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Research Article

Existence of Solution for Impulsive Differential Equations with Nonlinear Derivative Dependence via Variational Methods

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We use variational methods and iterative methods to investigate the solutions of impulsive differential equations with nonlinear derivative dependence. The conditions for the existence of solutions are established. The main results are also demonstrated with examples.

1. Introduction

Many dynamical systems have an impulsive dynamical behavior due to abrupt changes at certain instants during the evolution process. The mathematical description of these phenomena leads to impulsive differential equations. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics, and medicine [1–9].

We consider the following nonlinear Dirichlet boundary value problems for impulsive differential equations:

$$\begin{aligned} &-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + g(t)\left|u(t)\right|^{p-2}u(t) \\ &\quad = f\left(t, u(t), u'(t)\right), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ &\Delta u'\left(t_j\right) = \left|u'\left(t_j^+\right)\right|^{p-2}u'\left(t_j^+\right) - \left|u'\left(t_j^-\right)\right|^{p-2}u'\left(t_j^-\right) \\ &\quad = I_j\left(u\left(t_j\right)\right), \quad j = 1,2,\ldots,l, \\ &\quad u(0) = u(T) = 0, \end{aligned} \tag{1}$$

where $p \geq 2$, $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$, $g \in L^\infty[0,T]$, $f : [0,T] \times R \times R \to R$ is continuous, and $I_j : R \to R$, $j = 1,2,\ldots,l$, are continuous.

The characteristic of (1) is the presence of the first order derivative in the nonlinearity term. Most of the results concerning the existence of solutions of these equations are obtained using upper and lower solutions methods, coincidence degree theory, and fixed point theorems [10–14]. However, to the best of our knowledge, there are few papers concerned with the existence of solutions for impulsive boundary value problems like problem (1) by using variational methods. Motivated by [15, 16], in this paper we will fill the gap in this area.

When there is no derivative in the nonlinearity term, problem (1) has been extensively studied by [17–23], using variational methods. We know, contrary to these equations, (1) is not variational and the well-developed critical point theory is of no avail for, at least, a direct attack to problem (1). The technique used in this paper consists of, associating with problem (1), a family of the following Dirichlet boundary value problems with no dependence on the derivative of the solution. Namely, for each $w \in W_0^{1,p}$, we consider the problem

$$\begin{aligned} &-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + g(t)\left|u(t)\right|^{p-2}u(t) \\ &\quad = f\left(t, u(t), w'(t)\right), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \end{aligned}$$

$$\begin{aligned}\Delta u'(t_j) &= \left|u'(t_j^+)\right|^{p-2}u'(t_j^+) - \left|u'(t_j^-)\right|^{p-2}u'(t_j^-)\\ &= I_j(u(t_j)), \quad j = 1,2,\ldots,l,\\ &u(0) = u(T) = 0.\end{aligned} \tag{2}$$

Now problem (2) is variational and we can treat it by variational methods.

In this paper, we need the following conditions.

(f_1) $f : [0,T] \times R \times R \to R$ is measurable in $t \in [0,T]$ for every $(x,\zeta) \in R \times R$ and continuous in $(x,\zeta) \in R \times R$ for a.e. $t \in [0,T]$.

(f_2) $f(t,x,\zeta) = o(|x|^{p-1})$ as $x \to 0$ uniformly for $t \in [0,T]$ and $\zeta \in R$ and $f(t,x,0) \neq 0$ for $t \in [0,T]$ and $x \in R$.

(f_3) There exist constants $C > 0$ and $r \in (p,+\infty)$ such that

$$\left|f(t,x,\zeta)\right| \le C\left(1 + |x|^{r-1}\right), \qquad \forall t \in [0,T],\ x \in R,\ \zeta \in R. \tag{3}$$

(f_4) There exist constants $\mu > p$ and $x_0 > 0$ such that

$$\begin{aligned}&0 < \mu F(t,x,\zeta) \le x f(t,x,\zeta),\\ &\forall t \in [0,T],\ |x| \ge x_0,\ \zeta \in R,\end{aligned} \tag{4}$$

where $F(t,x,\zeta) = \int_0^x f(t,s,\zeta)ds$.

(f_5) There exist constants $a, b > 0$ such that

$$F(t,x,\zeta) \ge a|x|^{\mu} - b, \quad \forall t \in [0,T],\ x \in R,\ \zeta \in R. \tag{5}$$

(f_6) The function f satisfies the following conditions:

$$\begin{aligned}&\left|f(t,x,\zeta) - f(t,x',\zeta)\right| \le L_1\left|x - x'\right|^{p-1},\\ &\quad \forall t \in [0,T],\ x,x' \in [-\rho_1,\rho_1],\ \zeta \in R,\\ &\left|f(t,x,\zeta) - f(t,x,\zeta')\right| \le L_2\left|\zeta - \zeta'\right|^{p-1},\\ &\quad \forall t \in [0,T],\ x \in [-\rho_1,\rho_1],\ \zeta,\zeta' \in R,\end{aligned} \tag{6}$$

where ρ_1 is a constant.

(I_1) I_j $(j = 1,2,\ldots,l)$ are odd and nondecreasing, and there exist constants $a_j > 0$, $b_j > 0$, and $r_j \in [0, p-1)$, $j = 1,2,\ldots,l$ such that

$$\left|I_j(u)\right| \le a_j + b_j|u|^{r_j}, \quad \text{for } j = 1,2,\ldots,l. \tag{7}$$

(I_2) There exists $0 < \theta \le p$ such that

$$\theta\int_0^u I_j(s)\,ds \ge uI_j(u) > 0, \quad \forall u > 0. \tag{8}$$

(I_3) $|I_j(u) - I_j(v)| \le \alpha_j|u - v|^{p-1}$, for all $u, v \in [-\rho_1,\rho_1]$, $j = 1,2,\ldots,l$.

The paper is organized as follows: Section 2 is the preliminaries of this paper, Section 3 is devoted to show the solvability of problem (2), and Section 4 will show the solvability of problem (1).

2. Preliminaries

Firstly, we recall some facts which will be used in the proof of our main result. It has been shown, for instance, in [16] that the set of all eigenvalues of the following problem

$$\begin{aligned}&\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + \lambda|u(t)|^{p-2}u(t) = 0, \quad t \in [0,T],\\ &u(0) = u(T) = 0\end{aligned} \tag{9}$$

is given by the sequence of positive numbers

$$\lambda_k = (p-1)\left(\frac{k\pi_p}{T}\right)^p, \quad k = 1,2,\ldots, \tag{10}$$

where

$$\pi_p = 2\int_0^1 \frac{ds}{(1-s^p)^{1/p}} = \frac{2\pi}{p\sin(\pi/p)}. \tag{11}$$

Each eigenvalue λ_k is simple with the associated eigenfunction

$$\varphi_k(x) = \sin_p\left(\frac{k\pi_p x}{T}\right), \quad \text{for } 0 \le x \le T. \tag{12}$$

We recall the well-known characterization of λ_1 as the best constant in the Poincaré inequality

$$\int_0^T \left|u'(t)\right|^p dt \ge \lambda_1\int_0^T |u(t)|^p dt, \quad \forall u \in W_0^{1,p}(0,T). \tag{13}$$

Let $W_0^{1,p}(0,T)$ be the Sobolev space endowed with the norm

$$\|u\| = \left(\int_0^T \left|u'(t)\right|^p dt + \int_0^T g(t)\,|u(t)|^p dt\right)^{1/p}. \tag{14}$$

Throughout the paper, it will be assumed that $\operatorname{ess\,inf}_{t\in[0,T]} g(t) = m > -\lambda_1$. We also consider the norm

$$\begin{aligned}\|u\|_{W_0^{1,p}} &= \left(\int_0^T \left|u'(t)\right|^p dt\right)^{1/p},\\ \|u\|_p &= \left(\int_0^T |u(t)|^p dt\right)^{1/p}.\end{aligned} \tag{15}$$

We need the following Lemmas.

Lemma 1 (see [24, Lemma 2.4]). *If* $\operatorname{ess\,inf}_{t\in[0,T]} g(t) = m > -\lambda_1$, *then the norms* $\|\cdot\|$ *and* $\|\cdot\|_{W_0^{1,p}}$ *are equivalent.*

Lemma 2 (see [24, Lemma 2.5]). *There exists* $C_2 > 0$ *such that if* $u \in W_0^{1,p}$, *then*

$$\|u\|_\infty \le \left(\frac{T}{q+1}\right)^{1/q}\left(\int_0^T \left|u'(s)\right|^p ds\right)^{1/p}, \tag{16}$$

$$\|u\|_\infty \le C_2\|u\|. \tag{17}$$

Lemma 3 (see [24, Lemma 2.6]). *Let $u \in W_0^{1,p}(0,T)$; then there exists $C_1 \in (0,1)$ such that*

$$\|u\|_p \le \frac{1}{\sqrt[p]{\lambda_1 C_1}} \|u\|. \tag{18}$$

For $u \in W^{1,p}(0,T)$, we have that u and u' are both absolutely continuous. Hence $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in [0,T]$. If $u \in W_0^{1,p}(0,T)$, then u is absolutely continuous. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ may not hold for some $t \in [0,T]$. It leads to the impulsive effects. As a consequence, we need to introduce a different concept of solution.

Definition 4. A function

$$u \in \left\{u \in W_0^{1,p}(0,T) : \left|u'\right|^{p-2}u' \in W^{1,\infty}(0,T) \setminus \{t_1, t_2, \ldots, t_l\}\right\} \tag{19}$$

is said to be a classical solution of problem (1) if u satisfies the equation a.e. on $[0,T] \setminus \{t_1, t_2, \ldots, t_l\}$ and the limits $u'(t_j^+)$, $u'(t_j^-)$ and $j = 1, 2, \ldots, l$, exist and satisfy the impulsive condition

$$\Delta u'(t_j) = \left|u'(t_j^+)\right|^{p-2}u'(t_j^+) - \left|u'(t_j^-)\right|^{p-2}u'(t_j^-) = I_j(u(t_j)) \tag{20}$$

and the boundary condition $u(0) = u(T) = 0$.

We have the following fact. Take $v \in W_0^{1,p}(0,T)$ and multiply the equation in problem (1) by v and integrate from 0 to T:

$$-\int_0^T \left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\,dt + \int_0^T g(t)\,|u(t)|^{p-2}u(t)\,v(t)\,dt = \int_0^T f(t, u(t), u'(t))\,v(t)\,dt. \tag{21}$$

The first term is now

$$\begin{aligned} -\int_0^T \left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\,dt &= -\sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} \left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\,dt, \\ \int_{t_j}^{t_{j+1}} \left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\,dt &= \left|u'(t_{j+1}^-)\right|^{p-2}u'(t_{j+1}^-)\,v(t_{j+1}) - \left|u'(t_j^+)\right|^{p-2}u'(t_j^+)\,v(t_j) \\ &\quad - \int_{t_j}^{t_{j+1}} \left|u'(t)\right|^{p-2}u'(t)\,v'(t)\,dt. \end{aligned} \tag{22}$$

Hence

$$\begin{aligned} -\int_0^T \left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\,dt &= \sum_{j=1}^{l} \Delta u'(t_j)\,v(t_j) - \left|u'(T)\right|^{p-2}u'(T)\,v(T) \\ &\quad + \left|u'(0)\right|^{p-2}u'(0)\,v(0) + \int_0^T \left|u'(t)\right|^{p-2}u'(t)\,v'(t)\,dt \\ &= \sum_{j=1}^{l} I_j(u(t_j))\,v(t_j) + \int_0^T \left|u'(t)\right|^{p-2}u'(t)\,v'(t)\,dt. \end{aligned} \tag{23}$$

Definition 5. We say that a function $u \in W_0^{1,p}(0,T)$ is a weak solution of problem (1) if the identity

$$\int_0^T \left|u'(t)\right|^{p-2}u'(t)\,v'(t)\,dt + \int_0^T g(t)\,|u(t)|^{p-2}u(t)\,v(t)\,dt + \sum_{j=1}^{l} I_j(u(t_j))\,v(t_j) = \int_0^T f(t, u(t), u'(t))\,v(t)\,dt \tag{24}$$

holds for any $v \in W_0^{1,p}(0,T)$.

Proposition 6. *Under the hypotheses (f_1) and (f_2), the functional $\varphi_w : W_0^{1,p}(0,T) \to R$ defined by*

$$\varphi_w(u) = \frac{1}{p}\|u\|^p - \int_0^T F(t, u(t), w'(t))\,dt + \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(s)\,ds \tag{25}$$

is continuous and differentiable and

$$\varphi_w'(u)\,v = \int_0^T \left|u'(t)\right|^{p-2}u'(t)\,v'(t)\,dt + \int_0^T g(t)\,|u(t)|^{p-2}u(t)\,v(t)\,dt + \sum_{j=1}^{l} I_j(u(t_j))\,v(t_j) - \int_0^T f(t, u(t), w'(t))\,v(t)\,dt \tag{26}$$

for any $v \in W_0^{1,p}(0,T)$. Moreover, the critical point of φ_w is a classical solutions of problem (2).

Proof. Using the assumptions (f_1), we can obtain the continuity and differentiability of φ_w and that $\varphi_w' : W_0^{1,p}(0,T) \to (W_0^{1,p}(0,T))^*$ is defined by

$$\begin{aligned}\varphi_w'(u)v = &\int_0^T \left|u'(t)\right|^{p-2}u'(t)v'(t)\,dt \\ &+ \int_0^T g(t)|u(t)|^{p-2}u(t)v(t)\,dt \\ &+ \sum_{j=1}^{l} I_j\left(u\left(t_j\right)\right)v\left(t_j\right) \\ &- \int_0^T f\left(t,u(t),w'(t)\right)v(t)\,dt\end{aligned} \tag{27}$$

for any $v \in W_0^{1,p}(0,T)$. It follows that the critical point of φ_w is the weak solution of (2). Moreover, it is a classical solution of problem (2).

Evidently, $u(0) = u(T) = 0$ since $u \in W_0^{1,p}(0,T)$. By the definition of weak solution, we have

$$\begin{aligned}&\int_0^T \left|u'(t)\right|^{p-2}u'(t)v'(t)\,dt + \int_0^T g(t)|u(t)|^{p-2}u(t)v(t)\,dt \\ &+ \sum_{j=1}^{l} I_j\left(u\left(t_j\right)\right)v\left(t_j\right) - \int_0^T f\left(t,u(t),w'(t)\right)v(t)\,dt = 0.\end{aligned} \tag{28}$$

Choose $v \in W_0^{1,p}(0,T)$ with $v(t) = 0$ for every $t \in [0,t_j] \bigcup [t_{j+1},T]$; then

$$\begin{aligned}&\int_{t_j}^{t_{j+1}} \left|u'(t)\right|^{p-2}u'(t)v'(t)\,dt \\ &+ \int_{t_j}^{t_{j+1}} g(t)|u(t)|^{p-2}u(t)v(t)\,dt \\ &\quad = \int_{t_j}^{t_{j+1}} f\left(t,u(t),w'(t)\right)v(t)\,dt.\end{aligned} \tag{29}$$

This implies that

$$\begin{aligned}&-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + g(t)|u(t)|^{p-2}u(t) \\ &\quad = f\left(t,u(t),w'(t)\right), \quad \text{a.e. } t \in \left(t_j,t_{j+1}\right).\end{aligned} \tag{30}$$

Hence, $|u|^{p-2}u' \in W^{1,\infty}(t_j,t_{j+1})$ for every $j = 1,2,\ldots,l$. The impulsive condition in (2) is satisfied. This completes the proof. □

We will obtain the critical points of φ_w by using the Mountain Pass Theorem. Therefore, we state this theorem precisely.

Lemma 7 (see [25]). *Let X be a real Banach space and $I \in C'(X,R)$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *there exists constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(iii) *there exists $e \in X \setminus \overline{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\left(g(s)\right), \tag{31}$$

where $B_\rho(0)$ is an open ball in X of radius ρ centered at 0 and

$$\Gamma = \left\{g \in C([0,1],X) : g(0) = 0,\ g(1) = e\right\}. \tag{32}$$

3. The Solvability of (2)

Theorem 8. *Suppose that (f_1)–(f_5) and (I_1) hold; then there exist positive constants d_1 and d_2 such that, for each $w \in W_0^{1,p}(0,T)$, problem (2) has one solution u_w such that $d_1 \leq \|u_w\| \leq d_2$.*

Proof. (I) We show that φ_w satisfies the (PS)-condition.

Assume that $\{u_n\}_{n\in\mathbf{N}} \subset W_0^{1,p}(0,T)$ is a sequence such that $\{\varphi_w(u_n)\}_{n\in\mathbf{N}}$ is bounded and $\varphi_w'(u_n) \to 0$ as $n \to +\infty$. We will prove that the sequence $\{u_n\}_{n\in\mathbf{N}}$ is bounded. Obviously, there exists a constant $C_3 > 0$ such that

$$\left|\varphi_w(u_n)\right| \leq C_3, \qquad \left|\varphi_w'(u_n)\right| \leq C_3 \quad \text{for } n \in \mathbf{N}. \tag{33}$$

We set

$$m_0 = \max\{a_1,a_2,\ldots,a_l\}, \qquad M_0 = \max\{b_1,b_2,\ldots,b_l\}. \tag{34}$$

From (17), (25), (26), and (33), (f_4), and (I_1), we have

$$\begin{aligned}&\mu\varphi_w(u_n) - \varphi_w'(u_n)u_n \\ &= \left(\frac{\mu}{p} - 1\right)\|u_n\|^p - \int_0^T \left(\mu F\left(t,u_n,w'\right) - f\left(t,u_n,w'\right)u_n\right)dt \\ &\quad + \sum_{j=1}^{l}\left[\mu\int_0^{u_n(t_j)} I_j(s)\,ds - I_j\left(u_n\left(t_j\right)\right)u_n\left(t_j\right)\right] \\ &\geq \left(\frac{\mu}{p} - 1\right)\|u_n\|^p - \sum_{j=1}^{l}\left[\left(a_j + b_j\left|u_n\left(t_j\right)\right|^{r_j}\right)u_n\left(t_j\right)\right] \\ &\quad - \sum_{j=1}^{l}\left(a_j + b_j|s|^{r_j}\right)ds - C_4 \\ &\geq \left(\frac{\mu}{p} - 1\right)\|u_n\|^p \\ &\quad - (\mu+1)\left(C_2 l m_0\|u_n\| + M_0\sum_{j=1}^{l} C_2^{r_j+1}\|u_n\|^{r_j+1}\right) - C_4.\end{aligned} \tag{35}$$

By $p > r_j + 1$, we obtain that $\{u_n\}$ is bounded in $W_0^{1,p}(0,T)$.

Since $W_0^{1,p}(0,T)$ is a reflexive Banach space, passing to a subsequence if necessary, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W_0^{1,p}(0,T), \\ u_n &\longrightarrow u \quad \text{in } L^p(0,T), \\ u_n &\longrightarrow u \quad \text{uniformly in } C[0,T]. \end{aligned} \tag{36}$$

Hence

$$\int_0^T \left(f\left(t,u_n,w'\right) - f\left(t,u,w'\right)u_n\right)\left(u_n(t)-u(t)\right)dt \longrightarrow 0, \tag{37}$$

$$\sum_{j=1}^{l}\left[I_j\left(u_n\left(t_j\right)\right) - I_j\left(u\left(t_j\right)\right)\right]\left(u_n\left(t_j\right)-u\left(t_j\right)\right) \longrightarrow 0. \tag{38}$$

Notice that

$$\begin{aligned} &\left\langle \varphi_w'(u_n) - \varphi_w'(u), u_n - u\right\rangle \\ &\quad = \int_0^T \left(\left|u_n'(t)\right|^{p-2}u_n'(t) - \left|u'(t)\right|^{p-2}u'(t)\right) \\ &\qquad \times \left(u_n'(t)-u'(t)\right)dt \\ &\quad + \int_0^T \left(g(t)\left|u_n(t)\right|^{p-2}u_n(t) - |u(t)|^{p-2}u(t)\right) \\ &\qquad \times \left(u_n(t)-u(t)\right)dt \\ &\quad + \sum_{j=1}^{l}\left(I_j\left(u_n\left(t_j\right)\right) - I_j\left(u\left(t_j\right)\right)\left(u_n\left(t_j\right)-u\left(t_j\right)\right)\right) \\ &\quad + \int_0^T \left(f\left(t,u_n,w'\right) - f\left(t,u,w'\right)\right)\left(u_n(t)-u(t)\right)dt. \end{aligned} \tag{39}$$

From [26, Lemma 4.2], we see that there exists $C_p > 0$, for any $x, y \in R$:

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y) \geq C_p|x-y|^p, \quad p \geq 2. \tag{40}$$

Combining this inequality with (39), we have

$$\begin{aligned} &C_p\left(\int_0^T \left(\left|u_n'(t)-u'(t)\right|^p + g(t)\left|u_n(t)-u(t)\right|^p\right)dt\right) \\ &\quad \leq \left\|\varphi_w'(u_n) - \varphi_w'(u)\right\|\left\|u_n-u\right\| \\ &\qquad + \sum_{j=1}^{l}\left(I_j\left(u_n\left(t_j\right)\right) - I_j\left(u\left(t_j\right)\right)\left(u_n\left(t_j\right)-u\left(t_j\right)\right)\right) \\ &\qquad + \int_0^T \left(f\left(t,u_n,w'\right) - f\left(t,u,w'\right)\right)\left(u_n(t)-u(t)\right)dt. \end{aligned} \tag{41}$$

It follows from (37)–(39) that $u_n \to u$ in $W_0^{1,p}(0,T)$. Hence, φ_w satisfies (PS)-condition.

(II) We verify assumption (ii) of Lemma 7.

By (f_2), there exists $\delta \in (0,1)$ such that

$$\left|f(t,x,\zeta)\right| \leq \frac{\lambda_1 C_1}{2}|x|^{p-1} \quad \text{for } t \in [0,T],\ |x| \leq \delta,\ \zeta \in R. \tag{42}$$

Since $F(t,0,\zeta) = 0$, it follows that

$$\left|F(t,x,\zeta)\right| \leq \frac{\lambda_1 C_1}{2p}|x|^{p} \quad \text{for } t \in [0,T],\ |x| \leq \delta,\ \zeta \in R. \tag{43}$$

Using (I_1), we have

$$\sum_{j=1}^{l}\int_0^{u(t_j)} I_j(t)\,dt \geq 0. \tag{44}$$

Hence, from (18), (25), (35), (f_2), and (I_1), for $\|u\| \leq \delta/C_2$, we have

$$\begin{aligned} \varphi_w(u) &= \frac{1}{p}\|u\|^p - \int_0^T F\left(t,u(t),w'(t)\right)dt \\ &\quad + \sum_{j=1}^{l}\int_0^{u(t_j)} I_j(t)\,dt \\ &\geq \frac{1}{p}\|u\|^p - \frac{\lambda_1 C_1}{2p}\int_0^T |u(t)|^p dt \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\|u\|^p = \frac{1}{2p}\|u\|^p. \end{aligned} \tag{45}$$

Set $\alpha = (1/2p)(\delta/C_2)^p$, $\rho = \delta/C_2$. Equation (45) shows that $\|u\| = \rho$ implies that $\varphi_w(u) \geq \alpha$; that is, φ_w satisfies assumption (ii) of Lemma 7.

(III) We verify assumption (iii) of Lemma 7.

By (I_1) and (f_5), we know that for $s > 1$

$$\begin{aligned} \varphi_w(su) &= \frac{1}{p}\|su\|^p - \int_0^T F\left(t,su(t),w'(t)\right)dt \\ &\quad + \sum_{j=1}^{l}\int_0^{su(t_j)} I_j(t)\,dt \\ &\leq \frac{1}{p}\|u\|^p s^p + lC_2\|u\|\sum_{j=1}^{l}\left(a_j + b_j|s|^{r_j}\right)C_2^{r_j}\|u\|^{r_j} \\ &\quad - a|s|^{\mu}\int_0^T |u(t)|^{\mu}dt + bT. \end{aligned} \tag{46}$$

Take $v_0 \in W_0^{1,p}(0,T)$ such that $\|v_0\| = 1$. Since $\mu > p$, $a > 0$, and $0 \leq r_j < p-1$, (46) implies that there exists $\xi_1 > 1$ such that $\|e\| > \rho$, and $\varphi_w(e) < 0$ if we set $e = \xi_1 v_0$. By Lemma 7, φ_w possesses a critical value $c' \geq \alpha > 0$ given by

$$c' = \inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi_w\left(g(s)\right), \tag{47}$$

where $\Gamma = \{g \in C([0,1], W_0^{1,p}(0,T)) : g(0) = 0,\ g(1) = e\}$. Obviously, $\varphi_w(0) = 0$, so according to Lemma 7, there exists $u_w \neq 0$ and $u_w \in X$, such that

$$\varphi_w(u_w) = c', \qquad \varphi_w(u_w) = 0. \tag{48}$$

(IV) We prove that $d_1 \leq \|u_w\| \leq d_2$.

Since u_w is the solution of problem (2), we have

$$\int_0^T |u_w'|^p dt + \int_0^T g(t)|u_w|^p dt + \sum_{j=1}^{l} I_j(u_w(t_j))u_w(t_j) = \int_0^T f(t, u_w, w')u_w dt. \tag{49}$$

So

$$\|u_w\|^p + \sum_{j=1}^{l} I_j(u_w(t_j))u_w(t_j) = \int_0^T f(t, u_w, w')u_w dt. \tag{50}$$

It follows from (f_2) and (f_3) that, given $\epsilon > 0$, there exists a positive constant C_ϵ, independent of w, such that

$$|f(t,x,\zeta)| \leq \epsilon|x|^{p-1} + C_\epsilon|x|^{r-1}. \tag{51}$$

Hence, using (I_1), we have

$$\begin{aligned} \|u_w\|^p &\leq \|u_w\|^p + \sum_{j=1}^{l} I_j(u_w(t_j))u_w(t_j) \\ &= \int_0^T f(t, u_w, w')u_w dt \\ &\leq \epsilon \int_0^T |u_w|^p dt + C_\epsilon \int_0^T |u_w|^r dt. \end{aligned} \tag{52}$$

By Sobolev embedding theorem, we obtain

$$\|u_w\|^p \leq \epsilon \cdot \frac{1}{\lambda_1 C_1}\|u_w\|^p + C_\epsilon \cdot \frac{1}{(\lambda_1 C_1)^{r/p}}\|u_w\|^r. \tag{53}$$

Set $\epsilon/\lambda_1 C_1 = \epsilon'$ and $C_\epsilon/(\lambda_1 C_1)^{r/p} = C_\epsilon'$; then

$$(1-\epsilon')\|u_w\|^p \leq C_\epsilon'\|u_w\|^r, \tag{54}$$

which implies

$$\|u_w\| \geq d_1. \tag{55}$$

From the inf max characterization of u_w in (III), we obtain

$$\varphi_w(u_w) \leq \max \varphi_w(sv_0) \tag{56}$$

with v_0 chosen in (III). We estimate $\varphi_w(sv_0)$ using (f_5) and (I_1):

$$\begin{aligned} \varphi_w(sv_0) \leq{}& \frac{1}{p}s^p - a|s|^\mu \int_0^T |v_0|^\mu dt + bT - a_j T \\ &- b_j|s|^{r_j}\int_0^T |v_0|^{r_j} dt =: h(s), \end{aligned} \tag{57}$$

whose maximum is achieved at some $s_0 > 0$ and the value $h(s_0)$ can be taken as d_2. Clearly it is independent of w. Which implies

$$\|u_w\| \leq d_2. \tag{58}$$

This completes the proof. □

Lemma 9. *Suppose that (f_1)–(f_3) hold and (f_4), (f_5), and (I_1) hold only for positive x; then there exist positive constants d_3, and d_4 such that, for each $w \in W_0^{1,p}(0,T)$, problem (2) has a positive solution u_w such that $d_3 \leq \|u_w\| \leq d_4$.*

Proof. Set

$$\begin{aligned} \overline{f}(t,u,\zeta) &= \begin{cases} f(t,u,\zeta), & \text{if } u \geq 0, \\ 0, & \text{if } u < 0, \end{cases} \\ \overline{I}_j(u) &= \begin{cases} I_j(u), & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases} \end{aligned} \tag{59}$$

Consider the function

$$\begin{aligned} &-\left(|u'(t)|^{p-2}u'(t)\right)' + g(t)|u(t)|^{p-2}u(t) \\ &\quad = \overline{f}(t, u(t), u'(t)), \quad t \neq t_j, \quad \text{a.e. } t \in [0,T], \\ &\quad \Delta u'(t_j) = \overline{I}_j(u(t_j)), \quad j = 1,2,\ldots,l, \\ &\quad u(0) = u(T) = 0. \end{aligned} \tag{60}$$

Obviously, $\overline{f}$ satisfies (f_1)–(f_5), and $\overline{I}_j$ satisfies (I_1), so using Theorem 8, we obtained a solution u_w of (60). Multiplying the equation by u_w^- and integrating by parks, we conclude that $u_w^- \equiv 0$. So u_w is positive. □

Theorem 10. *Suppose that (f_1)–(f_5) and (I_2) hold; then there exist positive constants d_5 and d_6 such that, for each $w \in W_0^{1,p}(0,T)$, problem (2) has one solution u_w such that $d_5 \leq \|u_w\| \leq d_6$.*

Proof. (I) We show that φ_w satisfies the (PS)-condition.

Assume that $\{u_n\}_{n\in\mathbf{N}} \subset W_0^{1,p}(0,T)$ is a sequence such that $\{\varphi_w(u_n)\}_{n\in\mathbf{N}}$ is bounded and $\varphi_w'(u_n) \to 0$ as $n \to +\infty$. We will prove that the sequence $\{u_n\}_{n\in\mathbf{N}}$ is bounded.

It follows from (f_3), (f_4), and (I_2) that

$$\begin{aligned} &\mu\varphi_w(u_n) - \varphi_w'(u_n)u_n \\ &\quad = \left(\frac{\mu}{p} - 1\right)\|u_n\|^p - \int_0^T \left(\mu F(t, u_n, w') \right. \\ &\qquad\qquad \left. - f(t, u_n, w')u_n\right) dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\left[\mu\int_0^{u_n(t_j)} I_j(s)\,ds - I_j\left(u_n\left(t_j\right)\right)u_n\left(t_j\right)\right]\\
&\geq\left(\frac{\mu}{p}-1\right)\|u_n\|^p-\int_{|u_n|\geq x_0}\left(\mu F\left(t,u_n,w'\right)\right.\\
&\qquad\left.-f\left(t,u_n,w'\right)u_n\right)dt\\
&-\int_{|u_n|\leq x_0}\left(\mu F\left(t,u_n,w'\right)-f\left(t,u_n,w'\right)u_n\right)dt\\
&\geq\left(\frac{\mu}{p}-1\right)\|u_n\|^p-C_6.
\end{aligned}
\tag{61}
$$

Hence, $\{u_n\}$ is bounded in $W_0^{1,p}(0,T)$. The following proof of (PS)-condition is similar to that in (I) of Theorem 8.

(II) We verify assumption (ii) of Lemma 7.

Using (I_2), we have

$$
\sum_{j=1}^{l}\int_0^{u(t_j)} I_j(t)\,dt\geq 0. \tag{62}
$$

The proof is similar to that in (II) of Theorem 8.

(III) We verify assumption (iii) of Lemma 7.

Take $v_0\in W_0^{1,p}(0,T)$ such that $\|v_0\|=1$. By (I_2) and (f_5), using Sobolev embedding theorem, we have

$$
\begin{aligned}
\varphi_w\left(sv_0\right)&=\frac{1}{p}s^p-\int_0^T F\left(t,sv_0(t),w'(t)\right)\\
&\quad+\sum_{j=1}^{l}\int_0^{sv_0(t_j)} I_j(t)\,dt\\
&\leq\frac{1}{p}s^p+ls\|u\|_\infty^\theta-C_7|s|^\mu\int_0^T\left|v_0(t)\right|^\mu dt-C_8T\\
&\leq\frac{1}{p}s^p+C_9s^\theta-C_{10}|s|^\mu-C_8T.
\end{aligned}
\tag{63}
$$

Since $\mu>p\geq\theta$, (63) implies that there exists $\xi_2\in R\backslash\{0\}$ such that $\|e\|>\rho$ and $\varphi_w(e)<0$ if we set $e=\xi_2v_0$. By Lemma 7, φ_w possess a critical value $c''\geq\alpha>0$, given by

$$
c''=\inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi_w\left(g(s)\right), \tag{64}
$$

where $\Gamma=\{g\in C([0,1],W_0^{1,p}(0,T)):g(0)=0,\ g(1)=e\}$. Obviously, $\varphi_w(0)=0$, so according to Lemma 7, there exists $u_w\neq 0$ and $u_w\in X$, such that

$$
\varphi_w\left(u_w\right)=c'',\qquad\varphi_w\left(u_w\right)=0. \tag{65}
$$

(IV) We prove that $d_5\leq\|u_w\|\leq d_6$.

Like (IV) of Theorem 8, we can obtain that there exist $d_5>0$ such that

$$
\|u_w\|\geq d_5. \tag{66}
$$

It follows from u_w satisfying (65) that

$$
\begin{aligned}
\left(\frac{\mu}{p}-1\right)\|u_w\|^p&\leq\mu\varphi_w\left(u_w\right)-\varphi_w'\left(u_w\right)u_w+C_{11}\\
&=\mu\varphi_w\left(u_w\right)+C_{11}\\
&=\mu\inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi_w\left(g(s)\right)+C_{11}\\
&\leq\mu\max_{s\geq0}\varphi_w\left(sv_0\right)+C_{11}\\
&\leq\mu\left(\frac{1}{p}|s|^p+\sum_{j=1}^{l}\int_0^{sv_0(t_j)} I_j(t)\,dt\right.\\
&\qquad\left.-a|s|^\mu\int_0^T\left|v_0\right|^\mu dt-bT\right)+C_{11}\\
&\leq\mu\left(\frac{1}{p}|s|^p+|s|^\theta l\|v_0\|_\infty^\theta\right.\\
&\qquad\left.-a|s|^\mu\int_0^T\left|v_0\right|^\mu dt+bT\right)+C_{11}.
\end{aligned}
\tag{67}
$$

Let

$$
r(t)=\frac{1}{p}|t|^p+t^\theta l\|v_0\|_\infty^\theta-at^\mu\int_0^T\left|v_0\right|^\mu dt,\quad t\geq0. \tag{68}
$$

Since $\mu>p\geq\theta$, then $r(t)$ can achieve its maximum at some t, which implies there exist $d_6>0$ such that

$$
\|u_w\|\leq d_6. \tag{69}
$$

This completes the proof. □

Lemma 11. *Suppose that (f_1)–(f_3) hold and (f_4), (f_5), and (I_2) hold only for positive x, then there exist positive constants d_7, and d_8 such that, for each $w\in W_0^{1,p}(0,T)$, problem* (2) *has a positive solution u_w such that $d_7\leq\|u_w\|\leq d_8$.*

Proof. The proof is similar to that of Lemma 9, so we omit it. □

Remark 12. By the same method, we can discuss the negative solution of (2).

4. The Solvability of (1)

Theorem 13. *Assume that (f_1)–(f_6) and (I_1) and (I_3) hold; then problem* (1) *has one nontrivial solution provided*

$$
k=\frac{L_2\lambda_1^{-1/p}C_1^{-1-(1/p)}}{C_p-L_1\lambda_1^{-1}C_1^{-1}-C_2^p\sum_{j=1}^{l}\alpha_j} \tag{70}
$$

satisfying $0<k<1$.

Proof. We construct a sequence $\{u_n\} \in W_0^{1,p}(0,T)$ as solutions of the following problem:

$$\begin{aligned} &-\left(\left|u_n'(t)\right|^{p-2}u_n'(t)\right)' + g(t)\left|u_n(t)\right|^{p-2}u_n(t) \\ &\quad = f\left(t, u_n(t), u_{n-1}'(t)\right), \quad t \neq t_j, \quad \text{a.e. } t \in [0,T], \\ &\quad \Delta u_n'\left(t_j\right) = I_j\left(u_n\left(t_j\right)\right), \quad j = 1,2,\ldots,l, \\ &\quad u_n(0) = u_n(T) = 0 \end{aligned} \tag{1.1$_n$}$$

obtained in Theorem 8, starting with an arbitrary $u_0 \in W_0^{1,p}(0,T)$. It follows from the Sobolev embedding theorem that $\|u_n\|_\infty \le \rho_1$. Using (1.1_n) and (1.1_{n+1}), we obtain

$$\begin{aligned} &\int_0^T \left|u_n'(t)\right|^{p-2}u_n'(t)\left(u_{n+1}' - u_n'\right)dt \\ &\quad + \int_0^T g(t)\left|u_n(t)\right|^{p-2}u_n(t)\left(u_{n+1} - u_n\right)dt \\ &\quad + \sum_{j=1}^{l} I_j\left(u_n\left(t_j\right)\right)\left(u_{n+1}\left(t_j\right) - u_n\left(t_j\right)\right) \\ &\qquad = \int_0^T f\left(t, u_n, u_{n-1}'\right)\left(u_{n+1} - u_n\right)dt, \\ &\int_0^T \left|u_{n+1}'(t)\right|^{p-2}u_{n+1}'(t)\left(u_{n+1}' - u_n'\right)dt \\ &\quad + \int_0^T g(t)\left|u_{n+1}(t)\right|^{p-2}u_{n+1}(t)\left(u_{n+1} - u_n\right)dt \\ &\quad + \sum_{j=1}^{l} I_j\left(u_{n+1}\left(t_j\right)\right)\left(u_{n+1}\left(t_j\right) - u_n\left(t_j\right)\right) \\ &\qquad = \int_0^T f\left(t, u_{n+1}, u_n'\right)\left(u_{n+1} - u_n\right)dt. \end{aligned} \tag{71}$$

Hence,

$$\begin{aligned} &C_p\left(\int_0^T \left(\left|u_{n+1}' - u_n'\right|^p + g(t)\left|u_{n+1} - u_n\right|^p\right)dt\right) \\ &\quad \le \int_0^T \left[f\left(t, u_{n+1}, u_n'\right) - f\left(t, u_n, u_{n-1}'\right)\right]\left(u_{n+1} - u_n\right)dt \\ &\qquad + \sum_{j=1}^{l}\left[I_j\left(u_n\left(t_j\right)\right) - I_j\left(u_{n+1}\left(t_j\right)\right)\right] \\ &\qquad\qquad \times \left(u_{n+1}\left(t_j\right) - u_n\left(t_j\right)\right) \\ &\quad = \int_0^T \left[f\left(t, u_{n+1}, u_n'\right) - f\left(t, u_n, u_n'\right)\right] \\ &\qquad\qquad \times \left(u_{n+1} - u_n\right)dt \\ &\qquad + \int_0^T \left[f\left(t, u_n, u_n'\right) - f\left(t, u_n, u_{n-1}'\right)\right]\left(u_{n+1} - u_n\right)dt \\ &\qquad + \sum_{j=1}^{l}\left[I_j\left(u_n\left(t_j\right)\right) - I_j\left(u_{n+1}\left(t_j\right)\right)\right] \\ &\qquad\qquad \times \left(u_{n+1}\left(t_j\right) - u_n\left(t_j\right)\right) \\ &\quad \le L_1\int_0^T \left|u_{n+1} - u_n\right|^p dt + L_2\int_0^T \left|u_n' - u_{n-1}'\right|^{p-1} \\ &\qquad\qquad \times \left|u_{n+1} - u_n\right| dt \\ &\qquad + \sum_{j=1}^{l}\alpha_j\left|u_{n+1}\left(t_j\right) - u_n\left(t_j\right)\right|^p. \end{aligned} \tag{72}$$

Using (17) and (18) and by Hölder inequality, we have

$$\begin{aligned} &\left\|u_{n+1} - u_n\right\|^{p-1} \\ &\quad \le \frac{L_2\lambda_1^{-1/p}C_1^{-1/p}}{C_p - L_1\lambda_1^{-1}C_1^{-1} - C_2^p\sum_{j=1}^{l}\alpha_j}\left[\int_0^T \left|u_n' - u_{n-1}'\right|^p dt\right]^{(p-1)/p} \\ &\quad = \frac{L_2\lambda_1^{-1/p}C_1^{-1/p}}{C_p - L_1\lambda_1^{-1}C_1^{-1} - C_2^p\sum_{j=1}^{l}\alpha_j}\left\|u_n - u_{n-1}\right\|_{W_0^{1,p}}^{p-1} \\ &\quad \le \frac{L_2\lambda_1^{-1/p}C_1^{-1-(1/p)}}{C_p - L_1\lambda_1^{-1}C_1^{-1} - C_2^p\sum_{j=1}^{l}\alpha_j}\left\|u_n - u_{n-1}\right\|^{p-1}; \end{aligned} \tag{73}$$

that is,

$$\left\|u_{n+1} - u_n\right\| \le k^{1/(p-1)}\left\|u_n - u_{n-1}\right\|. \tag{74}$$

Since k is less than 1, then it follows that $\{u_n\}$ strongly converges in $W_0^{1,p}(0,T)$, as it easily follows proving that $\{u_n\}$ is a Cauchy sequence in $W_0^{1,p}(0,T)$. By Theorem 8, we know that $u \neq 0$. In this way we obtain a nontrivial solution of (1). □

Example 14. Let $t_1 \in (0,T)$ and $p = 2$; consider the following nonlinear Dirichlet impulsive problem:

$$\begin{aligned} &-u''(t) + (-1-t)u(t) = u^3(t)\left(1 + \sin u'(t)\right), \\ &\qquad\qquad \text{a.e. } t \in [0,T], \ t \neq t_1, \\ &\Delta u'(t_1) = \sqrt[3]{u(t_1)}, \\ &u(0) = u(T) = 0. \end{aligned} \tag{75}$$

Compared with (1), $f(t,u,u') = u^3(1 + \sin u')$ and $I_1(u) = u^{1/3}$. We can take $\mu = 3$ and $\theta = 5/3$. Then by simple computation, it is easy to verify that all conditions of Theorem 13 are satisfied. Hence, by Theorem 13, problem (75) has one nontrivial solution.

Theorem 15. *Assume that* (f_1)–(f_6) *and* (I_2) *and* (I_3) *hold; then problem* (1) *has one nontrivial solution provided*

$$k = \frac{L_2 \lambda_1^{-1/p} C_1^{-1-(1/p)}}{C_p - L_1 \lambda_1^{-1} C_1^{-1} - C_2^p \sum_{j=1}^{l} \alpha_j} \tag{76}$$

satisfying $0 < k < 1$.

Proof. We construct a sequence $\{u_n\} \in W_0^{1,p}(0,T)$ as solutions of the problem (1.1_n) obtained in Theorem 10, starting with an arbitrary $u_0 \in W_0^{1,p}(0,T)$. The rest of proof is similar to that of Theorem 13, so we omit it. □

Example 16. Let $t_1 \in (0,T)$ and $p = 4$; consider the following nonlinear Dirichlet impulsive problem:

$$\begin{gathered} -\left(\left|u'(t)\right|^2 u'(t)\right)' - t^2 |u(t)|^2 u(t) = u^7(t)\left(2 + \cos u'(t)\right), \\ \text{a.e. } t \in [0,T],\ t \neq t_1, \\ \Delta u'(t_1) = u^2(t_1), \\ u(0) = u(T) = 0. \end{gathered} \tag{77}$$

Compared with (1), $f(t,u,u') = u^7(2+\cos u')$ and $I_1(u) = u^2$. We can take $\mu = 6$ and $r_1 = 2$. Then by simple computation, it is easy to verify that all conditions of Theorem 15 are satisfied. Hence, by Theorem 15, problem (77) has one nontrivial solution.

Remark 17. In order to discuss the positive solutions of (1), we can argue as in Theorem 8, by replacing f by $\overline{f}$. By the same method, we can discuss the negative solutions of (1).

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References

[1] G. Zeng, F. Wang, and J. J. Nieto, "Complexity of a delayed predator-prey model with impulsive harvest and Holling type II functional response," *Advances in Complex Systems*, vol. 11, no. 1, pp. 77–97, 2008.

[2] R. K. George, A. K. Nandakumaran, and A. Arapostathis, "A note on controllability of impulsive systems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 2, pp. 276–283, 2000.

[3] G. Jiang, Q. Lu, and L. Qian, "Complex dynamics of a Holling type II prey-predator system with state feedback control," *Chaos, Solitons & Fractals*, vol. 31, no. 2, pp. 448–461, 2007.

[4] G. Jiang, Q. Lu, and L. Qian, "Chaos and its control in an impulsive differential system," *Chaos, Solitons & Fractals*, vol. 34, no. 4, pp. 1135–1147, 2007.

[5] J. Jiao, X. Yang, L. Chen, and S. Cai, "Effect of delayed response in growth on the dynamics of a chemostat model with impulsive input," *Chaos, Solitons & Fractals*, vol. 42, no. 4, pp. 2280–2287, 2009.

[6] S. I. Nenov, "Impulsive controllability and optimization problems in population dynamics," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 36, no. 7, pp. 881–890, 1999.

[7] S. Tang and L. Chen, "Density-dependent birth rate, birth pulses and their population dynamic consequences," *Journal of Mathematical Biology*, vol. 44, no. 2, pp. 185–199, 2002.

[8] M. Choisy, J.-F. Guégan, and P. Rohani, "Dynamics of infectious diseases and pulse vaccination: teasing apart the embedded resonance effects," *Physica D*, vol. 223, no. 1, pp. 26–35, 2006.

[9] S. Gao, L. Chen, J. J. Nieto, and A. Torres, "Analysis of a delayed epidemic model with pulse vaccination and saturation incidence," *Vaccine*, vol. 24, pp. 6037–6045, 2006.

[10] G.-l. Cai and W.-G. Ge, "Positive solutions for second order impulsive differential equations with dependence on first order derivative," *Journal of Mathematical Research and Exposition*, vol. 26, no. 4, pp. 725–734, 2006.

[11] D. J. Guo and J. X. Sun, *Functional Methods of Ordinary Differential Equation*, Shandong Science Technology Press, Jinan, China, 1995.

[12] Y. P. Guo and W. G. Ge, "Positive solutions for three-point boundary value problems with dependence on the first order derivative," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 1, pp. 291–301, 2004.

[13] J. H. Shen and W. B. Wang, "Impulsive boundary value problems with nonlinear boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 11, pp. 4055–4062, 2008.

[14] Y. Zhao and H. Chen, "Multiplicity of solutions to two-point boundary value problems for second-order impulsive differential equations," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 925–931, 2008.

[15] D. De Figueiredo, M. Girardi, and M. Matzeu, "Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques," *Differential and Integral Equations*, vol. 17, no. 1-2, pp. 119–126, 2004.

[16] K. Teng and C. Zhang, "Existence of solution to boundary value problem for impulsive differential equations," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 5, pp. 4431–4441, 2010.

[17] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 2, pp. 680–690, 2009.

[18] J. J. Nieto, "Variational formulation of a damped Dirichlet impulsive problem," *Applied Mathematics Letters*, vol. 23, no. 8, pp. 940–942, 2010.

[19] J. Xiao, J. J. Nieto, and Z. Luo, "Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 426–432, 2012.

[20] Y. Tian and W. Ge, "Applications of variational methods to boundary-value problem for impulsive differential equations," *Proceedings of the Edinburgh Mathematical Society*, vol. 51, no. 2, pp. 509–527, 2008.

[21] J. Sun and H. Chen, "Variational method to the impulsive equation with Neumann boundary conditions," *Boundary Value Problems*, vol. 2009, Article ID 316812, 17 pages, 2009.

[22] J. Sun, H. Chen, J. J. Nieto, and M. Otero-Novoa, "The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 12, pp. 4575–4586, 2010.

[23] Z. Zhang and R. Yuan, "An application of variational methods to Dirichlet boundary value problem with impulses," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 1, pp. 155–162, 2010.

[24] P. Chen and X. Tang, "Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems," *Applied Mathematics and Computation*, vol. 218, no. 24, pp. 11775–11789, 2012.

[25] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6, World Scientific Publishing Co. Inc., Singapore, 1989.

[26] P. Lindqvist, "On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|\nabla u|^{p-2}\nabla u = 0$," *Proceedings of the American Mathematical Society*, vol. 109, no. 1, pp. 157–164, 1990.