Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 861948, 6 pages
http://dx.doi.org/10.1155/2013/861948

Research Article

Inequalities Similar to Hilbert's Inequality

Chang-Jian Zhao

Department of Mathematics, China Jiliang University, Hangzhou 310018, China

Correspondence should be addressed to Chang-Jian Zhao; chjzhao@aliyun.com

Received 23 June 2013; Accepted 4 August 2013

Academic Editor: Wenchang Sun

In the present paper, we establish some new inequalities similar to Hilbert's type inequalities. Our results provide some new estimates to these types of inequalities.

1. Introduction

The well-known classical Hilbert's double-series inequality can be stated as follows [1, page 253].

Theorem A. *If $p_1, p_2 > 1$ such that $1/p_1 + 1/p_2 \geq 1$ and $0 < \lambda = 2 - 1/p_1 - 1/p_2 = 1/q_1 + 1/q_2 \leq 1$, where, as usual, q_1 and q_2 are the conjugate exponents of p_1 and p_2, respectively, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K\left(\sum_{m=1}^{\infty} a_m^{p_1}\right)^{1/p_1}\left(\sum_{n=1}^{\infty} b_n^{p_2}\right)^{1/p_2}, \quad (1)$$

where $K = K(p_1, p_2)$ depends on p_1 and p_2 only.

In recent years, several authors [1–18] have given considerable attention to Hilbert's double-series inequality together with its integral version, inverse version, and various generalizations. In particular, Pachpatte [11] established an inequality similar to inequality (1) as follows.

Theorem 1. *Let $p > 1$ be constant and $1/p + 1/q = 1$. If $a(s)$ and $b(t)$ are real-valued functions defined for $\{0, 1, \ldots, m\}$ and $\{0, 1, \ldots, n\}$, respectively, and $a(0) = b(0) = 0$. Moreover, define the operators ∇ by $\nabla u(t) = u(t) - u(t-1)$. Then,*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)|\,|b(t)|}{qs^{p-1} + pt^{q-1}} \leq \frac{1}{pq} m^{(p-1)/p} n^{(q-1)/q} \times \left(\sum_{s=1}^{m}(m-s+1)\,|\nabla a(s)|^p\right)^{1/p} \times \left(\sum_{t=1}^{n}(n-t+1)\,|\nabla b(t)|^q\right)^{1/q}. \quad (2)$$

The first aim of this paper is to establish a new inequality similar to Hilbert's type inequality. Our result provides new estimates to this type of inequality.

Theorem 2. *Let $p > 1$ be constants, and $1/p + 1/q = 1$. For $i = 1, 2$, let $a_i(s_i, t_i)$ be real-valued functions defined for (s_i, t_i), where $s_i = 1, 2, \ldots, m_i$; $t_i = 1, 2, \ldots, n_i$, and let m_i, n_i be natural numbers. Let $a_i(0, t_i) = a_i(s_i, 0) = 0$, and define the operators ∇_1, ∇_2 by*

$$\begin{aligned} \nabla_1 v_i(s_i, t_i) &= v_i(s_i, t_i) - v_i(s_i - 1, t_i), \\ \nabla_2 v_i(s_i, t_i) &= v_i(s_i, t_i) - v_i(s_i, t_i - 1). \end{aligned} \quad (3)$$

Then,

$$\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(|a_1(s_1,t_1)|^p + |a_2(s_2,t_2)|^q\right) \times \left(\Gamma_{p,q}(s_1,t_1,s_2,t_2) \cdot \max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}\right)^{-1}\right)$$

$$\leq \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \times \left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1) \times |\nabla_2\nabla_1 a_1(s_1,t_1)|^p\right)^{1/p} \times \left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}(m_2-s_2+1)(n_2-t_2+1) \times |\nabla_2\nabla_1 a_2(s_2,t_2)|^p\right)^{1/p}, \tag{4}$$

where

$$\Gamma_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p \times \left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{-1}\right), \tag{5}$$

$$S(h) = \frac{h^{1/(h-1)}}{e\log h^{1/(h-1)}}, \quad h \neq 1. \tag{6}$$

Remark 3. Inequality (4) is just a similar version of the following inequality established by Pachpatte [11]:

$$\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{|a(s,t)||b(k,r)|}{q(st)^{p-1}+p(kr)^{q-1}}\right) \leq \frac{1}{pq}(xy)^{1/q}(zw)^{1/p}\left(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1) \times (y-t+1) \times |\nabla_2\nabla_1 a(s,t)|^p\right)^{1/p} \times \left(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1) \times |\nabla_2\nabla_1 b(k,r)|^q\right)^{1/q}. \tag{7}$$

On the other hand, let $a_1(s_1,t_1)$ and $a_2(s_2,t_2)$ change to $a_1(s_1)$ and $a_2(s_2)$, respectively, and, with appropriate transformation, we have

$$\sum_{s_1=1}^{m_1}\sum_{s_2=1}^{m_2}\frac{|a_1(s_1)|^p+|a_2(s_2)|^q}{\Gamma_{p,q}(s_1,s_2)\cdot\max\{ps_1^{p/q},qs_2^{q/p}\}} \leq \frac{1}{pq}m_1^{1/q}m_2^{1/p} \times \left(\sum_{\tau_1=1}^{m_1}(m_1-\tau_1+1)|\nabla a_1(\tau_1)|^p\right)^{1/p} \times \left(\sum_{\tau_2=1}^{m_2}(m_2-\tau_2+1)|\nabla a_2(\tau_2)|^q\right)^{1/q}, \tag{8}$$

where

$$\Gamma_{p,q}(s_1,s_2) = pqS\left(\frac{\sum_{\tau_1=1}^{s_1}|\nabla a_1(\tau_1)|^p}{\sum_{\tau_2=1}^{s_2}|\nabla a_2(\tau_2)|^q}\right), \tag{9}$$

and $S(h)$ is as in (6). This is just a similar version of inequality (2) in Theorem 1.

The integral analogue of inequality (1) in Theorem A is as follows [1, page 254].

Theorem B. *Let p_1, p_2, q_1, q_2, and λ be as in Theorem A. If $f \in L^p(0,\infty)$ and $g \in L^q(0,\infty)$, then*

$$\iint_0^\infty \frac{f(x)g(x)}{(x+y)^\lambda}dx\,dy \leq K\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(y)\,dy\right)^{1/q}, \tag{10}$$

where $K = K(p,q)$ depends on p and q only.

In [11], Pachpatte also established a similar version of inequality (10) as follows.

Theorem 4. *Let $p > 1$ be constants, and $1/p+1/q = 1$. If $f(s)$ and $g(t)$ are real-valued continuous functions defined on $[0,x)$ and $[0,y)$, respectively, and let $f(0) = g(0) = 0$. Then,*

$$\int_0^x\int_0^y \frac{|f(s)||g(t)|}{qs^{p-1}+pt^{q-1}}dt\,ds \leq \frac{1}{pq}x^{(p-1)/p}y^{(q-1)/q} \times \left(\int_0^x (x-s)|f'(s)|^p ds\right)^{1/p} \times \left(\int_0^y (y-t)|g'(t)|^q dt\right)^{1/q}. \tag{11}$$

Another aim of this paper is to establish a new integral inequality similar to Hilbert's type inequality.

Theorem 5. *Let $p > 1$, and $1/p + 1/q = 1$. For $i = 1, 2$, let $h_i \geq 1$, $f_i(s_i, t_i)$ be real-valued differentiable functions defined on $[0, x_i) \times [0, y_i)$, where $x_i \in (0, \infty)$, $y_i \in (0, \infty)$, and $f_i(0, t_i) = f_i(s_i, 0) = 0$. As usual, partial derivatives of f_i are denoted by $D_1 f_i, D_2 f_i, D_{12} f_i = D_{21} f_i$, and so forth. Let*

$$D_{12}^* f_i(s_i, t_i) = D_2\left(h_i f_i^{h_i - 1}(s_i, t_i) \cdot D_1 f_i(s_i, t_i)\right). \tag{12}$$

Then,

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{L_{p,q}(s_1,t_1,s_2,t_2)\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}}ds_1dt_1\right)ds_2dt_2 \\
&\quad\leq \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p}\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\left|D_{12}^* f_1(s_1,t_1)\right|^p dt_1ds_1\right)^{1/p} \\
&\qquad\times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\left|D_{12}^* f_2(s_2,t_2)\right|^q dt_2ds_2\right)^{1/q},
\end{aligned} \tag{13}$$

where

$$L_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\frac{\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1}{\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2}\right), \tag{14}$$

and $S(h)$ is as in (6).

Remark 6. Inequality (13) is just a similar version of the following inequality established by Pachpatte [11]:

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1(s_1,t_1)\right|\left|f_2(s_2,t_2)\right|}{q(s_1t_1)^{p-1} + p(s_2t_2)^{q-1}}ds_1dt_1\right)ds_2dt_2 \\
&\quad\leq \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p} \\
&\qquad\times\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\right. \\
&\qquad\qquad\left.\times\left|D_2D_1f_1(s_1,t_1)\right|^p ds_1dt_1\right)^{1/p} \\
&\qquad\times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\right. \\
&\qquad\qquad\left.\times\left|D_2D_1f_2(s_2,t_2)\right|^q ds_2dt_2\right)^{1/q}.
\end{aligned} \tag{15}$$

On the other hand, let $f_1(s_1, t_1)$ and $f_2(s_2, t_2)$ change to $f_1(s_1)$ and $f_2(s_2)$, respectively, and, with appropriate transformation, we have

$$\begin{aligned}
&\int_0^{x_1}\int_0^{x_2}\frac{\left|f_1^{h_1}(s_1)\right|^p + \left|f_2^{h_2}(s_2)\right|^q}{L_{p,q}(s_1,s_2)\ \max\left\{ps_1^{p/q}, qs_2^{q/p}\right\}}ds_1dt_1 \\
&\quad\leq \frac{1}{pq}x_1^{1/q}x_2^{1/p}\left(\int_0^{x_1}(x_1-s_1)\left|f_1'(s_1)\right|^p ds_1\right)^{1/p} \\
&\qquad\times\left(\int_0^{x_2}(x_2-s_2)\left|f_2'(s_2)\right|^q ds_2\right)^{1/q},
\end{aligned} \tag{16}$$

where

$$L_{p,q}(s_1,s_2) = pqS\left(\frac{\int_0^{s_1}(s_1-\sigma_1)\left|f_1'(\sigma_1)\right|^p d\sigma_1}{\int_0^{s_2}(s_2-\sigma_2)\left|f_2'(\sigma_2)\right|^q d\sigma_2}\right). \tag{17}$$

This is just a similar version of inequality (11) in Theorem 4.

2. Proof of Theorems

Proof of Theorem 2. From the hypotheses of Theorem 2, we have

$$\begin{aligned}
\left|a_1(s_1,t_1)\right| &\leq \sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}\left|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)\right|, \\
\left|a_2(s_2,t_2)\right| &\leq \sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}\left|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)\right|.
\end{aligned} \tag{18}$$

By using Hölder's inequality and noticing the reverse Young's inequality [19],

$$s_1^{1/\alpha}s_2^{1/\beta}S\left(\frac{s_1}{s_2}\right) \geq \frac{s_1}{\alpha} + \frac{s_2}{\beta}, \tag{19}$$

for positive real numbers s_1, s_2 and $1/\alpha + 1/\beta = 1$, $\alpha > 1$, where $S(h)$ is as in (6). Hence,

$$\begin{aligned}
&\frac{|a_1(s_1,t_1)|^p + |a_2(s_2,t_2)|^q}{\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}} \\
&\quad\le \frac{1}{p}\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p \\
&\qquad + \frac{1}{q}\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q \\
&\quad\le S\left(\frac{\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p}{\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q}\right) \\
&\qquad\times\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad\times\left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}.
\end{aligned} \tag{20}$$

Dividing both sides of (20) by

$$\Gamma_{p,q}(s_1,t_1,s_2,t_2) = pqS\left(\frac{\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p}{\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q}\right), \tag{21}$$

taking the sum of both sides of (20) over t_i and s_i from 1 to m_i and n_i $(i = 1,2)$, respectively, and making use of Hölder's inequality, we have

$$\begin{aligned}
&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(|a_1(s_1,t_1)|^p + |a_2(s_2,t_2)|^q\right)\right. \\
&\qquad\times\left(\Gamma_{p,q}(s_1,t_1,s_2,t_2)\right. \\
&\qquad\left.\left.\cdot\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}\right)^{-1}\right) \\
&\quad\le \frac{1}{pq}\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\left(\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad\times\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\left(\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}. \\
&\quad\le \frac{1}{pq}(m_1n_1)^{1/q} \\
&\qquad\times\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\sum_{\xi_1=1}^{s_1}\sum_{\eta_1=1}^{t_1}|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad\times(m_2n_2)^{1/p} \\
&\qquad\times\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}\sum_{\xi_2=1}^{s_2}\sum_{\eta_2=1}^{t_2}|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q}. \\
&\quad= \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \\
&\qquad\times\left(\sum_{\xi_1=1}^{m_1}\sum_{\eta_1=1}^{n_1}(m_1-\xi_1+1)\right. \\
&\qquad\left.\times(n_1-\eta_1+1)|\nabla_2\nabla_1 a_1(\xi_1,\eta_1)|^p\right)^{1/p} \\
&\qquad\times\left(\sum_{\xi_2=1}^{m_2}\sum_{\eta_2=1}^{n_2}(m_2-\xi_2+1)(n_2-\eta_2+1)\right. \\
&\qquad\left.\times|\nabla_2\nabla_1 a_2(\xi_2,\eta_2)|^q\right)^{1/q} \\
&\quad= \frac{1}{pq}(m_1n_1)^{1/q}(m_2n_2)^{1/p} \\
&\qquad\times\left(\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}(m_1-s_1+1)(n_1-t_1+1)\right. \\
&\qquad\left.\times|\nabla_2\nabla_1 a_1(s_1,t_1)|^p\right)^{1/p} \\
&\qquad\times\left(\sum_{s_2=1}^{m_2}\sum_{t_2=1}^{n_2}(m_2-s_2+1)(n_2-t_2+1)\right. \\
&\qquad\left.\times|\nabla_2\nabla_1 a_2(s_2,t_2)|^p\right)^{1/p}.
\end{aligned} \tag{22}$$

This completes the proof. □

Proof of Theorem 5. From the hypotheses of Theorem 5, we obtain for $i = 1,2$:

$$\begin{aligned}
f_i^{h_i}(s_i,t_i) &= f_i^{h_i}(s_i,t_i) - f_i^{h_i}(0,t_i) - f_i^{h_i}(s_i,0) + f_i^{h_i}(0,0) \\
&= \int_0^{s_i} D_1 f_i^{h_i}(\xi_i,t_i)\,d\xi_i - \int_0^{s_i} D_1 f_i^{h_i}(\xi_i,0)\,d\xi_i \\
&= \int_0^{s_i}\left(D_1 f_i^{h_i}(\xi_i,t_i) - D_1 f_i^{h_i}(\xi_i,0)\right)d\xi_i
\end{aligned}$$

$$= \int_0^{s_i}\int_0^{t_i} D_2\left(h_i f_i^{h_i-1}(\xi_i,\eta_i)\cdot D_1 f_i(\xi_i,\eta_i)\right)d\eta_i d\xi_i$$
$$= \int_0^{s_i}\int_0^{t_i} D_{12}^* f_i(\xi_i,\eta_i)\,d\eta_i d\xi_i. \tag{23}$$

From (23), Hölder's integral inequality and in view of the reverse Young's inequality (19), we have

$$\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}} \le \frac{1}{p}\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1 + \frac{1}{q}\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2.$$
$$\le S\left(\frac{\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1}{\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2}\right) \times\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right)^{1/p} \times\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2\right)^{1/q}. \tag{24}$$

Integrating both sides of (24) over s_i and t_i from 1 to x_i and y_i $(i = 1, 2)$, respectively, and by using Hölder's integral inequality, we arrive at

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\left(\int_0^{x_2}\int_0^{y_2}\frac{\left|f_1^{h_1}(s_1,t_1)\right|^p + \left|f_2^{h_2}(s_2,t_2)\right|^q}{L_{p,q}(s_1,t_1,s_2,t_2)\max\left\{p(s_1t_1)^{p/q}, q(s_2t_2)^{q/p}\right\}}ds_1dt_1\right)ds_2dt_2\\
&\le \frac{1}{pq}\int_0^{x_1}\int_0^{y_1}\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right)^{1/p}ds_1dt_1\\
&\quad\times\int_0^{x_2}\int_0^{y_2}\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^q d\eta_2 d\xi_2\right)^{1/q}ds_2dt_2.\\
&\le \frac{1}{pq}(x_1y_1)^{1/q}\left(\int_0^{x_1}\int_0^{y_1}\left(\int_0^{s_1}\int_0^{t_1}\left|D_{12}^* f_1(\xi_1,\eta_1)\right|^p d\eta_1 d\xi_1\right)dt_1ds_1\right)^{1/p}\\
&\quad\times(x_2y_2)^{1/p}\left(\int_0^{x_2}\int_0^{y_2}\left(\int_0^{s_2}\int_0^{t_2}\left|D_{12}^* f_2(\xi_2,\eta_2)\right|^p d\eta_2 d\xi_2\right)dt_2ds_2\right)^{1/p}\\
&= \frac{1}{pq}(x_1y_1)^{1/q}(x_2y_2)^{1/p}\left(\int_0^{x_1}\int_0^{y_1}(x_1-s_1)(y_1-t_1)\left|D_{12}^* f_1(s_1,t_1)\right|^p dt_1ds_1\right)^{1/p}\\
&\quad\times\left(\int_0^{x_2}\int_0^{y_2}(x_2-s_2)(y_2-t_2)\left|D_{12}^* f_2(s_2,t_2)\right|^q dt_2ds_2\right)^{1/q}.
\end{aligned} \tag{25}$$

This completes the proof. □

Acknowledgment

The research is supported by the National Natural Science Foundation of China (11371334).

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, Mass, USA, 1934.

[2] B. G. Pachpatte, "On some new inequalities similar to Hilbert's inequality," *Journal of Mathematical Analysis and Applications*, vol. 226, no. 1, pp. 166–179, 1998.

[3] G. D. Handley, J. J. Koliha, and J. E. Pečarić, "New Hilbert-Pachpatte type integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 238–250, 2001.

[4] M. Gao and B. Yang, "On the extended Hilbert's inequality," *Proceedings of the American Mathematical Society*, vol. 126, no. 3, pp. 751–759, 1998.

[5] K. Jichang, "On new extensions of Hilbert's integral inequality," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 608–614, 1999.

[6] B. Yang, "On new generalizations of Hilbert's inequality," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 1, pp. 29–40, 2000.

[7] C. J. Zhao, "On inverses of disperse and continuous Pachpatte's inequalities," *Acta Mathematica Sinica*, vol. 46, no. 6, pp. 1111–1116, 2003.

[8] C. J. Zhao, "Generalization on two new Hilbert type inequalities," *Journal of Mathematics*, vol. 20, no. 4, pp. 413–416, 2000.

[9] C. J. Zhao and L. Debnath, "Some new inverse type Hilbert integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 1, pp. 411–418, 2001.

[10] G. D. Handley, J. J. Koliha, and J. Pečarić, "A Hilbert type inequality," *Tamkang Journal of Mathematics*, vol. 31, no. 4, pp. 311–315, 2000.

[11] B. G. Pachpatte, "Inequalities similar to certain extensions of Hilbert's inequality," *Journal of Mathematical Analysis and Applications*, vol. 243, no. 2, pp. 217–227, 2000.

[12] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, Germany, 1961.

[13] C. J. Zhao, J. Pecarić, and G. S. Leng, "Inverses of some new inequalities similar to Hilbert's inequalities," *Taiwanese Journal of Mathematics*, vol. 10, no. 3, pp. 699–712, 2006.

[14] S. S. Dragomir and Y.-H. Kim, "Hilbert-Pachpatte type integral inequalities and their improvement," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 16, 2003.

[15] G. A. Anastassiou, "Hilbert-Pachpatte type fractional integral inequalities," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1539–1550, 2009.

[16] J. Jin and L. Debnath, "On a Hilbert-type linear series operator and its applications," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 2, pp. 691–704, 2010.

[17] B. Yang, "A half-discrete Hilbert-type inequality with a non-homogeneous kernel and two variables," *Mediterranean Journal of Mathematics*, vol. 10, no. 2, pp. 677–692, 2013.

[18] K. Jichang and L. Debnath, "On Hilbert type inequalities with non-conjugate parameters," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 813–818, 2009.

[19] M. Tominaga, "Specht's ratio in the Young inequality," *Scientiae Mathematicae Japonicae*, vol. 55, no. 3, pp. 583–588, 2002.