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Research Article

Bounds of the Neuman-Sándor Mean Using Power and Identric Means

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In this paper we find the best possible lower power mean bounds for the Neuman-Sándor mean and present the sharp bounds for the ratio of the Neuman-Sándor and identric means.

1. Introduction

For $p \in \mathbb{R}$ the pth power mean $M_p(a,b)$, Neuman-Sándor Mean $M(a,b)$ [1], and identric mean $I(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad (1)$$

$$M(a,b) = \begin{cases} \dfrac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases} \quad (2)$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \quad (3)$$

respectively, where $\sinh^{-1}(x) = \log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

The main properties for $M_p(a,b)$ and $I(a,b)$ are given in [2]. It is well known that $M_p(a,b)$ is continuously and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Recently, the power, Neuman-Sándor, and identric means have been a subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [3–26].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of two positive numbers a and b with $a \neq b$, respectively. Then, it is well known that the inequalities

$$\begin{aligned} H(a,b) &= M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ &< P(a,b) < I(a,b) < A(a,b) = M_1(a,b) < M(a,b) \\ &< T(a,b) < Q(a,b) = M_2(a,b) < C(a,b), \end{aligned} \quad (4)$$

hold for all $a,b > 0$ with $a \neq b$.

The following sharp bounds for L, I, $(IL)^{1/2}$, and $(I+L)/2$ in terms of power means are presented in [27–32]:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < I^{1/2}(a,b)\,L^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b), \end{gathered} \quad (5)$$

for all $a,b > 0$ with $a \neq b$.

Pittenger [31] found the greatest value r_1 and the least value r_2 such that the double inequality

$$M_{r_1}(a,b) \le L_p(a,b) \le M_{r_2}(a,b), \tag{6}$$

holds for all $a,b > 0$, where $L_r(a,b)$ is the rth generalized logarithmic means which is defined by

$$L_r(a,b) = \begin{cases} \left[\dfrac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & a \neq b,\ r \neq -1,\ r \neq 0, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b,\ r = 0, \\ \dfrac{b-a}{\log b - \log a}, & a \neq b,\ r = -1, \\ a, & a = b. \end{cases} \tag{7}$$

The following sharp power mean bounds for the first Seiffert mean $P(a,b)$ are given in [10, 33]:

$$M_{\log 2/\log \pi}(a,b) < P(a,b) < M_{2/3}(a,b), \tag{8}$$

for all $a,b > 0$ with $a \neq b$.

In [17], the authors answered the question: for $\alpha \in (0,1)$, what are the greatest value p and the least value q such that the double inequality

$$M_p(a,b) < P^{\alpha}(a,b)\, G^{1-\alpha}(a,b) < M_q(a,b) \tag{9}$$

holds for all $a,b > 0$ with $a \neq b$?

Neuman and Sándor [1] established that

$$\begin{gathered} A(a,b) < M(a,b) < \frac{A(a,b)}{\log\left(1+\sqrt{2}\right)}, \\ \frac{\pi}{4}T(a,b) < M(a,b) < T(a,b), \\ M(a,b) < \frac{2A(a,b)+Q(a,b)}{3}, \end{gathered} \tag{10}$$

for all $a,b > 0$ with $a \neq b$.

Let $0 < a,b \le 1/2$ with $a \neq b$, $a' = 1-a$ and $b' = 1-b$. Then, the Ky Fan inequalities

$$\begin{aligned} \frac{G(a,b)}{G(a',b')} &< \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} \\ &< \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \end{aligned} \tag{11}$$

were presented in [1].

In [24], Li et al. found the best possible bounds for the Neuman-Sándor mean $M(a,b)$ in terms of the generalized logarithmic mean $L_r(a,b)$. Neuman [25] and Zhao et al. [26] proved that the inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha)A(a,b) \\ &\quad < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b), \\ &\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) \\ &\quad < \mu C(a,b) + (1-\mu)A(a,b), \\ &\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) \\ &\quad < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) \\ &\quad < \beta_2 G(a,b) + (1-\beta_2)Q(a,b) \end{aligned} \tag{12}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]$, $\beta \ge 1/3$, $\lambda \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$, $\mu \ge 1/6$, $\alpha_1 \ge 2/9$, $\beta_1 \le 1-1/[\sqrt{2}\log(1+\sqrt{2})]$, $\alpha_2 \ge 1/3$, and $\beta_2 \le 1-1/[\sqrt{2}\log(1+\sqrt{2})]$.

In [7], Sándor and Trif proved that the inequalities

$$\begin{aligned} &e^{((a-b)^2/6(a+b)^2)} < \frac{A(a,b)}{I(a,b)} < e^{((a-b)^2/24ab)}, \\ &e^{((a-b)^2/3(a+b)^2)} < \frac{I(a,b)}{G(a,b)} < e^{((a-b)^2/12ab)}, \\ &e^{((a-b)^4/30(a+b)^4)} < \frac{I(a,b)}{A^{2/3}(a,b)\,G^{1/3}(a,b)} \\ &\quad < e^{((a-b)^4/120ab(a+b)^4)} \end{aligned} \tag{13}$$

hold for all $a,b > 0$ with $a \neq b$.

Neuman and Sándor [15] and Gao [20] proved that $\alpha_1 = 1$, $\beta_1 = e/2$, $\alpha_2 = 1$, $\beta_2 = 2\sqrt{2}/e$, $\alpha_3 = 1$, $\beta_3 = 3/e$, $\alpha_4 = e/\pi$, $\beta_4 = 1$, $\alpha_5 = 1$, and $\beta_5 = 2e/\pi$ are the best possible constants such that the double inequalities $\alpha_1 < A(a,b)/I(a,b) < \beta_1$, $\alpha_2 < I(a,b)/M_{2/3}(a,b) < \beta_2$, $\alpha_3 < I(a,b)/He(a,b) < \beta_3$, $\alpha_4 < P(a,b)/I(a,b) < \beta_4$, and $\alpha_5 < T(a,b)/I(a,b) < \beta_5$ hold for all $a,b > 0$ with $a \neq b$, where $He(a,b) = (a+\sqrt{ab}+b)/3 = (2A(a,b)+G(a,b))/3$ is the Heronian mean of a and b.

In [34], Sándor established that

$$He(a,b) < M_{2/3}(a,b), \tag{14}$$

for all $a,b > 0$ with $a \neq b$.

It is not difficult to verify that the inequality

$$\frac{2A(a,b)+Q(a,b)}{3} < \left[He\left(a^2,b^2\right)\right]^{1/2} \tag{15}$$

holds for all $a,b > 0$ with $a \neq b$.

From inequalities (10), (14), and (15), one has

$$M(a,b) < \left[M_{2/3}\left(a^2,b^2\right)\right]^{1/2} = M_{4/3}(a,b), \tag{16}$$

for all $a,b > 0$ with $a \neq b$.

It is the aim of this paper to find the best possible lower power mean bound for the Neuman-Sándor mean $M(a,b)$ and to present the sharp constants α and β such that the double inequality

$$\alpha < \frac{M(a,b)}{I(a,b)} < \beta \tag{17}$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Results

Theorem 1. $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ *is the greatest value such that the inequality*

$$M(a,b) > M_{p_0}(a,b) \tag{18}$$

holds for all $a, b > 0$ with $a \neq b$.

Proof. From (1) and (2), we clearly see that both $M(a,b)$ and $M_p(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$.

Let $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})]$, then from (1) and (2) one has

$$\begin{aligned}&\log M(x,1) - \log M_{p_0}(x,1)\\&= \log\frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}.\end{aligned} \tag{19}$$

Let

$$f(x) = \log\frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}. \tag{20}$$

Then, simple computations lead to

$$\lim_{x\to 1^+} f(x) = 0, \tag{21}$$

$$\lim_{x\to+\infty} f(x) = \frac{1}{p_0}\log 2 - \log\left[2\sinh^{-1}(1)\right] = 0, \tag{22}$$

$$f'(x) = \frac{\left(1+x^{p_0-1}\right)f_1(x)}{(x-1)(x^{p_0}+1)\sinh^{-1}((x-1)/(x+1))}, \tag{23}$$

where

$$\begin{gathered}f_1(x) = -\frac{\sqrt{2}(x-1)(x^{p_0}+1)}{(x+1)(x^{p_0-1}+1)\sqrt{1+x^2}} + \sinh^{-1}\left(\frac{x-1}{x+1}\right),\\ f_1(1) = 0,\end{gathered} \tag{24}$$

$$\lim_{x\to+\infty} f_1(x) = -\sqrt{2} + \sinh^{-1}(1) = -0.5328\cdots < 0, \tag{25}$$

$$f_1'(x) = \frac{\sqrt{2}(x-1)f_2(x)}{(x+1)^2(x^{p_0-1}+1)^2(1+x^2)^{3/2}}, \tag{26}$$

where

$$\begin{aligned}f_2(x) = {}& 1 + x + 2x^2 + (p_0-1)x^{p_0-2} - x^{p_0-1} + x^{p_0+1}\\&-(p_0-1)x^{p_0+2} - 2x^{2p_0-2} - x^{2p_0-1} - x^{2p_0},\end{aligned} \tag{27}$$

$$\begin{gathered}f_2(1) = 0,\\ \lim_{x\to+\infty} f_2(x) = -\infty,\end{gathered} \tag{28}$$

$$\begin{aligned}f_2'(x) = {}& 1 + 4x + (p_0-1)(p_0-2)x^{p_0-3} - (p_0-1)x^{p_0-2}\\&+(p_0+1)x^{p_0} - (p_0-1)(p_0+2)x^{p_0+1}\\&-4(p_0-1)x^{2p_0-3} - (2p_0-1)x^{2p_0-2} - 2p_0x^{2p_0-1},\\ f_2'(1) = {}& 4(4-3p_0) > 0,\end{aligned} \tag{29}$$

$$\lim_{x\to+\infty} f_2'(x) = -\infty, \tag{30}$$

$$\begin{aligned}f_2''(x) = {}& 4 + (p_0-1)(p_0-2)(p_0-3)x^{p_0-4}\\&-(p_0-1)(p_0-2)x^{p_0-3} + p_0(p_0+1)x^{p_0-1}\\&-(p_0-1)(p_0+2)(p_0+1)x^{p_0}\\&-4(p_0-1)(2p_0-3)x^{2p_0-4}\\&-2(2p_0-1)(p_0-1)x^{2p_0-3}\\&-2p_0(2p_0-1)x^{2p_0-2},\\ f_2''(1) = {}& 4(2p_0-1)(4-3p_0) > 0,\end{aligned} \tag{31}$$

$$\lim_{x\to+\infty} f_2''(x) = -\infty, \tag{32}$$

$$f_2'''(x) = (p_0-1)x^{p_0-5}f_3(x), \tag{33}$$

where

$$\begin{aligned}f_3(x) = {}& -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x\\&+p_0(p_0+1)x^3 - p_0(p_0+1)(p_0+2)x^4\\&-8(3-2p_0)(2-p_0)x^{p_0} + 2(2p_0-1)(3-2p_0)x^{p_0+1}\\&-4p_0(2p_0-1)x^{p_0+2}\\ < {}& -(2-p_0)(3-p_0)(4-p_0)\\&-(2-p_0)(3-p_0)x + p_0(p_0+1)x^4\\&-p_0(p_0+1)(p_0+2)x^4 - 8(3-2p_0)(2-p_0)x^{p_0}\\&+2(2p_0-1)(3-2p_0)x^{p_0+2} - 4p_0(2p_0-1)x^{p_0+2}\\ = {}& -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x\\&-p_0(p_0+1)^2x^4 - 8(3-2p_0)(2-p_0)x^{p_0}\\&-2(2p_0-1)(4p_0-3)x^{p_0+2} < 0,\end{aligned} \tag{34}$$

for $x > 1$.

Equation (33) and inequality (34) imply that $f_2''(x)$ is strictly decreasing on $[1,+\infty)$. Then, the inequality (31) and (32) lead to the conclusion that there exists $x_1 > 1$, such that $f_2'(x)$ is strictly increasing on $[1, x_1]$ and strictly decreasing on $[x_1,+\infty)$.

From (29) and (30) together with the piecewise monotonicity of $f_2'(x)$, we clearly see that there exists $x_2 > x_1 > 1$, such that $f_2(x)$ is strictly increasing on $[1, x_2]$ and strictly decreasing on $[x_2,+\infty)$.

It follows from (26)–(28) and the piecewise monotonicity of $f_2(x)$ that there exists $x_3 > x_2 > 1$, such that $f_1(x)$, is strictly increasing on $[1, x_3]$ and strictly decreasing on $[x_3,+\infty)$.

From (23)–(25) and the piecewise monotonicity of $f_1(x)$ we see that there exists $x_4 > x_3 > 1$, such that $f(x)$ is strictly increasing on $(1, x_4]$ and strictly decreasing on $[x_4,+\infty)$.

Therefore, $M(x, 1) > M_{p_0}(x, 1)$ for $x > 1$ follows easily from (19)–(22) and the piecewise monotonicity of $f(x)$.

Next, we prove that $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ is the greatest value such that $M(x, 1) > M_{p_0}(x, 1)$ for all $x > 1$.

For any $\varepsilon > 0$ and $x > 1$, from (1) and (2), one has

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{M_{p_0+\varepsilon}(x,1)}{M(x,1)}\\
&=\lim_{x\to+\infty}\left[\left(\frac{1+x^{p_0+\varepsilon}}{2}\right)^{1/(p_0+\varepsilon)}\frac{2\sinh^{-1}((x-1)/(x+1))}{x-1}\right]\\
&=2^{-1/(p_0+\varepsilon)}\times 2\sinh^{-1}(1)\\
&=2^{\varepsilon/p_0(p_0+\varepsilon)}>1.
\end{aligned}\tag{35}$$

Inequality (35) implies that for any $\varepsilon > 0$, there exists $X = X(\varepsilon) > 1$, such that $M(x, 1) < M_{p_0+\varepsilon}(x, 1)$ for $x \in (X,+\infty)$. □

Remark 2. 4/3 is the least value such that inequality (16) holds for all $a, b > 0$ with $a \neq b$, namely, $M_{4/3}(a, b)$ is the best possible upper power mean bound for the Neuman-Sándor mean $M(a, b)$.

In fact, for any $\varepsilon \in (0, 4/3)$ and $x > 0$, one has

$$\begin{aligned}
&M_{4/3-\varepsilon}(1+x,1)-M(1+x,1)\\
&\quad=\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}.
\end{aligned}\tag{36}$$

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned}
&\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}\\
&\quad=\left[1+\frac{4-3\varepsilon}{6}x+\frac{(4-3\varepsilon)(1-3\varepsilon)}{36}x^2+o\left(x^2\right)\right]^{1/(4/3-\varepsilon)}\\
&\qquad-\frac{x}{x-(1/2)x^2+(5/24)x^3+o(x^3)}\\
&\quad=\left[1+\frac{1}{2}x+\frac{1-3\varepsilon}{24}x^2+o\left(x^2\right)\right]\\
&\qquad-\left[1+\frac{1}{2}x+\frac{1}{24}x^2+o\left(x^2\right)\right]=-\frac{\varepsilon}{8}x^2+o\left(x^2\right).
\end{aligned}\tag{37}$$

Equations (36) and (37) imply that for any $\varepsilon \in (0, 4/3)$ there exists $\delta = \delta(\varepsilon) > 0$, such that $M(1+x, 1) > M_{(4/3)-\varepsilon}(1+x, 1)$ for $x \in (0, \delta)$.

Theorem 3. *For all $a, b > 0$ with $a \neq b$, one has*

$$1<\frac{M(a,b)}{I(a,b)}<\frac{e}{2\log\left(1+\sqrt{2}\right)},\tag{38}$$

with the best possible constants 1 *and* $e/[2\log(1+\sqrt{2})] = 1.5419\ldots$.

Proof. From (2) and (3), we clearly see that both $M(a, b)$ and $I(a, b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$. Let

$$f(x)=\frac{M(x,1)}{I(x,1)}=\frac{e(x-1)}{2x^{x/(x-1)}\sinh^{-1}((x-1)/(x+1))}.\tag{39}$$

Then, simple computations lead to

$$\frac{f'(x)}{f(x)}=\frac{\log x}{(x-1)^2\sinh^{-1}((x-1)/(x+1))}f_1(x),\tag{40}$$

where

$$\begin{aligned}
f_1(x)&=\sinh^{-1}\left(\frac{x-1}{x+1}\right)-\frac{\sqrt{2}(x-1)^2}{(x+1)\sqrt{1+x^2}\log x},\\
&\lim_{x\to1^+}f_1(x)=0,
\end{aligned}\tag{41}$$

$$f_1'(x)=\frac{\sqrt{2}f_2(x)}{x(x+1)^2(1+x^2)^{3/2}\log^2x},\tag{42}$$

where

$$\begin{aligned}
f_2(x)&=x(x+1)\left(1+x^2\right)\log^2x\\
&\quad-x\left(3x^3-x^2+x-3\right)\log x\\
&\quad+(x-1)^2(x+1)\left(1+x^2\right),\\
&f_2(1)=0,
\end{aligned}\tag{43}$$

$$\begin{aligned} f_2'(x) = &\left(4x^3+3x^2+2x+1\right)\log^2 x \\ &+5\left(-2x^3+x^2+1\right)\log x+5x^4 \\ &-7x^3+x^2-x+2, \end{aligned} \tag{44}$$
$$f_2'(1)=0,$$

$$\begin{aligned} f_2''(x) = &\,2\left(6x^2+3x+1\right)\log^2 x \\ &+2\left(-11x^2+8x+2+x^{-1}\right)\log x+20x^3 \\ &-31x^2+7x-1+5x^{-1}, \end{aligned} \tag{45}$$
$$f_2''(1)=0,$$

$$\begin{aligned} f_2'''(x) = &\,6(4x+1)\log^2 x \\ &+2\left(-10x+14+2x^{-1}-x^{-2}\right)\log x \\ &+60x^2-84x+23+4x^{-1}-3x^{-2}, \end{aligned} \tag{46}$$
$$f_2'''(1)=0,$$

$$\begin{aligned} f_2^{(4)}(x) = &\,24\log^2 x+4\left(7+3x^{-1}-x^{-2}+x^{-3}\right)\log x \\ &+120x-104+28x^{-1}+4x^{-3}>0 \end{aligned} \tag{47}$$

for $x>1$.

From (46) and (47), we clearly see that $f_2''(x)$ is strictly increasing on $[1,+\infty)$. Then, (45) leads to the conclusion that $f_2'(x)$ is strictly increasing on $[1,+\infty)$.

Equations (43) and (44) together with the monotonicity of $f_2'(x)$ impliy that $f_2(x)>0$ for $x>1$. Then, (42) leads to the conclusion that $f_1(x)$ is strictly increasing on $[1,+\infty)$.

It follows from equations (40) and (41) together with the monotonicity of $f_1(x)$ that $f(x)$ is strictly increasing on $(1,+\infty)$.

Therefore, Theorem 3 follows from (39) and the monotonicity of $f(x)$ together with the facts that

$$\lim_{x\to+\infty} f(x)=\frac{e}{2\log\left(1+\sqrt{2}\right)}, \qquad \lim_{x\to 1^+} f(x)=1. \tag{48}$$

□

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