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Research Article

Optimal Simultaneous Approximation via $\mathscr{A}$-Summability

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We present optimal convergence results for the mth derivative of a function by sequences of linear operators. The usual convergence is replaced by $\mathscr{A}$-summability, with $\mathscr{A}$ being a sequence of infinite matrices with nonnegative real entries, and the operators are assumed to be m-convex. Saturation results for nonconvergent but almost convergent sequences of operators are stated as corollaries.

1. Introduction

The notion of almost convergence of a sequence introduced by Lorentz [1] in 1948 entered the Korovkin-type approximation theory (see [2]) through the papers of King and Swetits [3] and Mohapatra [4]. A step forward was given by Swetits [5] in 1979 who applied in the theory the more general notion of $\mathscr{A}$-summability that Bell [6] had introduced a few years earlier.

After Swetits, within a shape preserving approximation setting and using as well $\mathscr{A}$-summability, one finds in the literature two recent papers of the authors, [7, 8], where they studied, on one hand, qualitative and quantitative Korovkin-type results, and on the other, results on asymptotic formulae. In this paper we continue this line of work which naturally takes us to the topic of saturation. Indeed, after having established an asymptotic formula, a natural way to keep on is to study optimal results to control the goodness of the approximation errors. Here saturation enters the picture. Now, before detailing our specific aim, we present the general framework of the paper which includes the definition of $\mathscr{A}$-summability.

Let $\mathscr{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with nonnegative real entries; then a sequence of real numbers $\{x_j\}$ is said to be $\mathscr{A}$-summable to ℓ if (whenever the series below converges for all k and n)

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = \ell \quad \text{uniformly for } n \in \mathbb{N} = \{1, 2, \ldots\}. \tag{1}$$

Notice that $\mathscr{A}$-summability extends classical convergence, matrix summability, the Cesaro summability, and almost convergence amongst others.

Now, let $m \in \mathbb{N} \cup \{0\}$, let $C^m[a,b]$ denote the space of all m-times continuously differentiable functions on the real interval $[a,b]$, let D^m denote the usual mth differential operator, and finally, let $\mathscr{L} = \{L_j : C^m[a,b] \to C^m[a,b]\}$ be a sequence of linear operators fulfilling the following properties:

(P0) for each $f \in C^m[a,b]$ and $x \in [a,b]$, $D^m L_j f(x)$ is $\mathscr{A}$-summable to $D^m f(x)$, or equivalently

$$\mathscr{A}_{D^m\circ\mathscr{L}}^{k,n} f(x) := \sum_{j=1}^{\infty} a_{kj}^{(n)} D^m L_j f(x) \tag{2}$$

converges to $D^m f$ as k tends to infinity, uniformly in n, (P1) each L_j is m-convex; that is, it maps m-convex functions onto m-convex functions; recall that a function $f \in C^m[a,b]$ is said to be m-convex whenever $D^m f(t) \geq 0$ for all $t \in [a,b]$,

(P2) there exist a sequence of real positive numbers $\lambda_k \to +\infty$ and three strictly positive functions w_0, w_1, and w_2 defined on (a,b) with $w_i \in C^{2-i}(a,b)$ such that for $f \in C^m[a,b]$, $m+2$-times differentiable in some neighborhood of a point $x \in (a,b)$,

$$\lim_{k\to+\infty} \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x)\right) = \frac{1}{w_2} D^1 \left(\frac{1}{w_1} D^1 \left(\frac{1}{w_0} D^m f\right)\right)(x) \quad (3)$$

uniformly in n.

The asymptotic formula (3) informs us that the order of convergence of $\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x)$ towards $D^m f(x)$ is not better than λ_k^{-1} if the right-hand side of (3) is different from 0. Thus, λ_k^{-1} is called the optimal order of convergence, and those functions that possess it form the saturation class. As for our specific aim with this paper, the results of Section 2 give us information about this saturation class, while Section 3 is devoted to state a sort of converse result of asymptotic formulae. We follow the line of two respective papers of two of the authors, namely [9, 10], which at the same time have their foundations on two outstanding papers of Lorentz and Schumaker [11] and Berens [12]. The last section of the paper contains some applications. Now we close this one with some remarks and notation that we will use throughout the paper.

Firstly we point out that if (P1) fulfills and $D^m f \leq D^m g$ on $[a,b]$, then for all $t \in [a,b]$, $\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(t) \leq \mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} g(t)$.

Secondly, if we consider a bounded subinterval $J \subset [a,b]$ and fix a point $c \in J$, it is well known that the functions $u_0(t) = w_0(t)$, $u_1(t) = w_0(t) \int_c^t w_1(s)ds$, and $u_2(t) = w_0(t) \int_c^t w_1(t_1) \int_c^{t_1} w_2(t_2) dt_2 dt_1$ form in J an extended complete Tchebychev system $T = \{u_0, u_1, u_2\}$ (see [13]). Moreover $\{u_0, u_1\}$ is a fundamental system of solutions of the second-order differential equation in the unknown v (see the right-hand side of (3)) that follows:

$$\mathscr{D}v := w_2^{-1} D^1 \left(w_1^{-1} D^1 \left(w_0^{-1} v\right)\right) \equiv 0. \quad (4)$$

Besides $\mathscr{D}u_2 \equiv 1$.

In this respect, we refer the reader to [11] to recall the class $\mathrm{Lip}_M^T 1$, $M \geq 0$, formed by those functions f, differentiable on (a,b), fulfilling

$$\left|\Delta_T f(t_2) - \Delta_T f(t_1)\right| \leq M \int_{t_1}^{t_2} w_2(s)\, ds, \quad (5)$$

where $\Delta_T f = (1/w_1) D^1((1/w_0) f)$. Notice that if $w_2 \equiv 1$, then $f \in \mathrm{Lip}_M^T 1$ amounts to the fact that $\Delta_T f$ belongs to the classical class $\mathrm{Lip}_M 1$.

Finally, if $\alpha_k^{(n)}$ is a double sequence of real numbers such that $\lim_{k\to+\infty} \alpha_k^{(n)} = 0$ uniformly in $n \in \mathbb{N}$ and β_k is another sequence of real numbers with $\lim_{k\to+\infty} \beta_k = 0$, then we use the notation $\alpha_k^{(n)} = o^{(n)}(\beta_k)$ to indicate that

$$\lim_{k\to+\infty} \frac{\alpha_k^{(n)}}{\beta_k} = 0 \quad \text{unifomly in } n \in \mathbb{N}. \quad (6)$$

2. Saturation Results

In this section we obtain local saturation results in the approximation process of $\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x)$ towards $D^m f(x)$. Firstly we state without proof three lemmas; Lemma 1 coincides with [10, Lemma 1], Lemma 2 follows the same pattern as [10, Lemma 2], and finally Lemma 3 is a very direct consequence of (P1).

Lemma 1. *Let J be a bounded open subinterval of $[a,b]$. Let $g,h \in C(J)$ and $t_0, t_1, t_2 \in J$ such that $t_0 \in (t_1,t_2)$, $g(t_1) = g(t_2) = 0$ and $g(t_0) > 0$. Then there exist a real number $\epsilon < 0$, a solution of the differential equation (4) on J, say z, and a point $x \in (t_1,t_2)$ such that $\epsilon h(x) + z(x) = g(x)$ and, for all $t \in (t_1,t_2)$, $\epsilon h(t) + z(t) \geq g(t)$.*

Lemma 2. *Let $f, g \in C^m[a,b]$ and let $x \in (a,b)$. Assume that there exists a neighborhood N_x of x where $D^m f \leq D^m g$. Then*

$$\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) \leq \mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} g(x) + o^{(n)}\left(\lambda_k^{-1}\right). \quad (7)$$

Lemma 3. *$f \in C^m[a,b]$ is a solution of the differential equation (4) in some neighborhood of $x \in (a,b)$ if and only if*

$$\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x) = o^{(n)}\left(\lambda_k^{-1}\right). \quad (8)$$

The following two propositions, of interest by themselves, prepare the way to prove the announced results. An important role is played by the notion of convexity with respect to the extended complete Tchebychev system $\{u_0, u_1\}$ that here we relate to the monotonic convergence of the process and allows us to compare the degree of approximation for two different functions.

Proposition 4. *Let $f \in C^m[a,b]$; then*

(a) *$D^m f$ is convex with respect to $\{u_0,u_1\}$ on (a,b) if and only if for each $x \in (a,b)$*

$$\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) \geq D^m f(x) + o^{(n)}\left(\lambda_k^{-1}\right), \quad (9)$$

(b) *if $\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) \geq \mathscr{A}^{k+1,n}_{D^m\circ\mathscr{L}} f(x) + o^{(n)}(\lambda_k^{-1})$ for all $x \in (a,b)$, then $D^m f$ is convex with respect to $\{u_0,u_1\}$ on (a,b).*

Proof. (a) Let $x \in (a,b)$. Assume that $D^m f$ is convex with respect to $\{u_0,u_1\}$ on (a,b) and let $z \in \langle u_0,u_1\rangle$ such that

$$z(x) = D^m f(x), \qquad D^1 z(x) = D^1_+ (D^m f)(x) \quad (10)$$

(here D^1_+ denotes the right first derivative operator). Then, from [11, Lemma 2.2], we have that $z(t) \leq D^m f(t)$ for all

$t \in (a,b)$, and directly from Lemma 2, if we take $Z \in C^m[a,b]$ such that $D^m Z(t) = z(t)$ for all $t \in (a,b)$, we derive that

$$\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} Z(x) \le \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) + o^{(n)}\left(\lambda_k^{-1}\right), \tag{11}$$

or equivalently

$$\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} Z(x) - D^m Z(x) \le \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) - D^m f(x) + o^{(n)}\left(\lambda_k^{-1}\right). \tag{12}$$

Finally we apply Lemma 3 to the fuction Z and obtain the required inequality as follows:

$$\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) \ge D^m f(x) + o^{(n)}\left(\lambda_k^{-1}\right). \tag{13}$$

To prove the converse we assume the contrary; that is, that $D^m f$ is not convex with respect to $\{u_0, u_1\}$ on (a,b); then there exist three points x_1, x_2, and s such that

$$a < x_1 < s < x_2 < b, \qquad S\left(D^m f, x_1, x_2\right)(s) < D^m f(s), \tag{14}$$

where $S(D^m f, x_1, x_2)$ is the unique function of the space $\langle u_0, u_1\rangle$ which interpolates $D^m f$ at x_1 and x_2.

Now we apply Lemma 1 with $h = u_2$ and $g = D^m f - S(D^m f, x_1, x_2)$ and derive the existence of $\epsilon < 0$, a solution $\hat{z}$ of $\mathscr{D} z \equiv 0$ and $s_1 \in [x_1, x_2]$ satisfying

$$\epsilon u_2(t) + \hat{z}(t) \ge D^m f(t) - S\left(D^m f, x_1, x_2\right)(t), \quad t \in \left(x_1, x_2\right), \tag{15}$$

$$\epsilon u_2\left(s_1\right) + \hat{z}\left(s_1\right) = D^m f\left(s_1\right) - S\left(D^m f, x_1, x_2\right)\left(s_1\right). \tag{16}$$

Let us take $U_2, \widehat{Z}, \widehat{S} \in C^m[a,b]$ such that $D^m U_2 = u_2$, $D^m \widehat{Z} = \hat{z}$ and $D^m \widehat{S} = S(D^m f, x_1, x_2)$ on (a,b) and apply then Lemma 2 taking into account (15). This yields that

$$\begin{aligned} &\epsilon \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} U_2\left(s_1\right) + \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} \widehat{Z}\left(s_1\right) \\ &\quad \ge \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f\left(s_1\right) - \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} \widehat{S}\left(s_1\right) + o^{(n)}\left(\lambda_k^{-1}\right). \end{aligned} \tag{17}$$

After introducing equality (16) we get

$$\begin{aligned} &\epsilon\left(\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} U_2\left(s_1\right) - D^m U_2\left(s_1\right)\right) + \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} \widehat{Z}\left(s_1\right) - D^m \widehat{Z}\left(s_1\right) \\ &\quad \ge \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f\left(s_1\right) - D^m f\left(s_1\right) - \left(\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} \widehat{S}\left(s_1\right) - D^m \widehat{S}\left(s_1\right)\right) \\ &\quad\quad + o^{(n)}\left(\lambda_k^{-1}\right). \end{aligned} \tag{18}$$

Finally, multiplying by λ_k and applying (P2) we obtain the following inequality which contradicts our assumption:

$$\epsilon \ge \mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f\left(s_1\right) - D^m f\left(s_1\right) + o^{(n)}\left(\lambda_k^{-1}\right). \tag{19}$$

(b) If $\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) \ge \mathscr{A}^{k+1,n}_{D^m \circ \mathscr{L}} f(x) + o^{(n)}(\lambda_k^{-1})$ for $x \in (a,b)$, then directly from (P0) we have that $\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) \ge D^m f(x) + o^{(n)}(\lambda_k^{-1})$ and it suffices to use (a) to complete the proof. □

Proposition 5. *Let $M \ge 0$ and let $f, w \in C^m[a,b]$. Then the following items are equivalent*

(i) *$M D^m w \pm D^m f$ are convex with respect to $\{u_0, u_1\}$ on (a,b),*

(ii) *for each $x \in (a,b)$*

$$\left|\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) - D^m f(x)\right| \le M\left(\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} w(x) - D^m w(x)\right) + o^{(n)}\left(\lambda_k^{-1}\right). \tag{20}$$

Proof. It suffices to apply Proposition 4 replacing f by $Mw \pm f$. □

With appropriate choices of the function w and applying the results of [11], we give two saturation results; the first one is stated in terms of classic Lipschitz spaces, while the second one puts across the relationship with the asymptotic formula.

Theorem 6. *Let $f \in C^m[a,b]$. Then*

$$\lambda_k \left|\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) - D^m f(x)\right| \le \frac{M}{w_2(x)} + o^{(n)}(1), \quad x \in (a,b) \tag{21}$$

if only if, on (a,b),

$$\Delta_T D^m f = w_1^{-1} D^1\left(w_0^{-1}\left(D^m f\right)\right) \in \operatorname{Lip}_M 1. \tag{22}$$

Proof. Take $w \in C^m[a,b]$ such that $D^m w(t) = w_0(t) \int_c^t w_1(t_1) \int_c^{t_1} dt_2 dt_1$ and then apply Proposition 5. Thus the result follows directly after using (P2) and [11, Theorem 3.2] taking into account that $\mathscr{D}(D^m w) = w_2^{-1}$. □

Theorem 7. *Let $f \in C^m[a,b]$. Then*

$$\lambda_k \left|\mathscr{A}^{k,n}_{D^m \circ \mathscr{L}} f(x) - D^m f(x)\right| \le M + o^{(n)}(1), \quad x \in (a,b) \tag{23}$$

if and only if, almost everywhere on (a,b),

$$\frac{1}{w_2} D^1\left(\frac{1}{w_1} D^1\left(\frac{1}{w_0} D^m f\right)\right) \le M. \tag{24}$$

Proof. Take $w \in C^m[a,b]$ such that $D^m w(t) = w_0(t) \int_c^t w_1(t_1) \int_c^{t_1} w_2(t_2) dt_2 dt_1$ and then apply Proposition 5. Thus the result follows directly after using (P2) and [11, Theorem 3.2] taking into account that $\mathscr{D}(D^m w) \equiv 1$. □

3. Converse Result of the Asymptotic Formula

This section is devoted to give a converse result of the asymptotic formula stated in (3). It turns to be an extension of the results of [12]. A rough statement of the problem would read as follows: under the general framework of the paper, assume the existence of a function g such that for $f \in C^m[a,b]$,

$$\lim_{k\to+\infty} \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x)\right) = g(x) \quad \text{uniformly in } n. \tag{25}$$

Is f $m+2$-times differentiable at x? is it true that $\mathscr{D}(D^m f) = g$?

The answer, affirmative in certain sense, represents the content of this section. We will make use of two lemmas. We state them without proof as they resemble closely [10, Lemmas 3, 4].

Lemma 8. *Let $f \in C^m[a,b]$. If*

$$\lim_{k\to+\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x)\right) \geq 0, \quad x \in (a,b), \tag{26}$$

then $D^m f$ is convex with respect to $\{u_0, u_1\}$ on (a,b).

Lemma 9. *Let $x \in (a,b)$ and let $H \in C^m[a,b]$ such that for all $t \in (a,b)$ $D^m H(t) = w_0(t)\int_a^t h(s) w_1(s) ds$; then*

$$\begin{aligned}
&\lim_{t\to x}\inf \frac{h(t)-h(x)}{W_2(t)-W_2(x)} \\
&\quad \leq \lim_{k\to+\infty}\inf \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} H(x) - D^m H(x)\right), \\
&\lim_{k\to+\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} H(x) - D^m H(x)\right) \\
&\quad \leq \lim_{t\to x}\sup \frac{h(t)-h(x)}{W_2(t)-W_2(x)}.
\end{aligned} \tag{27}$$

Theorem 10. *Let $f \in C^k[a,b]$ and let ψ a finitely valued function in $L_1(a,b)$ such that for $x \in (a,b)$*

$$\begin{aligned}
&\lim_{k\to+\infty}\inf \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x)\right) \\
&\quad \leq \psi(x) \leq \lim_{k\to+\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} f(x) - D^m f(x)\right);
\end{aligned} \tag{28}$$

then almost everywhere on (a,b), $\psi = \mathscr{D}(D^m f)$.

Proof. It follows the same pattern as [10, Theorem 1]. We detail it however for the sake of completeness. Let $\Psi(t) = w_0(t)\int_a^t w_1(s)\int_a^s \psi(\nu) w_2(\nu) d\nu\, ds$, and let $G \in C^m[a,b]$ such that for all $t \in (a,b)$

$$D^m G(t) = D^m f(t) - \Psi(t). \tag{29}$$

For $q \in \mathbb{N}$, let m_q and M_q be, respectively, the minor and major functions of ψ with respect to w_2 such that

$$\begin{aligned}
&\left| m_q(t) - \int_a^t \psi(s) w_2(s) ds \right| < \frac{1}{q}, \quad t \in (a,b), \\
&\left| M_q(t) - \int_a^t \psi(s) w_2(s) ds \right| < \frac{1}{q}, \quad t \in (a,b),
\end{aligned} \tag{30}$$

whose existence is guaranteed from the theory of Lebesgue integration (see e.g., [14]). In particular it follows that

$$\lim_{t\to x}\sup \frac{m_q(t)-m_q(x)}{W_2(t)-W_2(x)} \leq \psi(x) \leq \lim_{t\to x}\inf \frac{M_q(t)-M_q(x)}{W_2(t)-W_2(x)}. \tag{31}$$

From the assumptions and Lemma 9, if we consider $\overline{m}_q \in C^m[a,b]$ such that for all $t \in (a,b)$ $D^m\overline{m}_q(t) = w_0(t)\int_a^t m_q(s) w_1(s) ds$, we have that

$$\begin{aligned}
&\lim_{k\to\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} \overline{m}_q - D^m \overline{m}_q\right)(x) \\
&\quad \leq \lim_{t\to x}\sup \frac{m_q(t)-m_q(x)}{W_2(t)-W_2(x)} \\
&\quad \leq \psi(x) \leq \lim_{k\to\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} L_n f - f\right)(x),
\end{aligned} \tag{32}$$

hence

$$\lim_{k\to\infty}\sup \lambda_k \left(\mathscr{A}^{k,n}_{D^m\circ\mathscr{L}} \left(f - \overline{m}_q\right) - D^m\left(f - \overline{m}_q\right)\right)(x) \geq 0. \tag{33}$$

Now Lemma 8 yields that for each $q \in \mathbb{N}$, $D^m(f - \overline{m}_q)$ is convex with respect to $\{u_0, u_1\}$ on (a,b). Letting q tend to infinity we derive that $D^m G$ is convex respect to $\{u_0, u_1\}$ on (a,b). If we proceed this way with M_q, we conclude that $-D^m G$ is convex respect to $\{u_0, u_1\}$ on (a,b) as well. Hence, in this interval $\mathscr{D}(D^m G) = 0$ and consequently, almost everywhere on (a,b)

$$\mathscr{D}\left(D^m f\right) = \mathscr{D}(\Psi), \tag{34}$$

from where the proof follows recalling the definition of Ψ at the top of the proof, the one of $\mathscr{D}$ in (4), and finally using (P2). □

4. Applications

In this section we illustrate the use of some of the results of the paper. We will make use of the asymptotic formulae obtained in [8, Section 3] to state some saturation results for the classical Bernstein operators and for a modification of them. Here we consider almost convergence, as a particular case of $\mathscr{A}$-summability. We refer the reader to [8, Subsections 3.1, 3.2] for further details.

4.1. Saturation of Bernstein Operators and Almost Convexity. Let $p_B(x) = x(1-x)$.

Corollary 11. *Let $M > 0$ and $f \in C^3[0,1]$; then*

$$\begin{aligned}
&\frac{k}{\log k}\left|\sum_{j=n}^{n+k-1} \frac{1}{k} D^3 B_j f(x) - D^3 f(x)\right| \\
&\quad \leq M \frac{1}{2(1-x)^2} + o^{(n)}(1), \quad x \in (0,1)
\end{aligned} \tag{35}$$

if and only if

$$p_B^3\left(\frac{1}{e_2}D^4 f + \frac{2}{e_3}D^3 f\right) \in \text{ Lip }_M 1 \quad \textit{on } (0,1). \tag{36}$$

Corollary 12. *Let $M > 0$ and $f \in C^3[0,1]$; then*

$$\frac{k}{\log k}\left|\sum_{j=n}^{n+k-1}\frac{1}{k}D^3 B_j f(x) - D^3 f(x)\right| \leq M + o^{(n)}(1), \quad x \in (a,b) \tag{37}$$

if and only if

$$\left|D^3\left(p_B D^2 f\right)\right| \leq M \quad \textit{a.e. on } (0,1). \tag{38}$$

4.2. Saturation of Modified Bernstein Operators and Almost Convexity. Here we consider the sequence of linear operators L_j given in [8, Subsection 3.2].

Corollary 13. *Let $M > 0$ and $f \in C[0,1]$; then*

$$\frac{k}{\log k}\left|\sum_{j=n}^{n+k-1}\frac{1}{k}L_j f(x) - f(x)\right| \leq M\frac{x(1-x)}{2} + o^{(n)}(1), \quad x \in (0,1) \tag{39}$$

if and only if

$$D^1 f \in \text{ Lip }_M 1 \quad \textit{on } (0,1). \tag{40}$$

Corollary 14. *Let $M > 0$ and $f \in C[0,1]$; then*

$$\frac{k}{\log k}\left|\sum_{j=n}^{n+k-1}\frac{1}{k}L_j f(x) - f(x)\right| \leq M + o^{(n)}(1), \quad x \in (0,1) \tag{41}$$

if and only if

$$\left|\left(p_B D^2 f\right)\right| \leq M \quad \textit{a.e. on } (0,1). \tag{42}$$

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References

[1] G. G. Lorentz, "A contribution to the theory of divergent sequences," *Acta Mathematica*, vol. 80, no. 1, pp. 167–190, 1948.

[2] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and Its Applications*, vol. 17 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1994.

[3] J. P. King and J. J. Swetits, "Positive linear operators and summability," *Australian Mathematical Society A*, vol. 11, pp. 281–290, 1970.

[4] R. N. Mohapatra, "Quantitative results on almost convergence of a sequence of positive linear operators," *Journal of Approximation Theory*, vol. 20, no. 3, pp. 239–250, 1977.

[5] J. J. Swetits, "On summability and positive linear operators," *Journal of Approximation Theory*, vol. 25, no. 2, pp. 186–188, 1979.

[6] H. T. Bell, "Order summability and almost convergence," *Proceedings of the American Mathematical Society*, vol. 38, pp. 548–552, 1973.

[7] F. Aguilera, D. Cárdenas-Morales, P. Garrancho, and J. M. Hernández, "Quantitative results in conservative approximation and summability," *Automation Computers Applied Mathematics*, vol. 17, no. 2, pp. 201–208, 2008.

[8] P. Garrancho, D. Cárdenas-Morales, and F. Aguilera, "On asymptotic formulae via summability," *Mathematics and Computers in Simulation*, vol. 81, no. 10, pp. 2174–2180, 2011.

[9] D. Cárdenas-Morales and P. Garrancho, "Local saturation of conservative operators," *Acta Mathematica Hungarica*, vol. 100, no. 1-2, pp. 83–95, 2003.

[10] P. Garrancho and D. Cárdenas-Morales, "A converse of asymptotic formulae in simultaneous approximation," *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2676–2683, 2010.

[11] G. G. Lorentz and L. L. Schumaker, "Saturation of positive operators," *Journal of Approximation Theory*, vol. 5, no. 4, pp. 413–424, 1972.

[12] H. Berens, "Pointwise saturation of positive operators," *Journal of Approximation Theory*, vol. 6, no. 2, pp. 135–146, 1972.

[13] S. J. Karlin and W. J. Studden, *Tchebycheff Systems*, Wiley-Interscience, New York, NY, USA, 1966.

[14] I. N. Pesin, *Classical and Modern Integration Theories*, Academic Press, New York, NY, USA, 1951.