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Research Article

A New Reversed Version of a Generalized Sharp Hölder's Inequality and Its Applications

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We present a new reversed version of a generalized sharp Hölder's inequality which is due to Wu and then give a new refinement of Hölder's inequality. Moreover, the obtained result is used to improve the well-known Popoviciu-Vasić inequality. Finally, we establish the time scales version of Beckenbach-type inequality.

1. Introduction

The classical Hölder's inequality states that if $a_k \geq 0$, $b_k \geq 0$ $(k = 1, 2, \ldots, n)$, $p > 1$, and $1/p + 1/q = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{1}$$

The inequality is reversed for $p < 1$ $(p \neq 0)$, (for $p < 0$, we assume that $a_k, b_k > 0$).

As is well known, Hölder's inequality plays an important role in different branches of modern mathematics such as classical real and complex analysis, probability and statistics, numerical analysis, and qualitative theory of differential equations and their applications. Various refinements, generalizations, and applications of inequality (1) and its series analogues in different areas of mathematics have appeared in the literature. For example, Abramovich et al. [1] presented a new generalization of Hölder's inequality and its reversed version in discrete and integral forms. Ivanković et al. [2] presented the properties of several mappings which have arisen from the Minkowski inequality and then gave some refinements of the Hölder inequality. Liu [3] obtained Hölder's inequality in fuzzy set theory and rough set theory. Nikolova and Varošanec [4] obtained some new refinements of the classical Hölder inequality by using a convex function. For detailed expositions, the interested reader may consult [1–18] and the references therein.

Among various refinements of (1), Hu in [9] established the following interesting sharpness of Hölder's inequality.

Theorem A. *Let $p \geq q > 0$, $1/p + 1/q = 1$, let $A_n, B_n \geq 0$, $\sum_n A_n^p < \infty$, and $\sum_n B_n^q < \infty$, and let $1 - e_n + e_m \geq 0$, $\sum_n |e_n| < \infty$. Then,*

$$\begin{aligned}\sum_n A_n B_n \leq{}& \left(\sum_n B_n^q\right)^{1/q-1/p} \\ &\times \left\{\left[\left(\sum_n B_n^q\right)\left(\sum_n A_n^p\right)\right]^2 \right. \\ &- \left[\left(\sum_n B_n^q e_n\right)\left(\sum_n A_n^p\right) \right. \\ &\left.\left. - \left(\sum_n B_n^q\right)\left(\sum_n A_n^p e_n\right)\right]^2\right\}^{1/2p}.\end{aligned} \tag{2}$$

In 2007, Wu [18] presented the generalization of Hu's result as follows.

Theorem B. *Let $A_r \geq 0, B_r > 0$ $(r = 1, 2, \ldots, n)$, let $1 - e_r + e_s \geq 0$ $(r, s = 1, 2, \ldots, n)$, and let $p \geq q > 0$, $\mu = \min\{1/p + 1/q, 1\}$. Then,*

$$\sum_{r=1}^{n} A_r B_r \leq n^{1-\mu} \left(\sum_{r=1}^{n} B_r^q \right)^{1/q-1/p} \times \left\{ \left[\left(\sum_{r=1}^{n} B_r^q \right) \left(\sum_{r=1}^{n} A_r^p \right) \right]^2 - \left[\left(\sum_{r=1}^{n} B_r^q e_r \right) \left(\sum_{r=1}^{n} A_r^p \right) - \left(\sum_{r=1}^{n} B_r^q \right) \left(\sum_{r=1}^{n} A_r^p e_r \right) \right]^2 \right\}^{1/2p}. \tag{3}$$

Theorem C. *Let $f(x)$, $g(x)$, and $e(x)$ be integrable functions defined on $[a, b]$ and $f(x) \geq 0$, $g(x) > 0$, $1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $p \geq q > 0$, $1/p + 1/q \leq 1$. Then,*

$$\int_a^b f(x) g(x) dx \leq (b-a)^{1-1/p-1/q} \left(\int_a^b g^q(x) dx \right)^{1/q-1/p} \times \left[\left(\int_a^b g^q(x) dx \int_a^b f^p(x) dx \right)^2 - \left(\int_a^b g^q(x) e(x) dx \int_a^b f^p(x) dx - \int_a^b g^q(x) dx \int_a^b f^p(x) e(x) dx \right)^2 \right]^{1/2p}. \tag{4}$$

Recently, Tian in [13] proved the following reversed versions of inequalities (3) and (4).

Theorem D. *Let $A_r > 0$, $B_r > 0$ $(r = 1, 2, \ldots, n)$, let $1 - e_r + e_s \geq 0$ $(r, s = 1, 2, \ldots, n)$, and let $q < 0$, $1/p + 1/q \geq 0$, $\mu = \max\{1/p + 1/q, 1\}$, $\lambda = \max\{1/q, -1\}$. Then,*

$$\sum_{r=1}^{n} A_r B_r \leq n^{1-\mu} \left(\sum_{r=1}^{n} A_r^p \right)^{1/p} \left(\sum_{r=1}^{n} B_r^q \right)^{1/q} \times \left[1 - \left(\frac{\sum_{r=1}^{n} A_r B_r e_r}{\sum_{r=1}^{n} A_r B_r} - \frac{\sum_{r=1}^{n} B_r^q e_r}{\sum_{r=1}^{n} B_r^q} \right)^2 \right]^{\lambda/2}. \tag{5}$$

Theorem E. *Let $f(x)$, $g(x)$, and $e(x)$ be integrable functions defined on $[a, b]$ and $f(x) > 0$, $g(x) > 0$, and $1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $q < 0$, $1/p + 1/q \geq 1$, $\lambda = \max\{-1, 1/q\}$. Then,*

$$\int_a^b f(x) g(x) dx \geq (b-a)^{1-1/p-1/q} \times \left(\int_a^b f^p(x) dx \right)^{1/p} \left(\int_a^b g^q(x) dx \right)^{1/q} \times \left[1 - \left(\frac{\int_a^b f(x) g(x) e(x) dx}{\int_a^b f(x) g(x) dx} - \frac{\int_a^b g^q(x) e(x) dx}{\int_a^b g^q(x) dx} \right)^2 \right]^{\lambda/2}. \tag{6}$$

The aim of this paper is to give new reversed versions of (3) and (4). Moreover, two applications of the obtained results are presented. The rest of this paper is organized as follows. In Section 2, we present reversed versions of (3) and (4). Moreover, we give a new refinement of Hölder's inequality. In Section 3, we apply the obtained result to improve the Popoviciu-Vasić inequality. Furthermore, we establish the time scales version of Beckenbach-type inequality.

2. A New Reversed Version of a Generalized Sharp Hölder's Inequality

In order to prove the main results, we need the following lemmas.

Lemma 1 (see, e.g., [11, page 12]). *Let $A_{kj} > 0$ $(j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n)$, $\sum_{j=1}^{m} 1/p_j \leq 1$. If $p_1 > 0$, $p_j < 0$ $(j = 2, 3, \ldots, m)$, then*

$$\sum_{k=1}^{n} \prod_{j=1}^{m} A_{kj} \geq \prod_{j=1}^{m} \left(\sum_{k=1}^{n} A_{kj}^{p_j} \right)^{1/p_j}. \tag{7}$$

Lemma 2 (see [19, page 12]). *If $x > -1$, $\alpha \geq 1$, or $\alpha < 0$, then*

$$(1 + x)^{\alpha} \geq 1 + \alpha x. \tag{8}$$

The inequality is reversed for $0 < \alpha \leq 1$.

Lemma 3 (see [7, page 27]). *If $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, \ldots, n$, and $0 < p \leq 1$, then*

$$\sum_{i=1}^{n} \lambda_i x_i^p \leq \left(\sum_{i=1}^{n} \lambda_i \right)^{1-p} \left(\sum_{i=1}^{n} \lambda_i x_i \right)^p. \tag{9}$$

The inequality is reversed for $p \geq 1$ or $p < 0$.

Next, we give a reversed version of inequality (3) as follows.

Theorem 4. *Let $A_r \geq 0, B_r > 0$ $(r = 1, 2, \ldots, n)$, let $1 - e_r + e_s \geq 0$ $(r, s = 1, 2, \ldots, n)$, and let $q < 0$, $p > 0$, $\rho = \max\{1/p + 1/q, 1\}$. Then,*

$$\begin{aligned}\sum_{r=1}^{n} A_r B_r \geq n^{1-\rho} &\left(\sum_{r=1}^{n} A_r^p\right)^{1/p-1/q} \\ &\times \left\{\left[\left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q\right)\right]^2 \right. \\ &\quad - \left[\left(\sum_{r=1}^{n} A_r^p e_r\right)\left(\sum_{r=1}^{n} B_r^q\right) \right. \\ &\quad \left.\left. - \left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q e_r\right)\right]^2\right\}^{1/2q}.\end{aligned} \tag{10}$$

Proof. We first consider the case (I) $1/p + 1/q \leq 1$. On one hand, performing some simple computations, we have

$$\begin{aligned}&\sum_{r=1}^{n} A_r B_r \sum_{s=1}^{n} A_s B_s \left(1 - e_r + e_s\right) \\ &\quad = \sum_{s=1}^{n}\sum_{r=1}^{n} A_r B_r A_s B_s - \sum_{s=1}^{n}\sum_{r=1}^{n} A_r B_r A_s B_s e_r \\ &\qquad + \sum_{s=1}^{n}\sum_{r=1}^{n} A_r B_r A_s B_s e_s \\ &\quad = \left(\sum_{k=1}^{n} A_k B_k\right)^2.\end{aligned} \tag{11}$$

On the other hand, by using inequality (9), we have

$$\begin{aligned}&\sum_{r=1}^{n} A_r B_r \sum_{s=1}^{n} A_s B_s \left(1 - e_r + e_s\right)^{1/p+1/q} \\ &\quad = \sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s \left(1 - e_r + e_s\right)^{1/p+1/q} \\ &\quad \leq \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s\right)^{1-1/p-1/q} \\ &\qquad \times \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s \left(1 - e_r + e_s\right)\right)^{1/p+1/q} \\ &\quad = \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s\right)^{1-1/p-1/q} \\ &\qquad \times \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s - \sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s e_r \right. \\ &\qquad\quad \left. + \sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s e_s\right)^{1/p+1/q} \\ &\quad = \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s\right)^{1-1/p-1/q} \\ &\qquad \times \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s\right)^{1/p+1/q} \\ &\quad = \sum_{r=1}^{n}\sum_{s=1}^{n} A_r B_r A_s B_s = \left(\sum_{r=1}^{n} A_r B_r\right)^2.\end{aligned} \tag{12}$$

By using inequality (7), we have

$$\begin{aligned}&\sum_{r=1}^{n} A_r B_r \sum_{s=1}^{n} A_s B_s \left(1 - e_r + e_s\right)^{1/p+1/q} \\ &\quad \geq \sum_{r=1}^{n} A_r B_r \left(\sum_{s=1}^{n} A_s^p \left(1 - e_r + e_s\right)\right)^{1/p} \\ &\qquad \times \left(\sum_{s=1}^{n} B_s^q \left(1 - e_r + e_s\right)\right)^{1/q} \\ &\quad = \sum_{r=1}^{n} \left[\left(\sum_{s=1}^{n} A_r^p A_s^p \left(1 - e_r + e_s\right)\right)^{1/p-1/q} \right. \\ &\qquad \times \left(\sum_{s=1}^{n} A_r^p B_s^q \left(1 - e_r + e_s\right)\right)^{1/q} \\ &\qquad \left. \times \left(\sum_{s=1}^{n} B_r^q A_s^p \left(1 - e_r + e_s\right)\right)^{1/q}\right].\end{aligned} \tag{13}$$

Consequently, according to $(1/p - 1/q) + 1/q + 1/q \leq 1$, by using inequality (7) on the right side of (13), we observe that

$$\begin{aligned}&\sum_{r=1}^{n} A_r B_r \sum_{s=1}^{n} A_s B_s \left(1 - e_r + e_s\right)^{1/p+1/q} \\ &\quad \geq \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r^p A_s^p \left(1 - e_r + e_s\right)\right)^{1/p-1/q} \\ &\qquad \times \left(\sum_{r=1}^{n}\sum_{s=1}^{n} A_r^p B_s^q \left(1 - e_r + e_s\right)\right)^{1/q} \\ &\qquad \times \left(\sum_{r=1}^{n}\sum_{s=1}^{n} B_r^q A_s^p \left(1 - e_r + e_s\right)\right)^{1/q} \\ &\quad = \left(\sum_{r=1}^{n} A_r^p\right)^{2/p-2/q}\end{aligned}$$

$$\times\left[\left(\sum_{r=1}^{n}A_r^p\sum_{s=1}^{n}B_s^q-\sum_{r=1}^{n}A_r^pe_r\sum_{s=1}^{n}B_s^q+\sum_{r=1}^{n}A_r^p\sum_{s=1}^{n}B_s^qe_s\right)\right.$$
$$\left.\times\left(\sum_{r=1}^{n}B_r^q\sum_{s=1}^{n}A_s^p-\sum_{r=1}^{n}B_r^qe_r\sum_{s=1}^{n}A_s^p+\sum_{r=1}^{n}B_r^q\sum_{s=1}^{n}A_s^pe_s\right)\right]^{1/q}$$
$$=\left(\sum_{r=1}^{n}A_r^p\right)^{2/p-2/q}$$
$$\times\left\{\left[\left(\sum_{r=1}^{n}A_r^p\right)\left(\sum_{r=1}^{n}B_r^q\right)\right]^2\right.$$
$$\left.-\left[\left(\sum_{r=1}^{n}A_r^pe_r\right)\left(\sum_{r=1}^{n}B_r^q\right)-\left(\sum_{r=1}^{n}A_r^p\right)\left(\sum_{r=1}^{n}B_r^qe_r\right)\right]^2\right\}^{1/q}. \tag{14}$$

Combining inequalities (12) and (14) leads to inequality (10) immediately.

Secondly, we consider the case (II) $1/p+1/q\geq 1$. Let $1/p+1/q=t$ $(t\geq 1)$, which implies that $1/pt+1/qt=1$. From Hölder's inequality and (7), we have

$$\begin{aligned}
&\sum_{r=1}^{n}A_rB_r\sum_{s=1}^{n}A_sB_s\left(1-e_r+e_s\right)\\
&\quad=\sum_{r=1}^{n}A_rB_r\sum_{s=1}^{n}A_sB_s\left(1-e_r+e_s\right)^{1/pt+1/qt}\\
&\quad\geq\sum_{r=1}^{n}A_rB_r\left[\left(\sum_{s=1}^{n}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/pt}\right.\\
&\qquad\left.\times\left(\sum_{s=1}^{n}B_s^{qt}\left(1-e_r+e_s\right)\right)^{1/qt}\right]\\
&\quad=\sum_{r=1}^{n}\left[\left(\sum_{s=1}^{n}A_r^{pt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/pt-1/qt}\right.\\
&\qquad\times\left(\sum_{s=1}^{n}B_r^{qt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/qt}\\
&\qquad\left.\times\left(\sum_{s=1}^{n}A_r^{pt}B_s^{qt}\left(1-e_r+e_s\right)\right)^{1/qt}\right]\\
&\quad\geq\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^{pt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/pt-1/qt}\\
&\qquad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}B_r^{qt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/qt}\\
&\qquad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^{pt}B_s^{qt}\left(1-e_r+e_s\right)\right)^{1/qt}.
\end{aligned} \tag{15}$$

Additionally, using Lemma 3 together with $t\geq 1$, we find

$$\begin{aligned}
&\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^{pt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/pt-1/qt}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}B_r^{qt}A_s^{pt}\left(1-e_r+e_s\right)\right)^{1/qt}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^{pt}B_s^{qt}\left(1-e_r+e_s\right)\right)^{1/qt}\\
&\quad\geq\left(\sum_{r=1}^{n}\sum_{s=1}^{n}\left(1-e_r+e_s\right)\right)^{(1-t)(1/pt-1/qt)}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^pA_s^p\left(1-e_r+e_s\right)\right)^{1/p-1/q}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}\left(1-e_r+e_s\right)\right)^{(1-t)/qt}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}B_r^qA_s^p\left(1-e_r+e_s\right)\right)^{1/q}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}\left(1-e_r+e_s\right)\right)^{(1-t)/qt}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^pB_s^q\left(1-e_r+e_s\right)\right)^{1/q}\\
&\quad=\left(\sum_{r=1}^{n}\sum_{s=1}^{n}\left(1-e_r+e_s\right)\right)^{1-t}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^pA_s^p\left(1-e_r+e_s\right)\right)^{1/p-1/q}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}B_r^qA_s^p\left(1-e_r+e_s\right)\right)^{1/q}\\
&\quad\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_r^pB_s^q\left(1-e_r+e_s\right)\right)^{1/q}\\
&\quad=n^{2-2t}\left(\sum_{r=1}^{n}A_r^p\right)^{2/p-2/q}\\
&\quad\times\left(\sum_{r=1}^{n}B_r^q\sum_{s=1}^{n}A_s^p-\sum_{r=1}^{n}B_r^qe_r\sum_{s=1}^{n}A_s^p+\sum_{r=1}^{n}B_r^q\sum_{s=1}^{n}A_s^pe_s\right)^{1/q}\\
&\quad\times\left(\sum_{r=1}^{n}A_r^p\sum_{s=1}^{n}B_s^q-\sum_{r=1}^{n}A_r^pe_r\sum_{s=1}^{n}B_s^q+\sum_{r=1}^{n}A_r^p\sum_{s=1}^{n}B_s^qe_s\right)^{1/q}\\
&\quad=n^{2(1-1/p-1/q)}\left(\sum_{r=1}^{n}A_r^p\right)^{2/p-2/q}\\
&\quad\times\left\{\left[\left(\sum_{r=1}^{n}A_r^p\right)\left(\sum_{r=1}^{n}B_r^q\right)\right]^2\right.\\
&\qquad\left.-\left[\left(\sum_{r=1}^{n}A_r^pe_r\right)\left(\sum_{r=1}^{n}B_r^q\right)-\left(\sum_{r=1}^{n}A_r^p\right)\left(\sum_{r=1}^{n}B_r^qe_r\right)\right]^2\right\}^{1/q}.
\end{aligned} \tag{16}$$

Combining inequalities (11), (15), and (16) leads to inequality (10) immediately.

The proof of Theorem 4 is completed. □

From Theorem 4 and Lemma 2, we obtain the refinement of Hölder's inequality (1) as follows.

Corollary 5. *Let $A_r > 0, B_r > 0$ $(r = 1, 2, \ldots, n)$, let $1 - e_r + e_s \geq 0$ $(r, s = 1, 2, \ldots, n)$, and let $q < 0$, $p > 0$, and $\rho = \max\{1/p + 1/q, 1\}$. Then,*

$$\sum_{r=1}^{n} A_r B_r \geq n^{1-\rho}\left(\sum_{r=1}^{n} A_r^p\right)^{1/p}\left(\sum_{r=1}^{n} B_r^q\right)^{1/q} \times\left[1-\frac{1}{2q}\left(\frac{\sum_{r=1}^{n} B_r^q e_r}{\sum_{k=1}^{n} B_r^q}-\frac{\sum_{r=1}^{n} A_r^p e_r}{\sum_{r=1}^{n} A_r^p}\right)^2\right]. \tag{17}$$

Proof. Since

$$\left|\frac{\sum_{r=1}^{n} B_r^q e_r}{\sum_{k=1}^{n} B_r^q}-\frac{\sum_{r=1}^{n} A_r^p e_r}{\sum_{r=1}^{n} A_r^p}\right| < 1, \tag{18}$$

by using Lemma 2 and Theorem 4, we obtain the assertion of the corollary. The proof of Corollary 5 is completed. □

Now, we give a reversed version of inequality (4) as follows.

Theorem 6. *Let $f(x)$, $g(x)$, and $e(x)$ be integrable functions defined on $[a, b]$ and $f(x)$, $g(x) > 0$, $1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$, and let $q < 0$, $1/p + 1/q \geq 1$. Then,*

$$\begin{aligned}\int_a^b f(x)\, g(x)\, dx &\geq (b-a)^{1-1/p-1/q}\left(\int_a^b f^p(x)\, dx\right)^{1/p-1/q} \\ &\quad\times\left[\left(\int_a^b f^p(x)\, dx \int_a^b g^q(x)\, dx\right)^2 - \left(\int_a^b f^p(x)\, e(x)\, dx \int_a^b g^q(x)\, dx - \int_a^b f^p(x)\, dx \int_a^b g^q(x)\, e(x)\, dx\right)^2\right]^{1/2q}.\end{aligned} \tag{19}$$

Proof. For any positive integer n, we choose an equidistant partition of $[a, b]$ as follows:

$$\begin{gathered} a < a+\frac{b-a}{n} < \cdots < a+\frac{b-a}{n}k < \cdots < a+\frac{b-a}{n}(n-1) < b, \\ x_k = a+\frac{b-a}{n}k, \quad \Delta x_k = \frac{b-a}{n}, \quad k = 0, 1, 2, \ldots, n. \end{gathered} \tag{20}$$

Applying Theorem 4, we obtain the following inequality:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k)\, g(x_k) &\geq n^{1-1/p-1/q}\left(\sum_{k=1}^{n} f^p(x_k)\right)^{1/p-1/q} \\ &\quad\times\left\{\left[\left(\sum_{k=1}^{n} f^p(x_k)\right)\left(\sum_{k=1}^{n} g^q(x_k)\right)\right]^2 - \left[\left(\sum_{k=1}^{n} f^p(x_k)\, e(x_k)\right)\left(\sum_{k=1}^{n} g^q(x_k)\right) - \left(\sum_{k=1}^{n} f^p(x_k)\right)\times\left(\sum_{k=1}^{n} g^q(x_k)\, e(x_k)\right)\right]^2\right\}^{1/2q},\end{aligned} \tag{21}$$

equivalently

$$\begin{aligned}\sum_{k=1}^{n} f(x_k)\, g(x_k)\, \frac{b-a}{n} &\geq (b-a)^{1-1/p-1/q}\left(\sum_{k=1}^{n} f^p(x_k)\, \frac{b-a}{n}\right)^{1/p-1/q} \\ &\quad\times\left\{\left[\left(\sum_{k=1}^{n} f^p(x_k)\, \frac{b-a}{n}\right)\left(\sum_{k=1}^{n} g^q(x_k)\, \frac{b-a}{n}\right)\right]^2 - \left[\left(\sum_{k=1}^{n} f^p(x_k)\, e(x_k)\, \frac{b-a}{n}\right)\left(\sum_{k=1}^{n} g^q(x_k)\, \frac{b-a}{n}\right) - \left(\sum_{k=1}^{n} f^p(x_k)\, \frac{b-a}{n}\right)\times\left(\sum_{k=1}^{n} g^q(x_k)\, e(x_k)\, \frac{b-a}{n}\right)\right]^2\right\}^{1/2q}.\end{aligned} \tag{22}$$

In view of the hypotheses that $f(x)$, $g(x)$, and $e(x)$ are positive Riemann integrable functions on $[a, b]$, we conclude that $f^p(x)$, $g^q(x)$, and $g^q(x)e(x)$ are also integrable on $[a, b]$. Passing the limit as $n \to \infty$ in both sides of inequality (22), we obtain inequality (19). The proof of Theorem 6 is completed. □

Remark 7. Making similar technique as in the proof of Corollary 5, from Theorem 6 we obtain

$$\int_a^b f(x)\,g(x)\,dx \geq (b-a)^{1-1/p-1/q}\left(\int_a^b f^p(x)\,dx\right)^{1/p}\left(\int_a^b g^q(x)\,dx\right)^{1/q} \times\left[1-\frac{1}{2q}\left(\frac{\int_a^b f^p(x)\,e(x)\,dx}{\int_a^b f^p(x)\,dx}-\frac{\int_a^b g^q(x)\,e(x)\,dx}{\int_a^b g^q(x)\,dx}\right)^2\right]. \tag{23}$$

3. Applications

In this section, we show two applications of the inequalities newly obtained in Section 2.

Firstly, we provide an application of the obtained result to improve the Popoviciu-Vasić inequality. In 1956, Aczél [20] established the following inequality.

Theorem F. *If a_i, b_i $(i = 1, 2, \ldots, n)$ are positive numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then*

$$\left(a_1^2-\sum_{i=2}^n a_i^2\right)\left(b_1^2-\sum_{i=2}^n b_i^2\right)\leq\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2. \tag{24}$$

Inequality (24) is the well-known Aczél's inequality, which has many applications in the theory of functional equations in non-Euclidean geometry. Due to the importance of Aczél's inequality, this inequality has been given considerable attention by mathematicians and has motivated a large number of research papers involving different proofs, various generalizations, improvements and applications (see, e.g., [21–24] and the references therein).

One of the most important results in the references mentioned above is the exponential generalization of (24) asserted by Theorem G.

Theorem G. *Let p and q be real numbers such that $p, q \neq 0$ and $1/p + 1/q = 1$, and let a_i, b_i $(i = 1, 2, \ldots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then, for $p > 1$, one has*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\leq a_1b_1-\sum_{i=2}^n a_ib_i. \tag{25}$$

If $p < 1$ $(p \neq 0)$, one has the reverse inequality.

Remark 8. The case $p > 1$ of Theorem G was proved by Popoviciu [21]. The case $p < 1$ was given in [24] by Vasić and Pečarić.

Now, we give a refinement of inequality (25) by Theorem 4 and Theorem B.

Theorem 9. *Let $a_i, b_i \geq 0$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, and $b_1^q - \sum_{i=2}^n b_i^q > 0$, let $1 - e_i + e_j \geq 0$ $(i, j = 1, 2, \ldots, n)$, and let $\mu = \min\{1/p + 1/q, 1\}$, $\rho = \max\{1/p + 1/q, 1\}$. Then, for $p \geq q > 0$, one has*

$$\begin{aligned}\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}&\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\\ &\leq n^{1-\mu}\times b_1^{1-q/p}\\ &\quad\times\left\{b_1^{2q}a_1^{2p}\right.\\ &\qquad-\left[a_1^p\left(b_1^qe_1+\sum_{i=2}^n b_i^q(e_i-e_1)\right)\right.\\ &\qquad\left.\left.-\,b_1^q\left(a_1^pe_1+\sum_{i=2}^n a_i^p(e_i-e_1)\right)\right]^2\right\}^{1/2q}\\ &\quad-\sum_{i=2}^n a_ib_i.\end{aligned} \tag{26}$$

If $q < 0$, $p > 0$, one has

$$\begin{aligned}\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}&\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\\ &\geq n^{1-\rho}\times a_1^{1-p/q}\\ &\quad\times\left\{a_1^{2p}b_1^{2q}\right.\\ &\qquad-\left[b_1^q\left(a_1^pe_1+\sum_{i=2}^n a_i^p(e_i-e_1)\right)\right.\\ &\qquad\left.\left.-\,a_1^p\left(b_1^qe_1+\sum_{i=2}^n b_i^q(e_i-e_1)\right)\right]^2\right\}^{1/2q}\\ &\quad-\sum_{i=2}^n a_ib_i.\end{aligned} \tag{27}$$

Proof. By substituting

$$\begin{aligned}&A_1^p\longrightarrow a_1^p-\sum_{i=2}^n a_i^p,\qquad B_1^q\longrightarrow b_1^q-\sum_{i=2}^n b_i^q,\\ &A_i\longrightarrow a_i,\quad B_i\longrightarrow b_i\quad(i=2,3,\ldots,n),\end{aligned} \tag{28}$$

in (3) and (10), respectively, we get Theorem 9. □

Remark 10. Let $a_1 \neq 0, b_1 \neq 0$, and let $1/p + 1/q = 1$. If $p \geq q > 0$, then we conclude from Theorem 9:

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{1/q} \leq a_1 b_1 \left\{1 - \left(\frac{b_1^q e_1 + \sum_{i=2}^{n} b_i^q (e_i - e_1)}{b_1^q} - \frac{a_1^p e_1 + \sum_{i=2}^{n} a_i^p (e_i - e_1)}{a_1^p}\right)^2\right\}^{1/2q} - \sum_{i=2}^{n} a_i b_i. \tag{29}$$

Inequality (29) is reversed for $q < 0$.

Next, we are to establish the time scales version of Beckenbach-type inequality which is due to Wang [25]. In 1983, Wang [25] established the following Beckenbach-type inequality.

Theorem H. *Let $f(x)$, and $g(x)$ be positive integrable functions defined on $[s,t]$, and let $1/p + 1/q = 1$. If $0 < p < 1$, then, for any of the positive numbers: a, b, or c, the inequality*

$$\frac{\left(a + c\int_s^t k^p(x)\,dx\right)^{1/p}}{b + c\int_s^t k(x)\,g(x)\,dx} \geq \frac{\left(a + c\int_s^t f^p(x)\,dx\right)^{1/p}}{b + c\int_s^t f(x)\,g(x)\,dx} \tag{30}$$

holds, where $k(x) = (ag(x)/b)^{q/p}$. The sign of inequality in (30) *is reversed if $p > 1$.*

In order to present the time scales version of (30), we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{31}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ is called right scattered, right dense, left scattered, and left dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, and $\rho(t) = t$ hold, respectively.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if the left-sided limit exists at every left dense point. The set of all rd-continuous functions is denoted by $C_{\mathrm{rd}}[\mathbb{T}, \mathbb{R}]$.

Let

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - m, & \text{if } \mathbb{T} \text{ has left scattered point in } M, \\ \mathbb{T}, & \text{otherwise.} \end{cases} \tag{32}$$

Let f be a function defined on $\mathbb{R}$. Then f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^\Delta(t)$ if given $\epsilon > 0$; there exists a neighbourhood $\mathbb{N}$ of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s| \tag{33}$$

for all $s \in \mathbb{N}$.

Remark 11. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative; that is, $f^\Delta(t) = f'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t)$ reduces to the usual forward difference; that is, $f^\Delta(t) = \Delta f(t)$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\Delta = f(t)$ holds for all $t \in \mathbb{T}$. In this case, we define the integral of f by

$$\int_s^t f(\tau)\,\Delta\tau = F(t) - F(s), \tag{34}$$

where $s, t \in \mathbb{T}$.

Remark 12. If $\mathbb{T} = \mathbb{R}$, then the time scale integral is an ordinary integral. If $\mathbb{T} = \mathbb{Z}$, then the time-scale integral is a sum.

For more details on time scales theory, the readers may consult [26–29] and the references therein. Now, we present the time scales version of (30) by using Corollary 5.

Theorem 13. *Let $f(x)$, $g(x)$, and $h(x) \in C_{rd}([s,t],[0,+\infty))$, where $C_{rd}([s,t],[0,+\infty))$ denotes the set of rd-continuous functions defined by $C_{rd}([s,t],[0,+\infty)) = \{\alpha \mid \alpha : [s,t] \to [0,+\infty)$ and $\alpha(t)$ is an rd-continuous function$\}$ and let $1/p + 1/q = 1$. If $p > 1$, then, for any of the positive numbers a, b, or c, the inequality*

$$\frac{\left(a + c\int_s^t h(x)\,k^p(x)\,\Delta x\right)^{1/p}}{b + c\int_s^t h(x)\,k(x)\,g(x)\,\Delta x} \leq \frac{\left(a + c\int_s^t h(x)\,f^p(x)\,\Delta x\right)^{1/p}}{b + c\int_s^t h(x)\,f(x)\,g(x)\,\Delta x} \times \left[1 - \frac{1}{2q}\left(\frac{c\int_s^t h(x)\,f^p(x)\,\Delta x}{a + c\int_s^t h(x)\,f^p(x)\,\Delta x} - \frac{c\int_s^t h(x)\,g^q(x)\,\Delta x}{a^{-q/p}b^q + c\int_s^t h(x)\,g^q(x)\,\Delta x}\right)^2\right] \tag{35}$$

holds, where $k(x) = (ag(x)/b)^{q/p}$. The sign of inequality in (35) *is reversed if $0 < p < 1$.*

Proof. We only consider the case $0 < p < 1$. Noting that $1 + q/p = q$, the left-hand side of (35) becomes

$$\frac{\left[a + c\int_s^t h(x)\,(ag(x)/b)^q \Delta x\right]^{1/p}}{b + c\int_s^t h(x)\,(ag(x)/b)^{q/p} g(x)\,\Delta x} = \frac{(a/b)^{q/p}\left[a(b/a)^q + c\int_s^t h(x)\,g^q(x)\,\Delta x\right]^{1/p}}{(a/b)^{q/p}\left[b(b/a)^{q/p} + c\int_s^t h(x)\,g^q(x)\,\Delta x\right]} = \left(a^{-q/p}b^q + c\int_s^t h(x)\,g^q(x)\,\Delta x\right)^{-1/q}. \tag{36}$$

On the other hand, by using Hölder's inequality and inequality (17) for $e_1 = 0$, $e_2 = 1$, we obtain

$$\begin{aligned} & b + c\int_s^t h(x) f(x) g(x) \Delta x \\ & \geq b + c\left(\int_s^t h(x) f^p(x) \Delta x\right)^{1/p} \\ & \quad \times \left(\int_s^t h(x) g^q(x) \Delta x\right)^{1/q} \\ & = a^{1/p}\left(ba^{-1/p}\right) + \left(c\int_s^t h(x) f^p(x) \Delta x\right)^{1/p} \\ & \quad \times \left(c\int_s^t h(x) g^q(x) \Delta x\right)^{1/q} \\ & \geq \left(a + c\int_s^t h(x) f^p(x) \Delta x\right)^{1/p} \\ & \quad \times \left(a^{-q/p}b^q + c\int_s^t h(x) g^q(x) \Delta x\right)^{1/q} \\ & \quad \times \left[1 - \frac{1}{2q}\left(\frac{c\int_s^t h(x) f^p(x) \Delta x}{a + c\int_s^t h(x) f^p(x) \Delta x} \right.\right. \\ & \qquad \left.\left. - \frac{c\int_s^t h(x) g^q(x) \Delta x}{a^{-q/p}b^q + c\int_s^t h(x) g^q(x) \Delta x}\right)^2\right]. \end{aligned} \tag{37, 38}$$

Combining inequalities (36) and (38) yields inequality (35). The proof of Theorem 13 is completed. □

In (35), taking $c\int_s^t h(x)f^p(x)\Delta x/(a + c\int_s^t h(x)f^p(x)\Delta x) = c\int_s^t h(x)g^q(x)\Delta x/(a^{-q/p}b^q + c\int_s^t h(x)g^q(x)\Delta x)$, from Theorem 13 we obtain the time scales version of Beckenbach-type inequality as follows.

Corollary 14. *Let $f(x)$, $g(x)$, and $h(x) \in C_{rd}([s,t],[0,+\infty))$, and let $1/p + 1/q = 1$. If $p > 1$, then, for any of the positive numbers a, b, or c, the inequality*

$$\frac{\left(a + c\int_s^t h(x) k^p(x) \Delta x\right)^{1/p}}{b + c\int_s^t h(x) k(x) g(x) \Delta x} \leq \frac{\left(a + c\int_s^t h(x) f^p(x) \Delta x\right)^{1/p}}{b + c\int_s^t h(x) f(x) g(x) \Delta x} \tag{39}$$

holds, where $k(x) = (ag(x)/b)^{q/p}$. The sign of inequality in (39) *is reversed if* $0 < p < 1$.

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