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Research Article

Solutions to the System of Operator Equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ on Hilbert C^*-Modules

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We study the solvability of the system of the adjointable operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ over Hilbert C^*-modules. We give necessary and sufficient conditions for the existence of a solution and a positive solution of the system. We also derive representations for a general solution and a positive solution to this system. The above results generalize some recent results concerning the equations for operators with closed ranges.

1. Introduction

Many results have been made on the study of solvability of equations for operators on Hilbert spaces and Hilbert C^*-modules. In 1966, Douglas presented the famous Douglas theorem in [1]. He gave the conditions of the existence of the solution to the equation $AX = B$ for operators on a Hilbert space. By using the generalized inverse of operators, Dajić and Koliha [2] got the existence of the common Hermitian and positive solution to the equations $AX = C$, $XB = D$ for operators on a Hilbert space.

Hilbert C^*-module is a natural generalization both of Hilbert space and C^*-algebra and it has been an important tool in the theory of C^*-algebra, especially in the study of KK-groups and induced representations (see [3–9]). Therefore it is meaningful to put forward a generalized version of the previous results about operator equations in the context of Hilbert C^*-modules.

By using the generalized inverses of adjointable operators on a Hilbert C^*-module, Wang and Dong recently obtained the necessary and sufficient conditions for the existence of a positive solution to the system of adjointable operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ for operators on Hilbert C^*-modules in [10].

To use the generalized inverse, the authors mentioned above have to focus their attentions on those adjointable operators whose ranges are closed. However, closed range is a very strong condition in infinite dimensional case which general bounded (adjointable) linear operators may not satisfy. In fact an operator with closed range is also called a generalized Fredholm operator. In [6, 11], Fang et al. generalize the famous Douglas theorem from the case of the Hilbert spaces to the one of the Hilbert C^*-modules and get some results about solutions to some equations and some systems of equations without the assumption of the closed ranges by use of some new approaches. They put the attention to the operators whose adjoint operators' range closures are orthogonally complemented, which is automatically satisfied in Hilbert space case.

In this paper, along the same way as in [6, 11], we obtain the existence of the solution to the system of equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ which was studied in [10] and then two theorems about the existence of the positive solution to this system, which extend the main results in [10] from operators with closed range to operators whose adjoint operators' range closures are orthogonally complemented.

2. Preliminaries

First of all, we recall some knowledge about Hilbert C^*-modules.

Throughout this paper, $\mathscr{A}$ is a C^*-algebra. An inner-product $\mathscr{A}$-module is a linear space H which is a right $\mathscr{A}$-module, together with a map $(x, y) \rightarrow \langle x, y\rangle : H\times H \rightarrow \mathscr{A}$ such that, for any $x, y, z \in H$, $\alpha, \beta \in \mathbb{C}$, and $a \in \mathscr{A}$, the following conditions hold:

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;

(2) $\langle x, ya\rangle = \langle x, y\rangle a$;

(3) $\langle x, y\rangle = \langle y, x\rangle^*$;

(4) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner-product $\mathscr{A}$-module H which is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$ is called a (right) Hilbert $\mathscr{A}$-module.

Suppose that H_1 and H_2 are two Hilbert $\mathscr{A}$-modules; let $L_{\mathscr{A}}(H_1, H_2)$ be the set of all maps $T : H_1 \rightarrow H_2$ for which there is a map $T^* : H_2 \rightarrow H_1$ such that

$$\langle Tx, y\rangle = \langle x, T^* y\rangle, \quad \text{for each } x \in H_1,\ y \in H_2. \tag{1}$$

It is known that any element T of $L_{\mathscr{A}}(H_1, H_2)$ must be a bounded linear operator, which is also $\mathscr{A}$-linear in the sense that $T(xa) = T(x)a$ for $x \in H_1$ and $a \in \mathscr{A}$. For any $T \in L_{\mathscr{A}}(H_1, H_2)$, the range and the null space of T are denoted by $R(T)$ and $N(T)$, respectively. We call $L_{\mathscr{A}}(H_1, H_2)$ the set of adjointable operators from H_1 to H_2. We denote by $B_{\mathscr{A}}(H_1, H_2)$ the set of all bounded $\mathscr{A}$-linear maps, and therefore we have $L_{\mathscr{A}}(H_1, H_2) \subseteq B_{\mathscr{A}}(H_1, H_2)$. In case $H_1 = H_2$, $L_{\mathscr{A}}(H_1)$, to which we abbreviate $L_{\mathscr{A}}(H_1, H_2)$, is a C^*-algebra. Then for $A \in L_{\mathscr{A}}(H)$, A is Hermitian (self-adjointable) if and only if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for any $x, y \in H$, and positive if and only if $\langle Ax, x\rangle \geq 0$ for any $x \in H$, in which case, we denote by $A^{1/2}$ the unique positive element B such that $B^2 = A$ in the C^*-algebra $L_{\mathscr{A}}(H)$ and then $\overline{R(A)} = \overline{R(A^{1/2})}$. Let $L_{\mathscr{A}}(H)_{sa}, L_{\mathscr{A}}(H)_+$ be the sets of Hermitian and positive elements of $L_{\mathscr{A}}(H)$, respectively. For any $A, B \in L_{\mathscr{A}}(H)_{sa}$, we say $A \geq B$ if $\langle (A - B)x, x\rangle \geq 0$ for any $x \in H$. For $\mathscr{A}_+$, the set of positive elements of the C^*-algebra $\mathscr{A}$ is a positive cone; we could easily verify that "$\geq$" is a partial order on $L_{\mathscr{A}}(H)$. For an operator $T \in L_{\mathscr{A}}(H)$, set $\mathrm{Re}(T) = T + T^*$, and T is called real positive if $\mathrm{Re}(T) \geq 0$.

We say that a closed submodule H_1 of H is topologically complemented if there is a closed submodule H_2 of H such that $H_1 + H_2 = H$ and $H_1 \cap H_2 = 0$ and briefly denote the sum by $H = H_1 \dot{\oplus} H_2$, called the direct sum of H_1 and H_2. Moreover, if $H_2 = H_1^{\perp}$ where $H_1^{\perp} = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in H_1\}$, we say H_1 is orthogonally complemented and briefly denote the sum by $H = H_1 \oplus H_2$, called the orthogonal sum of H_1 and H_2. In this case, $H_1 = H_1^{\perp\perp}$ and there exists unique orthogonal projection (i.e., idempotent and self-adjointable operator in $L_{\mathscr{A}}(H)$) onto H_1. For two submodules H_1 and H_2 of H, if $H_1 \subseteq H_2$, then $H_1^{\perp} \supseteq H_2^{\perp}$.

Let $T \in L_{\mathscr{A}}(H_1, H_2)$; then (1) $N(T) = R(T^*)^{\perp}$ and $N(T)^{\perp} \supseteq \overline{R(T^*)}$; (2) if $R(T)$ is closed, then so is $R(T^*)$, and in this case both $R(T)$ and $R(T^*)$ are orthogonally complemented and $R(T)^{\perp} = N(T^*)$, $R(T^*)^{\perp} = N(T)$ (see [7], Theorem 3.2).

Any element T^- of $\{X \in L_{\mathscr{A}}(H_1, H_2) : TXT = T\}$ is called the inner inverse of T and $R(TT^-) = R(T)$. $R(T)$ is closed if and only if T has a inner inverse. The Moore-Penrose inverse T^+ of T is the unique inner inverse of T which satisfies

$$\begin{gathered} T^+TT^+ = T^+, \qquad TT^+ = \left(TT^+\right)^*, \\ T^+T = \left(T^+T\right)^*. \end{gathered} \tag{2}$$

In this case, $(T^+)^* = (T^*)^+$, $R(T^+) = R(T^*)$ and $T^+|_{R(T)^{\perp}} = 0$. Thus TT^+ and T^+T are the projections onto $R(T)$ and $R(T^*)$, respectively.

Throughout this paper, H_1, H_2, H_3, H_4, and H_5 are Hilbert $\mathscr{A}$-modules. For an operator $T \in L_{\mathscr{A}}(H_1, H_2)$, if $\overline{R(T^*)}$ is orthogonally complemented, then $\overline{R(T^*)}^{\perp} = N(T)$ and there exists an orthogonal decomposition $E = \overline{R(T^*)} \oplus N(T)$. Let P_{T^*} denote the orthogonal projection of H_1 onto $\overline{R(T^*)}$ and N_T the projection $I - P_{T^*}$; then $P_{T^*} + N_T = I_{H_1}$.

Lemma 1 (see [6, Theorem 1.1]). *Let $T' \in L_{\mathscr{A}}(H_1, H_2)$ and $T \in L_{\mathscr{A}}(H_3, H_2)$ with $\overline{R(T^*)}$ being orthogonally complemented. The following statements are equivalent:*

(1) *$T'T'^* \leq \lambda TT^*$ for some $\lambda \geq 0$;*

(2) *there exists $\mu \geq 0$ such that $\|T'^* z\| \leq \mu\|T^* z\|$ for all $z \in H_2$;*

(3) *there exists $D \in L_{\mathscr{A}}(H_3, H_1)$ such that $T' = TD$; that is, $TX = T'$ has a solution;*

(4) *$R(T') \subseteq R(T)$.*

Moreover there exists a unique operator D which satisfies the conditions

$$T' = TD, \qquad R(D) \subseteq N(T)^{\perp}. \tag{3}$$

In this case,

$$\begin{gathered} \|D\|^2 = \inf\left\{\lambda : T'T'^* \leq \lambda TT^*\right\}, \\ R(D) \subseteq \overline{R(T^*)}; \qquad N(D) \subseteq N\left(T'\right), \end{gathered} \tag{4}$$

and D is called the reduced solution of the equation $TX = T'$.

The general solution to $TX = T'$ is of the form

$$X = D + N_T K, \tag{5}$$

where $K \in L_{\mathscr{A}}(H_3, H_1)$ is arbitrary.

Lemma 2 (see [6, Theorem 2.1]). *Let $A \in L_{\mathscr{A}}(H_1, H_2)$, $C \in L_{\mathscr{A}}(H_3, H_2)$, $B \in L_{\mathscr{A}}(H_4, H_3)$, and $D \in L_{\mathscr{A}}(H_4, H_1)$, suppose $\overline{R(A^*)}$ and $\overline{R(B)}$ are orthogonally complemented submodules in H_1 and H_3, respectively, Then $AX = C$ and $XB = D$ have a common solution $X \in L_{\mathscr{A}}(H_3, H_1)$ if and only if*

$$R(C) \subseteq R(A), \quad R\left(D^*\right) \subseteq R\left(B^*\right), \quad AD = CB. \tag{6}$$

In this case, the general solution is of the form:

$$X = D_1 + N_A D_2^* + N_A V N_{B^*}, \tag{7}$$

where D_1 and D_2 are the reduced solutions of $AX = C$ and $B^* X = D^*$, respectively, and $V \in L_{\mathscr{A}}(H_3, H_1)$ is arbitrary.

Lemma 3 (see [11, Theorem 3.1]). *Let $A \in L_{\mathscr{A}}(H_1, H_2)$, $B \in L_{\mathscr{A}}(H_3, H_4)$, and $C \in L_{\mathscr{A}}(H_3, H_2)$.*

(1) *If the equation $AXB = C$ has a solution $X \in L_{\mathscr{A}}(H_4, H_1)$, then*

$$R(C) \subseteq R(A), \qquad R(C^*) \subseteq R(B^*). \tag{8}$$

(2) *Suppose $\overline{R(B)}$ and $\overline{R(A^*)}$ are orthogonally complemented submodules of H_4 and H_1, respectively. If*

$$R(C) \subseteq R(A), \qquad \overline{R(C^*)} \subseteq R(B^*) \tag{9}$$

or

$$\overline{R(C)} \subseteq R(A), \qquad R(C^*) \subseteq R(B^*), \tag{10}$$

then $AXB = C$ has a unique solution $D \in L_{\mathscr{A}}(H_4, H_1)$ such that

$$R(D) \subseteq N(A)^{\perp}, \qquad R(D^*) \subseteq N(B^*)^{\perp}, \tag{11}$$

which is called the reduced solution, and the general solution to $AXB = C$ is of the form

$$X = D + N_A V_1 + V_2 N_{B^*}, \tag{12}$$

where $V_1, V_2 \in L_{\mathscr{A}}(H_4, H_1)$.

Then, from $R(D) \subseteq N(A)^{\perp}$ and $R(D^*) \subseteq N(B^*)^{\perp}$, one knows that $D = P_{A^*} D P_B$. As in [11], one can obtain that $AXB = 0 \Leftrightarrow P_{A^*} X P_B = 0$.

Lemma 4 (see [6, Theorem 1.3]). *Let $A, C \in L_{\mathscr{A}}(H_1, H_2)$ such that $\overline{R(A^*)}$ is orthogonally complemented. Then $AX = C$ has a positive solution $X \in L_{\mathscr{A}}(H_1)$ if and only if $R(C) \subseteq R(A), CA^* \geq 0$.*

In this case, $D \geq 0, R(D) \subseteq N(A)^{\perp}$, and the general positive solution is of the form

$$X = D + N_A K N_A, \tag{13}$$

where D is the positive reduced solution and $K \in L_{\mathscr{A}}(H_1)_+$ is an arbitrary positive operator.

Lemma 5 (see [6, Lemma 2.1]). *Let $U \in L_{\mathscr{A}}(H_1), V \in L_{\mathscr{A}}(H_2, H_1)$, and $L \in L_{\mathscr{A}}(H_2)$. Then $\begin{pmatrix} U & V \\ V^* & L \end{pmatrix} \geq 0$ if and only if $U \geq 0$, $L \geq 0$, and $\varphi(\langle x, Vy\rangle)\varphi(\langle Vy, x\rangle) \leq \varphi(\langle Ux, x\rangle)\varphi(\langle Ly, y\rangle)$ for any $x \in H_1$, $y \in H_2$, and any state $\varphi \in S(\mathscr{A})$.*

Lemma 6 (see [6, Proposition 2.2]). *Let $A_1, C_1 \in L_{\mathscr{A}}(H_1, H_2)$ and $B_2, C_2 \in L_{\mathscr{A}}(H_3, H_1)$,*

$$\begin{gathered} D = \begin{pmatrix} A_1 \\ B_2^* \end{pmatrix}, \qquad E = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix}, \\ F = ED^* = \begin{pmatrix} C_1 A_1^* & C_1 B_2 \\ C_2^* A_1^* & C_2^* B_2 \end{pmatrix} \end{gathered} \tag{14}$$

such that $\overline{R(D^*)}$ is orthogonally complemented. Then $A_1 X = C_1$ and $XB_2 = C_2$ have a common positive solution $X \in L_{\mathscr{A}}(H_1)$, if and only if $F \geq 0$ and $R(E) \subseteq R(D)$.

In this case, the general positive solution can be expressed as $X = Y_0 + N_D Y N_D$, where $Y_0 \in L_{\mathscr{A}}(H_1)_+$ is the positive reduced solution and $Y \in L_{\mathscr{A}}(H_1)_+$ is an arbitrary positive operator.

Lemma 7 (see [11, Corollary 3.3(iii)]). *Let $A \in L_{\mathscr{A}}(H_1, H_2)$ and $C \in L_{\mathscr{A}}(H_2)$ such that $\overline{R(A)}$ and $\overline{R(A^*)}$ are orthogonally complemented, and A or C has the closed range; the equation $AXA^* = C$ has a positive solution if and only if $C \geq 0$ and $R(C) \subseteq R(A)$. In this case, the operator*

$$X = D + DV_1 N_A + N_A V_1^* D + N_A V_1^* D V_1 N_A + N_A V_2 N_A \tag{15}$$

is a positive solution for any $V_1 \in L_{\mathscr{A}}(H_1)$ and $V_2 \in L_{\mathscr{A}}(H_1)_+$, where D is the reduced solution.

Let $A \in L_{\mathscr{A}}(H_1, H_2), B \in L_{\mathscr{A}}(H_3, H_1)$, and $C \in L_{\mathscr{A}}(H_3, H_2)$. Suppose $\overline{R(A^*)}$ and $\overline{R(B)}$ are orthogonally complemented submodules of H_1; the equation $AXB = C$ has the reduced solution $D \in L_{\mathscr{A}}(H_1)$. Set $T = N_A B$ and $Z = N_T B^* \operatorname{Re}(D) B N_T$, and assume that $\overline{R(T)}$ and $\overline{R(T^*)}$ are orthogonally complemented submodules of H_1 and H_3, respectively. Set

$$\begin{aligned} S\mathscr{Y}(A,B,C) = \{ & Z + ZV_1 P_{T^*} + P_{T^*} V_1^* Z + P_{T^*} V_1^* Z V_1 P_{T^*} \\ & + P_{T^*} V_2 P_{T^*} : V_1 \in L_{\mathscr{A}}(H_3) \\ & \text{such that } N_T B^* D B P_{T^*} \\ & = N_T B^* D B N_T V_1 P_{T^*}, V_2 \in L_{\mathscr{A}}(H_3)_+ \}, \\ S\Sigma(A,B,C)_1 = \{ & X_1 \in L_{\mathscr{A}}(H_1) : X_1 B \\ & = \frac{1}{2} P_T T^{*-1} P_{T^*} \\ & \times (Y - B^* \operatorname{Re}(D) B)(N_T + I) \\ & \text{for some } Y \in S\mathscr{Y}(A,B,C), \ X_1 N_{B^*} = 0 \}, \\ S\Sigma(A,B,C)_2 = \{ & X_2 \in L_{\mathscr{A}}(H_1) : X_2 B \\ & = \frac{1}{2} P_T T^{*-1} (P_{T^*} B^* (D^* - D) B P_{T^*}) \\ & + N_{T^*} (V_3 N_T + i V_4 T) \\ & \text{for some } V_4 \in L_{\mathscr{A}}(E), V_3 \in L_{\mathscr{A}}(H_3, H_1) \\ & \text{with } R(V_3^* N_{T^*}) \subseteq R(B^*) \}, \\ S\Sigma(A,B,C) = \{ & X_1 + X_2 : X_1 \in S\Sigma(A,B,C)_1, \\ & X_2 \in S\Sigma(A,B,C)_2 \}, \\ SS(A,B,C) = \{ & D - D^* N_{B^*} + N_A W P_B \\ & - P_B W^* N_A N_{B^*} + V N_{B^*} : V \in L_{\mathscr{A}}(H_1), \\ & W \in S\Sigma(A,B,C) \}. \end{aligned} \tag{16}$$

It is clear that $SS(A, B, C)$ is a subset of the solution space to the equation $AXB = C$.

Lemma 8 (see [11, Theorem 4.7]). *Let $A \in L_{\mathscr{A}}(H_1, H_2)$, $B \in L_{\mathscr{A}}(H_3, H_1)$, and $C \in L_{\mathscr{A}}(H_3, H_2)$. Suppose that $\overline{R(A^*)}$*

and $\overline{R(B)}$ are orthogonally complemented submodules of H_1, $AXB = C$ has the reduced solution $D \in L_{\mathscr{A}}(H_1)$, $T = N_A B$ has the closed range, and $N_T B^* DBN_T$ is self-adjoint. Let $X \in SS(A,B,C)$ with $X = D - D^* N_{B^*} + N_A W P_B - P_B W^* N_A N_{B^*} + VN_{B^*}$ for some $V \in L_{\mathscr{A}}(H_1)$ and $W \in S\Sigma(A,B,C)$. Then $X = X^*$ if and only if there exist $V_1 \in L_{\mathscr{A}}(H_1)$ and $V_2 \in L_{\mathscr{A}}(H_1)_{sa}$ such that

$$\begin{aligned} V = -\frac{1}{2}\,&(P_B(D + N_A W P_B) \\ &- (D^* + P_B W^* N_A)(I + 3N_{B^*})) \\ &- iP_B V_1{}^* P_B - N_{B^*} V_2, \end{aligned} \tag{17}$$

in which case, $X = D + D^* N_{B^*} + N_A W P_B + P_B W^* N_A N_{B^*} + N_{B^*}(-V_2)N_{B^*}$.

As a consequence,

$$\begin{aligned} SS(A,B,C)_{sa} = \{&D + D^* N_{B^*} + N_A W P_B + P_B W^* N_A N_{B^*} \\ &+ N_{B^*} V N_{B^*} : V \in L_{\mathscr{A}}(H_1)_{sa}, \\ &W \in S\Sigma(A,B,C)\}. \end{aligned} \tag{18}$$

Lemma 9 (see [11, Theorem 4.12]). *Let $A \in L_{\mathscr{A}}(H_1,H_2)$, $B \in L_{\mathscr{A}}(H_3,H_1)$, and $C \in L_{\mathscr{A}}(H_3,H_2)$ such that $\overline{R(B)} \subseteq \overline{R(A^*)}$, and let $\overline{R(A^*)}$ and $\overline{R(B)}$ be orthogonally complemented submodules of H_1. Suppose that $AXB = C$ has the reduced solution $D \in L_{\mathscr{A}}(H_1)$.*

(1) *If $AXB = C$ has a positive solution $X \in L_{\mathscr{A}}(H_1)$, then $B^* DB \geq 0$ and there exists a positive number λ such that*

$$B^* D^* N_{B^*} DB \leq \lambda B^* DB. \tag{19}$$

(2) *Suppose that $\overline{R(P_B D P_B)}$ is orthogonally complemented in H_1. If $B^* DB \geq 0$ and*

$$B^* D^* N_{B^*} DB \leq \lambda B^* DB, \tag{20}$$

for some positive number λ, then $AXB = C$ has a positive solution $X \in SS(A,B,C)_{sa}$.

3. Main Results

Theorem 10. *Let H_i $(i = 1,2,3,4,5)$ be Hilbert $\mathscr{A}$-modules, $A_1 \in L_{\mathscr{A}}(H_1,H_2)$, $A_3 \in L_{\mathscr{A}}(H_1,H_4)$, $B_2 \in L_{\mathscr{A}}(H_3,H_1)$, $B_3 \in L_{\mathscr{A}}(H_5,H_1)$, $C_1 \in L_{\mathscr{A}}(H_1,H_2)$, $C_2 \in L_{\mathscr{A}}(H_3,H_1)$, and $C_3 \in L_{\mathscr{A}}(H_5,H_4)$.*

(1) *If the system of operator equations $A_1 X = C_1$, $XB_2 = C_2$, and $A_3 X B_3 = C_3$ has a solution $X \in L_{\mathscr{A}}(H_1)$, then*

$$\begin{gathered} R(C_1) \subseteq R(A_1), \qquad R(C_2^*) \subseteq R(B_2^*), \\ R(C_3) \subseteq R(A_3), \qquad R(C_3^*) \subseteq R(B_3^*), \\ A_1 C_2 = C_1 B_2. \end{gathered} \tag{21}$$

(2) *Set $A_4 = A_3 N_{A_1}$, $B_4 = N_{B_2^*} B_3$. Suppose $\overline{R(A_1^*)}$, $\overline{R(B_2)}$, $\overline{R(A_4^*)}$, and $\overline{R(B_4)}$ are orthogonally complemented submodules of H_1. If*

$$\begin{gathered} R(C_1) \subseteq R(A_1), \qquad R(C_2^*) \subseteq R(B_2^*), \\ R(C_3) \subseteq R(A_3), \qquad \overline{R(C_3^*)} \subseteq R(B_3^*) \end{gathered} \tag{22}$$

or

$$\begin{gathered} R(C_1) \subseteq R(A_1), \qquad R(C_2^*) \subseteq R(B_2^*), \\ \overline{R(C_3)} \subseteq R(A_3), \qquad R(C_3^*) \subseteq R(B_3^*), \end{gathered} \tag{23}$$

and $A_1 C_2 = C_1 B_2$, then the system of operator equations $A_1 X = C_1$, $XB_2 = C_2$, and $A_3 X B_3 = C_3$ has a unique solution $D \in L_{\mathscr{A}}(H_1)$ such that

$$\begin{gathered} R(D) \subseteq N(A_1)^{\perp} \bigcap N(B_2^*)^{\perp} \bigcap N(A_4)^{\perp}, \\ R(D^*) \subseteq N(A_1^*)^{\perp} \bigcap N(B_2)^{\perp} \bigcap N(B_4^*)^{\perp}. \end{gathered} \tag{24}$$

In this case, the general solution is of the form

$$X = D_1 + N_{A_1} D_2^* + N_{A_1}\left(D_3 + N_{A_4} V_1 + V_2 N_{B_4^*}\right) N_{B_2^*}, \tag{25}$$

where D_1, D_2, and D_3 are the reduced solutions of $A_1 X = C_1$, $B_2^* X = C_2^*$, and $A_4 X B_4 = C_3 - A_3 D_1 B_3 - A_3 N_{A_1} D_2^* B_3$ respectively. $V_1, V_2 \in L_{\mathscr{A}}(H_1)$ are arbitrary.

Proof. (1) If the system of operator equations $A_1 X = C_1$, $XB_2 = C_2$, and $A_3 X B_3 = C_3$ has a solution $X \in L_{\mathscr{A}}(H_1)$, it is easy to know that

$$\begin{gathered} R(C_1) \subseteq R(A_1), \qquad R(C_2^*) \subseteq R(B_2^*), \\ R(C_3) \subseteq R(A_3), \qquad R(C_3^*) \subseteq R(B_3^*), \\ A_1 C_2 = A_1 X B_2 = C_1 B_2. \end{gathered} \tag{26}$$

(2) Since $\overline{R(A_1^*)}$ and $\overline{B_2}$ are orthogonally complemented, $R(C_1) \subseteq R(A_1)$, $R(C_2^*) \subseteq R(B_2^*)$, and $A_1 C_2 = C_1 B_2$. By Lemma 2, we know that $A_1 X = C_1$ and $XB_2 = C_2$ have a common solution X_0 and it has the form:

$$X_0 = D_1 + N_{A_1} D_2^* + N_{A_1} Y N_{B_2^*}, \tag{27}$$

where D_1 and D_2 are the reduced solutions of $A_1 X = C_1$, $B_2^* X = C_2^*$ respectively.

Take $X_0 = D_1 + N_{A_1} D_2^* + N_{A_1} Y N_{B_2^*}$ into $A_3 X B_3 = C_3$, then we can get

$$A_3 N_{A_1} Y N_{B_2^*} B_3 = C_3 - A_3 D_1 B_3 - A_3 N_{A_1} D_2^* B_3. \tag{28}$$

$$A_4 Y B_4 = C_3 - A_3 D_1 B_3 - A_3 N_{A_1} D_2^* B_3. \tag{29}$$

Since $R(C_3) \subseteq R(A_3)$ and $\overline{R(C_3^*)} \subseteq R(B_3^*)$ (or $\overline{R(C_3)} \subseteq R(A_3)$ and $R(C_3^*) \subseteq R(B_3^*)$), then

$$\begin{gathered} R\left(C_3 - A_3 D_1 B_3 - A_3 N_{A_1} D_2^* B_3\right) \subseteq R(A_3), \\ \overline{R(C_3^* - B_3^* D_1^* A_3^* - B_3^* D_2 N_{A_1} A_3^*)} \subseteq R(B_3^*), \end{gathered} \tag{30}$$

or

$$\overline{R\left(C_3 - A_3D_1B_3 - A_3N_{A_1}D_2^*B_3\right)} \subseteq R\left(A_3\right),$$
$$R\left(C_3^* - B_3^*D_1^*A_3^* - B_3^*D_2N_{A_1}A_3^*\right) \subseteq R\left(B_3^*\right). \quad (31)$$

By Lemma 3(2), we know that the operator equation $A_4YB_4 = C_3 - A_3D_1B_3 - A_3N_{A_1}D_2^*B_3$ has a unique reduced solution $D_3 \in L_{\mathscr{A}}(H_1)$ and the general solution has the form

$$Y = D_3 + N_{A_4}V_1 + V_2N_{B_4^*}, \quad (32)$$

where V_1 and $V_2 \in L_{\mathscr{A}}(H_1)$ are arbitrary.

Hence the system of operator equations $A_1X = C_1, XB_2 = C_2$, and $A_3XB_3 = C_3$ has a solution and it has the form

$$X = D_1 + N_{A_1}D_2^* + N_{A_1}\left(D_3 + N_{A_4}V_1 + V_2N_{B_4^*}\right)N_{B_2^*}. \quad (33)$$

And it is easy to see that $D = D_1 + N_{A_1}D_2^* + N_{A_1}D_3N_{B_2^*}$ is the reduced solution to the system of operator equations $A_1X = C_1$, $XB_2 = C_2$ and $A_3XB_3 = C_3$; $R(D) \subseteq N(A_1)^{\perp}\bigcap N(B_2^*)^{\perp}\bigcap N(A_4)^{\perp}$ and $R(D^*) \subseteq N(A_1^*)^{\perp}\bigcap N(B_2)^{\perp}\bigcap N(B_4^*)^{\perp}$. □

If $\overline{R(A_3^*)}$ and $\overline{R(B_3)}$ are orthogonally complemented submodules, then $\overline{R(A_4^*)}$ and $\overline{R(B_4)}$ are orthogonally complemented submodules.

Remark 11. Let H_i $(i = 1,2,3,4,5)$ be Hilbert $\mathscr{A}$-modules, $A_1 \in L_{\mathscr{A}}(H_1,H_2)$, $A_3 \in L_{\mathscr{A}}(H_1,H_4)$, $B_2 \in L_{\mathscr{A}}(H_3,H_1)$, $B_3 \in L_{\mathscr{A}}(H_5,H_1)$, $C_1 \in L_{\mathscr{A}}(H_1,H_2)$, $C_2 \in L_{\mathscr{A}}(H_3,H_1)$, and $C_3 \in L_{\mathscr{A}}(H_5,H_4)$. Set D, E, and F as Lemma 6, and suppose $\overline{R(D^*)}$, $\overline{R(A_3^*)}$, and $\overline{R(B_3^*)}$ are orthogonally complemented submodules of H_1. If $R(E) \subseteq R(D)$, $A_1C_2 = C_1B_2$, $R(C_3) \subseteq R(A_3)$, $\overline{R(C_3^*)} \subseteq R(B_3^*)$ (or, $R(E) \subseteq R(D)$, $A_1C_2 = C_1B_2$, $\overline{R(C_3)} \subseteq R(A_3)$, $R(C_3^*) \subseteq R(B_3^*)$), then the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, $A_3XB_3 = C_3$ has a solution from Theorem 10. Let $C \in L_{\mathscr{A}}(H_1)$ be the reduced solution to the system, and then we can obtain the next theorem about the positive solution to the system.

Theorem 12. *Let the notions and conditions as Remark 11. Set $A_4 = A_3N_D$, $B_4 = N_DB_3$, and $R(B_3) \subseteq \overline{R(A_3^*)}$.*

(1) *If the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ has a positive solution, then $F \geq 0$, $B_4^*CB_4 \geq 0$, and there exists a positive number λ such that $B_4^*C^*N_{B_4^*}CB_4 \leq \lambda B_4^*CB_4$.*

(2) *Suppose $\overline{R(P_{B_4}CP_{B_4})}$ is orthogonally complemented submodule in H_1. If $F \geq 0$, $B_4^*CB_4 \geq 0$, and $B_4^*C^*N_{B_4^*}CB_4 \leq \lambda B_4^*CB_4$, for some $\lambda > 0$, then the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ has a positive solution.*

Proof. (1) If the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ has a positive solution $X_0 \in L_{\mathscr{A}}(H_1)_+$, then X_0 is the positive solution to the system of equations $A_1X = C_1$, $XB_2 = C_2$. by Lemma 6, we know that $F \geq 0$ and X_0 can be expressed as

$$X_0 = Y_0 + N_DYN_D, \quad Y \in L_{\mathscr{A}}(H_1)_+, \quad (34)$$

where Y_0 is the positive reduced solution to the system of equations $A_1X = C_1$, $XB_2 = C_2$. Taking $X_0 = Y_0 + N_DYN_D$ into $A_3XB_3 = C_3$ yields that $A_4YB_4 = C_3 - A_3Y_0B_3$. And it has a positive solution. From Lemma 9, we know that $B_4^*CB_4^* \geq 0$, and there exists a positive number λ such that $B_4^*C^*N_{B_4^*}CB_4 \leq \lambda B_4^*CB_4$.

(2) If $F \geq 0$, we can get that the system of equations $A_1X = C_1$, $XB_2 = C_2$ has a positive solution $X_0 \in L_{\mathscr{A}}(H_1)_+$, and it has the form

$$X_0 = Y_0 + N_DYN_D, \quad Y \in L_{\mathscr{A}}(H_1)_+, \quad (35)$$

where Y_0 is the positive reduced solution to the system of equations $A_1X = C_1$, $XB_2 = C_2$. Take $X_0 = Y_0 + N_DYN_D$ into $A_3XB_3 = C_3$; then we can obtain that $A_4YB_4 = C_3 - A_3Y_0B_3$.

If $\overline{R(P_{B_4}CP_{B_4})}$ is orthogonally complemented submodule in H_1, $B_4^*CB_4^* \geq 0$ and $B_4^*C^*N_{B_4^*}CB_4 \leq \lambda B_4^*CB_4$ for some $\lambda > 0$, by Lemma 9(2), we can easily get that the equation $A_4YB_4 = C_3 - A_3Y_0B_3$ has a positive solution. Therefore the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ has a positive solution. □

Next, we give another theorem about the existence of the positive solution to the system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$. First we propose a lemma as follows.

Lemma 13. *Suppose that $M \in L_{\mathscr{A}}(H_1,H_2)$, $N \in L_{\mathscr{A}}(H_3,H_1)$, and $\overline{R(M^*)}$ and $\overline{R(N)}$ are orthogonally complemented submodules of H_1. Let $T = P_{M^*} + P_N$, and suppose T has a closed range; then $TX = P_N$ and $YT = P_{M^*}$ both have positive solutions.*

In this case, the general positive solutions are of the forms

$$X = P + N_TUN_T, \qquad Y = Q + N_TVN_T, \quad (36)$$

where P and Q are the reduced positive solutions, respectively, U and $V \in L_{\mathscr{A}}(H_1)_+$ are arbitrary. Furthermore, on has $(Q + P)P_{M^*} = P_{M^*}$ and $P_N(Q + P) = P_N$.

Proof. It is clear that $P_{M^*} \geq 0$, $P_N \geq 0$, so $T \geq 0$, $P_NP_{M^*} \geq 0$ and $P_NP_{M^*} \geq 0$. Consider $R(P_N) \subseteq R(T)$, $P_NT^* = P_NT = P_N(P_{M^*} + P_N) = P_NP_{M^*} + P_N \geq 0$, $R(P_{M^*}) \subseteq R(T)$, $P_{M^*}^*T = P_{M^*}T = P_{M^*}(P_{M^*} + P_N) = P_{M^*} + P_{M^*}P_N \geq 0$, then it follows from Lemma 4 that $P \geq 0$, $Q \geq 0$, $R(P), R(Q) \subseteq N(T)^{\perp}$ and the general positive solutions are of the forms

$$X = P + N_TUN_T, \qquad Y = Q + N_TVN_T, \quad (37)$$

where $U, V \in L_{\mathscr{A}}(H_1)_+$ are arbitrary.

Consider $TP = P_N$, $QT = P_{M^*}$, $T, P, Q, P_N, P_{M^*} \geq 0$; then $PT = P_N$, $TQ = P_{M^*}$, and $T(Q+P) = T$, $T(Q+P)P_{M^*} = TP_{M^*}$, $T((Q + P)P_{M^*} - P_{M^*}) = 0$, and $(Q + P)P_{M^*} - P_{M^*} \subseteq N(T)$. By Lemma 4, we know that $(Q + P)P_{M^*} - P_{M^*} \subseteq N(T)^{\perp}$, so $(Q+P)P_{M^*} = P_{M^*}$. Similarly, we can obtain that $(Q+P)P_N = P_N$; then $P_N(Q + P) = P_N$. □

For simplicity, put

$$D = \begin{pmatrix} A_1 \\ B_2^* \end{pmatrix}, \quad E = \begin{pmatrix} C_1 \\ C_2^* \end{pmatrix},$$
$$F = \begin{pmatrix} C_1 A_1^* & C_1 B_2 \\ C_2^* A_1^* & C_2^* B_2 \end{pmatrix}. \tag{38}$$

Y_0 is the common reduced positive solution to $A_1X = C_1$, $XB_2 = C_2$. Consider $M = A_3N_D$, $N = N_DB_3$; L is the reduced solution to $MYN = C_3 - A_3Y_0B_3$. Consider $T = P_{M^*} + P_N$. P is the positive reduced solution to $TX = P_N$. Q is the positive reduced solution to $XT = P_{M^*}$. Y_1 is the positive solution to $XP = LQ$. Y_2 is the positive solution to $QX = PL$. Take

$$R = L + L^* + Y_1 + Y_2, \qquad S = PLQ. \tag{39}$$

By Lemma 6, we know that Y_0 uniquely exists when $\overline{R(D^*)}$ is orthogonally complemented, $F \geq 0$, and $R(E) \subset R(D)$. From Lemma 3, we know that L uniquely exists when $\overline{R(N)}$ and $\overline{R(M^*)}$, are orthogonally complemented, $R(C_3 - A_3Y_0B_3) \subset R(M)$ and $\overline{R((C_3 - A_3Y_0B_3)^*)} \subset R(M^*)$. From Lemma 13, we know that P and Q uniquely exist when $\overline{R(M^*)}$ and $\overline{R(N)}$ are orthogonally complemented, T has a closed range. By Lemmas 4 and 13, we know that Y_1 and Y_2 exist. So S uniquely exists. In fact, if the conditions in the next theorem are satisfied, we can easily get that Y_0, L, P, Q, and S uniquely exist.

Theorem 14. *Let H_i ($i = 1,2,3,4,5$) be Hilbert $\mathscr{A}$-modules, $A_1 \in L_{\mathscr{A}}(H_1,H_2)$, $A_3 \in L_{\mathscr{A}}(H_1,H_4)$, $B_2 \in L_{\mathscr{A}}(H_3,H_1)$, $B_3 \in L_{\mathscr{A}}(H_5,H_1)$, $C_1 \in L_{\mathscr{A}}(H_1,H_2)$, $C_2 \in L_{\mathscr{A}}(H_3,H_1)$, and $C_3 \in L_{\mathscr{A}}(H_5,H_4)$. Suppose $\overline{R(D^*)}$, $\overline{R(M^*)}$, and $\overline{R(N)}$ are orthogonally complemented submodules of H_1, $\overline{R(C_3)} \subseteq R(A_3)$, $R(C_3^*) \subseteq R(B_3^*)$ (or $R(C_3) \subseteq R(A_3)$, $\overline{R(C_3^*)} \subseteq R(B_3^*)$), and T has closed range. The system of operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ has a positive solution $X \in L_{\mathscr{A}}(H_1)$ if and only if*

$$F \geq 0, \qquad S \geq 0, \qquad R(E) \subseteq R(D),$$
$$R(PL) \subseteq R(S), \qquad R(QL^*) \subseteq R(S), \tag{40}$$

in which case the general common positive solution to $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$ can be expressed as

$$X = Y_0 + N_D \left(C + CV_1N_T + N_TV_1^*C + N_TV_1^*CV_1N_T + N_TV_2N_T\right) N_D, \tag{41}$$

where Y_0 is the common positive reduced solution to $A_1X = C_1$, $XB_2 = C_2$, C is the positive reduced solution to $TXT^* = R$, Y_1 and Y_2 are arbitrary positive solutions to $Y_1P = LQ$, $QY_2 = PL$, respectively, such that R is positive, V_1 is arbitrary, and V_2 is arbitrary positive operator in $L_{\mathscr{A}}(H_1)$.

Proof. Suppose X_0 is a positive solution to the system of the adjointable operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$; then X_0 is a positive solution to the operator equations $A_1X = C_1$, $XB_2 = C_2$. It follows from Lemma 6 that $F \geq 0$, $R(E) \subseteq R(D)$, and X_0 has the form

$$X_0 = Y_0 + N_DYN_D, \tag{42}$$

where Y_0 is the positive reduced solution of $A_1X = C_1$, $XB_2 = C_2$, and $Y \in L_{\mathscr{A}}(H_1)_+$ is an arbitrary operator.

Take $X_0 = Y_0 + N_DYN_D$ into $A_3XB_3 = C_3$; we can get that

$$MYN = C_3 - A_3Y_0B_3 \tag{43}$$

has a positive solution. Consider

$$S = PLQ = PP_{M^*}LP_NQ = PP_{M^*}YP_NQ = PYQ. \tag{44}$$

By Lemma 13, we know that $P, Q \geq 0$. Hence, if Y is positive, so is S.

For all $x \in H_1$, $S^*x = 0$; that is, $QL^*Px = 0$; then $L^*Px = 0$ since that $Q \geq 0$. We can get $N(S^*) \subseteq N(L^*P) = N((PL)^*)$; then $R(PL) \subseteq R(S)$. Similarly, $R(QL^*) \subseteq R(S)$.

If $F \geq 0$ and $R(E) \subseteq R(D)$, then equations $A_1X = C_1$, $XB_2 = C_2$ have a positive solution by Lemma 6 and this positive solution can be expressed as

$$X_0 = Y_0 + N_DYN_D, \tag{45}$$

where Y_0 is the positive reduced solution of the system of the adjointable operator equations $A_1X = C_1$, $XB_2 = C_2$, and $Y \in L_{\mathscr{A}}(H_1)_+$ is an arbitrary operator.

Taking $X_0 = Y_0 + N_DYN_D$ into $A_3XB_3 = C_3$, we can get

$$MYN = C_3 - A_3Y_0B_3. \tag{46}$$

Now, we want to show that the equation $MYN = C_3 - A_3Y_0B_3$ has a positive solution. By Lemma 3, we know that the equation $MYN = C_3 - A_3Y_0B_3$ has a solution; then L exists.

We rewrite $Y_1P = LQ$, $QY_2 = PL$ as $Y_1P = LQ = L_1$, $QY_2 = PL = L_2$; then $L_2Q^* = PLQ = S \geq 0$, $L_1^*P = Q^*L^*P = S^* = S \geq 0$, and $R(S) = R(L_1^*) = R(L_2)$. Consider $R(P^*L) \subseteq R(S)$, $R(QL^*) \subseteq R(S)$; the equations $Y_1P = L_1$, $QY_2 = L_2$ both have positive solutions by Lemma 4 and they can be expressed as

$$Y_1 = D_1 + N_PVN_P, \qquad Y_2 = D_2 + N_QWN_Q, \tag{47}$$

where D_1 and D_2 are the positive reduced solutions to the equations $Y_1P = L_1$, $QY_2 = L_2$, respectively, and $V, W \in L_{\mathscr{A}}(H_1)_+$ are arbitrary.

For operator equations $SX = L_2$, $XS = L_1$, we can obtain that $L_2S^* = PLQL^*P \geq 0$, $L_1^*S^* = QL^*PLQ^* \geq 0$ and $R(L_2) \subseteq R(S)$, $R(L_1^*) \subseteq R(S^*)$. By Lemma 4, we can get that

$SX = L_2$ and $XS = L_1$ both have positive solutions. Let Z_1 and Z_2 be the positive reduced solutions, respectively. Hence

$$\begin{aligned}
SZ_1 &= L_2, \qquad PLQZ_1 = PL, \\
PL_1Z_1 &= L, \qquad P(L - L_1Z_1) = 0, \\
Z_2S &= L_1, \qquad Z_2PLQ = LQ, \\
Z_2L_2Q &= LQ, \qquad (L - Z_2L_2)Q = 0, \\
R &= L + L^* + Y_1 + Y_2 \\
&= L + L^* + D_1 + N_PVN_P + D_2 + N_QWN_Q \\
&= L_1Z_1 + Z_1^*L_1^* + D_1 + D_2 + N_PVN_P \\
&\quad + N_QWN_Q + L - L_1Z_1 + L^* - Z_1^*L_1^* \\
&= Z_2SZ_1 + Z_1^*SZ_2^* + D_1 + D_2 \\
&\quad + \begin{pmatrix} N_P & N_Q \end{pmatrix} \begin{pmatrix} V & L - L_1Z_1 \\ L^* - Z_1^*L_1^* & W \end{pmatrix} \begin{pmatrix} N_P \\ N_Q \end{pmatrix} \\
&= (Z_2 + Z_1^*) S (Z_1 + Z_2^*) + D_1 + D_2 \\
&\quad + \begin{pmatrix} N_P & N_Q \end{pmatrix} \begin{pmatrix} V & L - L_1Z_1 \\ L^* - Z_1^*L_1^* & W \end{pmatrix} \begin{pmatrix} N_P \\ N_Q \end{pmatrix}.
\end{aligned} \tag{48}$$

Let $V = I$ and $W = (L^* - Z_1^*L_1^*)(L - L_1Z_1)$. By Lemma 5, $R \geq 0$.

For the operator equation $TXT^* = R$, by Lemma 7, $TXT^* = R$ has a positive solution U and U has the form

$$U = C + CV_1N_T + N_TV_1^*C + N_TV_1^*CV_1N_T + N_TV_2N_T, \tag{49}$$

where C is the positive reduced solution to the operator equation $TXT^* = R$ and $V_1 \in L_{\mathscr{A}}(H_1)$ and $V_2 \in L_{\mathscr{A}}(H_1)_+$ are arbitrary.

Then we claim that $X = Y_0 + N_D(C + CV_1N_T + N_TV_1^*C + N_TV_1^*CV_1N_T + N_TV_2N_T)N_D$ is the positive solution to $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$. In fact, we only need to prove that $MUN = C_3 - A_3Y_0B_3$. Consider

$$\begin{aligned}
MUN &= MQTUT^*PN = MQRPN \\
&= MQ(L + L^* + Y_1 + Y_2)PN \\
&= MQLPN + MQL^*PN + MQY_1PN + MQY_2PN \\
&= MQLPN + MPLQN + MQLQN + MPLPN \\
&= M(Q + P)L(Q + P)N \\
&= M(Q + P)P_{M^*}LP_N(Q + P)N = MP_{M^*}LP_NN \\
&= MLN = C_3 - A_3Y_0B_3.
\end{aligned} \tag{50}$$

Suppose that $\overline{X}$ is a positive solution to the system of the operator equations $A_1X = C_1$, $XB_2 = C_2$, and $A_3XB_3 = C_3$. It follows from Lemma 6 that $\overline{X}$ can be expressed as

$$\overline{X} = Y_0 + N_DYN_D, \quad Y \in L_{\mathscr{A}}(H_1)_+, \tag{51}$$

where Y_0 is the positive reduced solution of the system of the operator equations $A_1X = C_1$, $XB_2 = C_2$. Hence there is a positive operator U such that

$$\overline{X} - Y_0 = N_DUN_D. \tag{52}$$

Let $Y_1 = P_{M^*}UP_{M^*}$, $Y_2 = P_NUP_N$; it follows from

$$\begin{aligned}
Y_1P &= P_{M^*}UP_{M^*}P = P_{M^*}U(P_N - P_NP) \\
&= P_{M^*}UP_N - P_{M^*}UP_NP = P_{M^*}LP_N - P_{M^*}LP_NP \\
&= P_{M^*}L(P_N - P_NP) = P_{M^*}LP_NQ = LQ, \\
QY_2 &= QP_NUP_N = (P_{M^*} - QP_{M^*})UP_N \\
&= P_{M^*}UP_N - QP_{M^*}UP_N = P_{M^*}LP_N - QP_{M^*}LP_N \\
&= (P_{M^*} - QP_{M^*})LP_N = PP_{M^*}LP_N = PL
\end{aligned} \tag{53}$$

that Y_1 and Y_2 are positive solutions to the operator equations $Y_1P = LQ$, $QY_2 = PL$, respectively. Consider

$$\begin{aligned}
R &= L + L^* + Y_1 + Y_2 \\
&= P_{M^*}LP_N + P_NLP_{M^*} + P_{M^*}UP_{M^*} + P_NUP_N \\
&= P_{M^*}UP_N + P_NUP_{M^*} + P_{M^*}UP_{M^*} + P_NUP_N \\
&= (P_{M^*} + P_N)U(P_{M^*} + P_N) \\
&= TUT = TUT^*.
\end{aligned} \tag{54}$$

By Lemma 7, U can be expressed as

$$U = C + CV_1N_T + N_TV_1^*C + N_TV_1^*CV_1N_T + N_TV_2N_T, \tag{55}$$

where C is the positive reduced solution to $TXT^* = R$ and $V_1 \in L_{\mathscr{A}}(H_1)$ and $V_2 \in L_{\mathscr{A}}(H_1)_+$ are arbitrary.

Take $U = C + CV_1N_T + N_TV_1^*C + N_TV_1^*CV_1N_T + N_TV_2N_T$ into $\overline{X} - Y_0 = N_DUN_D$; we know that $\overline{X}$ has the form that is expressed as (41). □

Let $A \in L_{\mathscr{A}}(H_1, H_2)$, $B \in L_{\mathscr{A}}(H_3, H_1)$, and $C \in L_{\mathscr{A}}(H_3, H_1)$, and suppose that $\overline{R(A^*)}$, $\overline{R(B)}$ are orthogonally complemented submodules of H_1. If $\overline{R(C)} \subseteq R(A)$ and $R(C^*) \subseteq R(B^*)$ (or $R(C) \subseteq R(A)$, $\overline{R(C^*)} \subseteq R(B^*)$) we can obtain that the operator equation $AXB = C$ has a solution by Lemma 3. Let L be the reduced solution to $AXB = C$ and $T = P_{A^*} + P_B$ and suppose it has a closed range, where P is the positive reduced solution to $TX = P_B$. Q is the positive reduced solution to $XT = P_{A^*}$. Y_1 is the positive solution to $Y_1P = LQ$. Y_2 is the positive solution to $QY_2 = PL$.

Consider

$$R = L + L^* + Y_1 + Y_2, \qquad S = PLQ. \tag{56}$$

By Theorem 14, we can give necessary and sufficient conditions for the existence of a positive solution to operator equation $AXB = C$.

Corollary 15. *Let the notions and conditions be as described above. Then the operator equation $AXB = C$ has a positive solution in $L_{\mathscr{A}}(H_1)_+$ if and only if*

$$S \geq 0, \qquad R(PL) \subseteq R(S), \qquad R(QL^*) \subseteq R(S), \tag{57}$$

in this case the form of general positive solution to $AXB = C$ is

$$X = D + DV_1N_T + N_TV_1^*D + N_TV_1^*DV_1N_T + N_TV_2N_T, \tag{58}$$

where D is the positive reduced solution to $TXT^* = R$, Y_1 and Y_2 are arbitrary positive solutions to $Y_1P = LQ$, $QY_2 = PL$, respectively, such that R is positive, and $V_1 \in L_{\mathscr{A}}(H_1)$ and $V_2 \in L_{\mathscr{A}}(H_1)_+$ are arbitrary.

The results obtained in Lemma 9 give us the condition of the existence of the positive solution to $AXB = C$, but they are restricted by the assumption that $R(B) \subseteq R(A^*)$. The result in Corollary 15 does not have the constraint mentioned above. Clearly Theorem 14 extends Theorem 3.6 in [10], and Corollary 15 extends Corollary 4.1 in [10].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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