18]). Let (X, d) be a cone b-metric space, $\{x_n\}$ a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. One denotes this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

The following lemma is useful in our work.

Lemma 10 (see [24]).

(1) *If E is a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) *If $c \in \operatorname{int} P$, $\theta \preceq a_n$, and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

(3) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

(4) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

2. Fixed Point Results

In this section, we prove some fixed point theorems on ordered cone b-metric space. We begin with a simple but a useful lemma.

Lemma 11. *Let $\{x_n\}$ be a sequence in a cone b-metric space (X, d) with the coefficient $s \geq 1$ relative to a solid cone P such that*

$$d(x_n, x_{n+1}) \preceq h d(x_{n-1}, x_n), \tag{6}$$

where $h \in [0, 1/s)$ and $n = 1, 2, \ldots$. Then $\{x_n\}$ is a Cauchy sequence in (X, d).

Proof. Let $m > n \geq 1$. It follows that

$$\begin{aligned} d(x_n, x_m) \preceq\ & s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) \\ & + \cdots + s^{m-n} d(x_{m-1}, x_m). \end{aligned} \tag{7}$$

Now, (6) and $sh < 1$ imply that

$$\begin{aligned} d(x_n, x_m) &\preceq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + s^{m-n} d(x_{m-1}, x_m) \\ &\preceq s h^n d(x_0, x_1) + s^2 h^{n+1} d(x_0, x_1) \\ &\quad + \cdots + s^{m-n} h^{m-1} d(x_0, x_1) \\ &= \left(s h^n + s^2 h^{n+1} + \cdots + s^{m-n} h^{m-1}\right) d(x_0, x_1) \\ &= s h^n \left(1 + sh + (sh)^2 + \cdots + (sh)^{m-n-1}\right) d(x_0, x_1) \\ &\preceq \frac{s h^n}{1 - sh} d(x_0, x_1) \longrightarrow \theta \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{8}$$

According to Lemma 10(2), and for any $c \in E$ with $c \gg \theta$, there exists $N_0 \in \mathbb{N}$ such that for any $n > N_0$, $(sh^n/(1 - sh)) d(x_0, x_1) \ll c$. Furthermore, from (8) and for any $m > n > N_0$, Lemma 10(3) shows that

$$d(x_n, x_m) \ll c. \tag{9}$$

Hence, by Definition 9(2) $\{x_n\}$ is a Cauchy sequence in X. □

Theorem 12. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist a_i, $i = 1, \ldots, 5$, such that $2sa_1 + (s + 1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2$ with $\sum_{i=1}^{5} a_i < 1$,*

$$\begin{aligned} d(fx, fy) \preceq\ & a_1 d(x, y) + a_2 d(fx, x) + a_3 d(fy, y) \\ & + a_4 d(fx, y) + a_5 d(fy, x) \end{aligned} \tag{10}$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$.

Proof. If $x_0 = fx_0$, then the proof is finished. Suppose that $x_0 \neq fx_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing with respect

to $\sqsubseteq$, we obtain by induction that $x_0 \sqsubseteq fx_0 = x_1 \sqsubseteq f^1 x_0 = x_2 \sqsubseteq \cdots \sqsubseteq f^{n-1}x_0 = x_n \sqsubseteq f^n x_0 = x_{n+1} \sqsubseteq \cdots$. Then we have,

$$\begin{aligned}
d(x_{n+1}, x_n) &= d\left(f^n x_0, f^{n-1}x_0\right) \\
&= d\left(f\left(f^{n-1}x_0\right), f\left(f^{n-2}x_0\right)\right) \\
&\preceq a_1 d\left(f^{n-1}x_0, f^{n-2}x_0\right) + a_2 d\left(f^{n-1}x_0, f^{n-2}x_0\right) \\
&\quad + a_3 d\left(f^n x_0, f^{n-1}x_0\right) \\
&\quad + a_4 d\left(f^n x_0, f^{n-2}x_0\right) + a_5 d\left(f^{n-1}x_0, f^{n-1}x_0\right) \\
&= a_1 d(x_n, x_{n-1}) + a_2 d(x_{n+1}, x_n) + a_3 d(x_n, x_{n-1}) \\
&\quad + a_4 d(x_{n+1}, x_{n-1}) + a_5 d(x_n, x_n) \\
&\preceq a_1 d(x_n, x_{n-1}) + a_2 d(x_{n+1}, x_n) + a_3 d(x_n, x_{n-1}) \\
&\quad + s a_4 \left(d(x_{n+1}, x_n) + d(x_n, x_{n-1})\right).
\end{aligned} \tag{11}$$

Then, one can assert that

$$\begin{aligned}
d(x_{n+1}, x_n) \preceq{} & (a_1 + a_3 + s a_4) d(x_n, x_{n-1}) \\
& + (a_2 + s a_4) d(x_{n+1}, x_n).
\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d\left(f^{n-1}x_0, f^n x_0\right) \\
&= d\left(f\left(f^{n-2}x_0\right), f\left(f^{n-1}x_0\right)\right) \\
&\preceq a_1 d\left(f^{n-2}x_0, f^{n-1}x_0\right) + a_2 d\left(f^{n-2}x_0, f^{n-1}x_0\right) \\
&\quad + a_3 d\left(f^{n-1}x_0, f^n x_0\right) \\
&\quad + a_4 d\left(f^{n-1}x_0, f^{n-1}x_0\right) + a_5 d\left(f^n x_0, f^{n-2}x_0\right) \\
&= a_1 d(x_n, x_{n-1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n+1}, x_n) \\
&\quad + a_4 d(x_n, x_n) + a_5 d(x_{n+1}, x_{n-1}) \\
&\preceq a_1 d(x_n, x_{n-1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n+1}, x_n) \\
&\quad + s a_5 \left(d(x_{n+1}, x_n) + d(x_n, x_{n-1})\right).
\end{aligned} \tag{13}$$

Then, one can assert that

$$\begin{aligned}
d(x_{n+1}, x_n) \preceq{} & (a_1 + a_2 + s a_5) d(x_n, x_{n-1}) \\
& + (a_3 + s a_5) d(x_{n+1}, x_n).
\end{aligned} \tag{14}$$

Adding (12) and (14), we get

$$\begin{aligned}
d(x_{n+1}, x_n) &\preceq \frac{2a_1 + a_2 + a_3 + s a_4 + s a_5}{2 - (a_2 + a_3 + s a_4 + s a_5)} d(x_n, x_{n-1}) \\
&= \lambda d(x_n, x_{n-1}),
\end{aligned} \tag{15}$$

where $\lambda = (2a_1 + a_2 + a_3 + s a_4 + s a_5)/(2 - (a_2 + a_3 + s a_4 + s a_5)) < 1/s$. According to Lemma 11, we have $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since f is continuous, then $x^* = \lim x_{n+1} = \lim f^n x_0 = \lim f(f^{n-1}x_0) = f(\lim f^{n-1}x_0) = f(\lim x_n) = f(x^*)$. Therefore, x^* is a fixed point of f. □

If we use condition (iii) instead of the continuity of f in Theorem 12, we have the following result.

Theorem 13. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P. Let $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist a_i, $i = 1, \ldots, 5$, such that $2sa_1 + (s+1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2$ with $\sum_{i=1}^{5} a_i < 1$,*

$$\begin{aligned}
d(fx, fy) \preceq{} & a_1 d(x, y) + a_2 d(fx, x) + a_3 d(fy, y) \\
& + a_4 d(fx, y) + a_5 d(fy, x)
\end{aligned} \tag{16}$$

for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*

(iii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f has a fixed point $x^* \in X$.

Proof. As in the Theorem 12, we can construct an increasing sequence $\{x_n\}$ and prove that there exists $x^* \in X$ such that $x_n \to x^*$. Now, condition (iii) implies $x_n \sqsubseteq x^*$ for all n. Therefore, we can use condition (i) and so

$$\begin{aligned}
d(fx_n, fx^*) \preceq{} & a_1 d(x_n, x^*) + a_2 d(fx_n, x_n) + a_3 d(fx^*, x^*) \\
& + a_4 d(fx_n, x^*) + a_5 d(fx^*, x_n).
\end{aligned} \tag{17}$$

Taking $n \to \infty$, we have $d(x^*, fx^*) \preceq (a_3 + a_5) d(x^*, fx^*)$ $d(x^*, fx^*)$. Since $(a_3 + a_5) < 1$, Lemma 10(1) shows that $d(x^*, fx^*) = \theta$; that is, $x^* = fx^*$. Therefore x^* is a fixed point of f. □

3. Common Fixed Point Results

Now, we give two common fixed point theorems on ordered cone b-metric spaces. We need the following definition.

Definition 14 (see [9]). Let $(X, \sqsubseteq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ hold for all $x \in X$.

Theorem 15. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist* a_i, $i = 1, \ldots, 5$, *such that* $2sa_1 + (s+1)(a_2 + a_3) + (s^2+s)(a_4+a_5) < 2$ *with* $\sum_{i=1}^{5} a_i < 1$,

$$\begin{aligned} d(fx, gy) \preceq{} & a_1 d(x,y) + a_2 d(x, fx) + a_3 d(y, gy) \\ & + a_4 d(y, fx) + a_5 d(x, gy) \end{aligned} \tag{18}$$

for all comparative $x, y \in X$;

(ii) f *or* g *is continuous.*

Then f *and* g *have a common fixed point* $x^* \in X$.

Proof. Let x_0 be an arbitrary point of X and define a sequence $\{x_n\}$ in X as follows: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n > 0$. Note that, since f and g are weakly increasing, we have $x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2$ and $x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3$, and continuing this process we have $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$. That is, the sequence $\{x_n\}$ is nondecreasing. Now, since x_{2n} and x_{2n+1} are comparative, we can use the inequality (18), and then we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\preceq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, fx_{2n}) \\ &\quad + a_3 d(x_{2n+1}, gx_{2n+1}) \\ &\quad + a_4 d(x_{2n+1}, fx_{2n}) + a_5 d(x_{2n}, gx_{2n+1}) \\ &\preceq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) \\ &\quad + a_3 d(x_{2n+1}, x_{2n+2}) \\ &\quad + a_4 d(x_{2n+1}, x_{2n+1}) + a_5 d(x_{2n}, x_{2n+2}) \\ &\preceq (a_1 + a_2) d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) \\ &\quad + sa_5 (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \\ &= (a_1 + a_2 + sa_5) d(x_{2n}, x_{2n+1}) \\ &\quad + (a_3 + sa_5) d(x_{2n+1}, x_{2n+2}). \end{aligned} \tag{19}$$

Hence,

$$\begin{aligned} & (1 - (a_3 + sa_5)) d(x_{2n+1}, x_{2n+2}) \\ & \quad \preceq (a_1 + a_2 + sa_5) d(x_{2n}, x_{2n+1}). \end{aligned} \tag{20}$$

On the other hand and by symmetry we have

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(gx_{2n+1}, fx_{2n}) \\ &\preceq a_1 d(x_{2n+1}, x_{2n}) + a_2 d(x_{2n+1}, gx_{2n+1}) \\ &\quad + a_3 d(x_{2n}, fx_{2n}) \\ &\quad + a_4 d(x_{2n}, gx_{2n+1}) + a_5 d(x_{2n+1}, fx_{2n}) \\ &\preceq a_1 d(x_{2n+1}, x_{2n}) + a_2 d(x_{2n+1}, x_{2n+2}) \\ &\quad + a_3 d(x_{2n}, x_{2n+1}) \\ &\quad + a_4 d(x_{2n}, x_{2n+2}) + a_5 d(x_{2n+1}, x_{2n+1}) \\ &\preceq (a_1 + a_3) d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n+1}, x_{2n+2}) \\ &\quad + sa_4 (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \\ &= (a_1 + a_3 + sa_4) d(x_{2n}, x_{2n+1}) \\ &\quad + (a_2 + sa_4) d(x_{2n+1}, x_{2n+2}). \end{aligned} \tag{21}$$

Hence,

$$\begin{aligned} & (1 - (a_2 + sa_4)) d(x_{2n+2}, x_{2n+1}) \\ & \quad \preceq (a_1 + a_3 + sa_4) d(x_{2n}, x_{2n+1}). \end{aligned} \tag{22}$$

Adding inequalities (20) and (22), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preceq \frac{(2a_1 + a_2 + a_3 + sa_4 + sa_5)}{2 - (a_2 + a_3 + sa_4 + sa_5)} d(x_{2n}, x_{2n+1}) \\ &= \lambda d(x_{2n}, x_{2n+1}), \end{aligned} \tag{23}$$

where $\lambda = (2a_1 + a_2 + a_3 + sa_4 + sa_5)/(2 - (a_2 + a_3 + sa_4 + sa_5)) < 1/s$. Similarly, it can be shown that

$$d(x_{2n+3}, x_{2n+2}) \preceq \lambda d(x_{2n+}, x_{2n+2}). \tag{24}$$

Therefore,

$$d(x_{n+1}, x_{n+2}) \preceq \lambda d(x_n, x_{n+1}). \tag{25}$$

According to Lemma 11, we have $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Suppose that f is continuous. Then, $x^* = \lim x_{n+1} = \lim f^n x_0 = \lim f(f^{n-1} x_0) = f(\lim f^{n-1} x_0) = f(\lim x_n) = f(x^*)$. Therefore, x^* is a fixed point of f. Now, we need to show that x^* is a fixed point of g. Since $x^* \sqsubseteq x^*$, we can use the inequality (18) for $x = y = x^*$. Then we have

$$\begin{aligned} d(fx^*, gx^*) &\preceq a_1 d(x^*, x^*) + a_2 d(x^*, fx^*) + a_3 d(x^*, gx^*) \\ &\quad + a_4 d(x^*, fx^*) + a_5 d(x^*, gx^*) \\ &= a_1 d(x^*, x^*) + a_2 d(x^*, x^*) + a_3 d(x^*, gx^*) \\ &\quad + a_4 d(x^*, x^*) + a_5 d(x^*, gx^*) \\ &= a_3 d(x^*, gx^*) + a_5 d(x^*, gx^*) \\ &= (a_3 + a_5) d(x^*, gx^*). \end{aligned} \tag{26}$$

Hence,

$$d(x^*, gx^*) \preceq (a_3 + a_5) d(x^*, gx^*). \tag{27}$$

Since $(a_3 + a_5) < 1$, Lemma 10(1) shows that $d(x^*, gx^*) = \theta$; that is, $x^* = gx^*$. Therefore x^* is a fixed point of g. Therefore, f and g have a common fixed point. The proof is similar when g is a continuous mapping. □

Theorem 16. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone b-metric d in X such that the cone b-metric space (X, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist a_i, $i = 1, \ldots, 5$ such that $2sa_1 + (s+1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2$ with $\sum_{i=1}^{5} a_i < 1$,*

$$d(fx, gy) \preceq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, gy) + a_4 d(y, fx) + a_5 d(x, gy), \tag{28}$$

for all comparative $x, y \in X$;

(ii) *if an increasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all n.*

Then f and g have a common fixed point $x^* \in X$.

Proof. As in Theorem 15, we can construct an increasing sequence $\{x_n\}$ and prove that there exists $x^* \in X$ such that $x_n \to x^*$, also; by the construction of x_n, $gx_n \to x^*$. Now, condition (iii) implies $x_n \sqsubseteq x^*$ for all n. Putting $x = x^*$ and $y = x_n$ in (28), we get

$$\begin{aligned} d(fx^*, gx_n) &\preceq a_1 d(x^*, x_n) + a_2 d(x^*, fx^*) \\ &\quad + a_3 d(x_n, gx_n) \\ &\quad + a_4 d(x_n, fx^*) + a_5 d(x^*, gx_n) \\ &= a_1 d(x_n, x^*) + a_2 d(fx^*, x^*) \\ &\quad + a_3 d(x_n, gx_n) \\ &\quad + a_4 d(x_n, fx^*) + a_5 d(gx_n, x^*) \\ &\preceq a_1 d(x_n, x^*) \\ &\quad + a_2 (d(fx^*, gx_n) + d(gx_n, x^*)) \\ &\quad + a_3 (d(x_n, x^*) + d(x^*, gx_n)) \\ &\quad + a_4 (d(x_n, x^*) + d(x^*, gx_n) \\ &\qquad + d(gx_n, fx^*)) \\ &\quad + a_5 d(gx_n, x^*) \\ &= (a_1 + a_3 + a_4) d(x_n, x^*) \\ &\quad + (a_2 + a_3 + a_4 + a_5) d(gx_n, x^*) \\ &\quad + (a_2 + a_4) d(fx^*, gx_n). \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} d(fx^*, gx_n) &\preceq \frac{a_1 + a_3 + a_4}{1 - (a_2 + a_4)} d(x_n, x^*) \\ &\quad + \frac{a_2 + a_3 + a_4 + a_5}{1 - (a_2 + a_4)} d(gx_n, x^*). \end{aligned} \tag{30}$$

Since $x_n \to x^*$ and $gx_n \to x^*$, then by Definition 9(1) and for $c \gg \theta$ there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $d(x_n, x^*) \ll c(1 - (a_2 + a_4))/2(a_1 + a_3 + a_4)$, and $d(gx_n, x^*) \ll c(1 - (a_2 + a_4))/2(a_2 + a_3 + a_4 + a_5)$. Then we have

$$\begin{aligned} d(gx_n, fx^*) &= d(fx^*, gx_n) \\ &\preceq \frac{a_1 + a_3 + a_4}{1 - (a_2 + a_4)} d(x_n, x^*) \\ &\quad + \frac{a_2 + a_3 + a_4 + a_5}{1 - (a_2 + a_4)} d(gx_n, x^*) \\ &\ll \frac{a_1 + a_3 + a_4}{1 - (a_2 + a_4)} \frac{c(1 - (a_2 + a_4))}{2(a_1 + a_3 + a_4)} \\ &\quad + \frac{a_2 + a_3 + a_4 + a_5}{1 - (a_2 + a_4)} \frac{c(1 - (a_2 + a_4))}{2(a_2 + a_3 + a_4 + a_5)} \\ &= \frac{c}{2} + \frac{c}{2} \\ &= c. \end{aligned} \tag{31}$$

Now again, according to Definition 9(1) it follows that $gx_n \to fx^*$. It follows that $fx^* = x^*$. In a similar way and using that $x^* \sqsubseteq x^*$, we can prove that $gx^* = x^*$. Therefore, f and g have a common fixed point. □

Now, we present two examples to illustrate our results. In the first example (the case of a normal cone), the conditions of Theorem 12 are fulfilled, but Theorem 2 of Altun et al. [9, Theorem 12] cannot be applied. In the second example (the case of a nonnormal cone), the conditions of Theorem 12 are fulfilled, but Theorem 3 of Altun et al. [9, Theorem 13] cannot be applied.

Example 17. Let $X = [0, 1]$ endowed with the standard order and $E = \mathbb{R}^2$ and let $P = \{(x, y) : x, y \geq 0\}$. Define $d : X \times X \to E$ as in Example 8. Define $f : X \to X$ by $f(x) = x^2/6$. Then f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Then we have

$$\begin{aligned} d(fx, fy) &= d\left(\frac{x^2}{6}, \frac{y^2}{6}\right) \\ &= \left(\left|\frac{x^2}{6} - \frac{y^2}{6}\right|^2, \left|\frac{x^2}{6} - \frac{y^2}{6}\right|^2\right) \\ &= \frac{1}{36}|x + y|^2 \left(|x - y|^2, |x - y|^2\right) \\ &\preceq \frac{4}{36}\left(|x - y|^2, |x - y|^2\right) \\ &\preceq \frac{4}{36} d(x, y), \end{aligned} \tag{32}$$

where $a_1 = 4/36$, $a_2 = a_3 = a_4 = a_5 = 0$. It is clear that the conditions of Theorem 12 are satisfied. Therefore, f has a fixed point $x = 0$.

Example 18. Let $X = [0, \infty)$ endowed with the standard order and $E = C^1_{\mathbb{R}}[0,1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 \text{ on } [0,1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone metric $d : X \times X \to E$ by $d(x,y)(t) = |x-y|^2 e^t$. Then (X,d) is a complete cone b-metric space with the coefficient $s = 2$. Let us define $f : X \to X$ by $f(x) = x/2$. Then f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Then we have f is an increasing mapping; also we have

$$\begin{aligned} d(fx, fy)(t) &= \left|\frac{1}{2}x - \frac{1}{2}y\right|^2 e^t \\ &= \frac{1}{4}|x-y|^2 e^t \\ &\preceq \frac{1}{4}|x-y|^2 e^t + \frac{1}{5}\left|\frac{x}{2}\right|^2 e^t \\ &\preceq \frac{1}{4} d(x,y)(t) + \frac{1}{5} d(fx, x)(t), \end{aligned} \tag{33}$$

where $a_1 = 1/4$, $a_2 = 1/5$, $a_3 = a_4 = a_5 = 0$. It is clear that the conditions of Theorem 12 are satisfied. Therefore, f has a fixed point $x = 0$.

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