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Research Article

Best Proximity Point Results for Modified α-ψ-Proximal Rational Contractions

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We first introduce certain new concepts of α-η-proximal admissible and α-η-ψ-rational proximal contractions of the first and second kinds. Then we establish certain best proximity point theorems for such rational proximal contractions in metric spaces. As an application, we deduce best proximity and fixed point results in partially ordered metric spaces. The presented results generalize and improve various known results from best proximity point theory. Several interesting consequences of our obtained results are presented in the form of new fixed point theorems which contain famous Banach's contraction principle and some of its generalizations as special cases. Moreover, some examples are given to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

Let (X, d) be a metric space and T be a self-mapping defined on a subset of X. Fixed point theory is an important tool for solving equations of the kind $Tx = x$, whose solutions are the fixed points of the mapping T. Many problems arising in different areas of mathematics, such as optimization, variational analysis, and differential equations, can be modeled as fixed point equations of the form $Tx = x$. On the other hand, if T is not a self-mapping, the equation $Tx = x$ could have no solutions and, in this case, it is of a certain interest to determine an element x that is in some sense closest to Tx. One of the most interesting results in this direction is due to Fan [1] and can be stated as follows.

Theorem F. *Let K be a nonempty compact convex subset of a normed space E and $T : K \to E$ be a continuous non-self-mapping. Then there exists an x such that $\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}$.*

Many generalizations and extensions of this result appeared in the literature (see [2–6] and, references therein).

Let A and B be two nonempty subsets of a metric space (X, d). A best proximity point of a nonself-mapping $T : A \to B$ is a point $x \in A$ satisfying the equality $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Though best approximation theorems ensure the existence of approximation solutions, such results need not yield optimal solutions. But best proximity point theorems provide sufficient conditions that assure the existence of approximate solutions which are optimal as well. For more details on this approach, we refer the reader to [5, 7–32].

The aim of this paper is to introduce certain new concepts of α-η-proximal admissible and α-η-ψ-rational proximal contractions of the first and second kinds. Then we establish certain best proximity point theorems for such rational proximal contractions. As an application, we deduce best proximity and fixed point results in partially ordered metric spaces. The presented results generalize and improve various known results from best proximity point theory. Several interesting consequences of our obtained results are presented in the form of new fixed point theorems which contain famous Banach's contraction principle and some of its generalizations as special cases. Moreover, some examples are given to illustrate the usability of the obtained results.

Now we give some basic notations and definitions that will be used in the sequel.

Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}, \tag{1}$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. For the map $T : A \to B$, we define the set of all best proximity points of T by

$$B_{\text{est}}(T) = \{x \in A : d(x, Tx) = d(A, B)\}. \tag{2}$$

Definition 1. Let (X, d) be a metric space and let A and B be two nonempty subsets of X. Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $d(x, y_n) \to d(x, B)$ for some x in A, has a convergent subsequence.

Obviously, every set is approximatively compact with respect to itself.

Very recently, Nashine et al. [22] introduced rational proximal contraction of the first and second kinds as follows.

Definition 2. Let A and B be nonempty subsets of a metric space (X, d). Then $T : A \to B$ is said to be a rational proximal contraction of the first kind if there exist nonnegative real numbers a, b, c, and d with $a + b + 2c + 2d < 1$, such that, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{aligned} &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B) \\ &\quad\Longrightarrow d(u_1, u_2) \\ &\quad\leq ad(x_1, x_2) \\ &\quad\quad + b\frac{[1 + d(x_1, u_1)]\, d(x_2, u_2)}{1 + d(x_1, x_2)} \\ &\quad\quad + c[d(x_1, u_1) + d(x_2, u_2)] \\ &\quad\quad + d[d(x_1, u_2) + d(x_2, u_1)]. \end{aligned} \tag{3}$$

Definition 3. Let A and B be nonempty closed subsets of a metric space (X, d). Then $T : A \to B$ is said to be a rational proximal contraction of the second kind if there exist nonnegative real numbers a, b, c, and d with $a+b+2c+2d < 1$, such that, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{aligned} &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B) \\ &\quad\Longrightarrow d(Tu_1, Tu_2) \\ &\quad\leq ad(Tx_1, Tx_2) \\ &\quad\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\ &\quad\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\ &\quad\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)]. \end{aligned} \tag{4}$$

Note that a rational proximal contraction of the second kind is not necessarily a rational proximal contraction of the first kind; for examples, see [22].

Definition 4 (see [28]). Let T be a self-mapping on a metric space (X, d) and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{5}$$

Recently, Jleli and Samet [15] introduced new concepts of α-proximal admissible and α-ψ-proximal contractive type mappings as follows.

Definition 5 (see [15]). Let $T : A \to B$, $\alpha : A \times A \to [0, \infty)$. We say that T is α-proximal admissible if

$$\begin{aligned} &\alpha(x_1, x_2) \geq 1, \\ &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B), \\ &\qquad\qquad \Longrightarrow \alpha(u_1, u_2) \geq 1 \end{aligned} \tag{6}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $A = B$, then α-proximal admissible map T reduces to α-admissible map.

Definition 6 (see [15]). A nonself-mapping $T : A \to B$ is said to be α-ψ-proximal contraction if

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)) \tag{7}$$

for all $x, y \in A$, $\alpha : A \times A \to [0, \infty)$, $\psi \in \Psi$.

Salimi et al. [27] modified the concept of α-admissible mappings as follows.

Definition 7. Let T be a self-mapping on a metric space (X, d) and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty). \tag{8}$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 4. Also, if we take $\alpha(x, y) = 1$, then we say that T is η-subadmissible mapping.

For the examples of α-admissible mappings with respect to η, we refer to [27] and the examples in the next section.

2. Best Proximity and Fixed Point Results in Metric Spaces

First we modify the notion of α-proximal admissible mapping as follows.

Definition 8. Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be functions. We say that T is α-proximal admissible with respect to η if, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{gathered} \alpha(x_1, x_2) \geq \eta(x_1, x_2), \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \\ \Longrightarrow \alpha(u_1, u_2) \geq \eta(u_1, u_2). \end{gathered} \tag{9}$$

Note that if we take $\eta(x, y) = 1$ for all $x, y \in A$, then this definition reduces to Definition 5. In case $\alpha(x, y) = 1$ for all $x, y \in A$, then we shall say that T is η-proximal subadmissible mapping.

Clearly, if $A = B$, then the previous definition reduces to Definition 7.

Definition 9. Let A and B be nonempty subsets of a metric space (X, d). Then $T : A \to B$ is said to be α-η-rational proximal contraction of the first kind if there exist nonnegative real numbers a, b, c, and d with $a + b + 2c + 2d < 1$, such that, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{aligned} &\alpha(x_1, x_2) \geq \eta(x_1, x_2) \\ &d(u_1, Tx_1) = d(A, B) \\ &d(u_2, Tx_2) = d(A, B) \\ &\Longrightarrow d(u_1, u_2) \\ &\leq ad(x_1, x_2) \\ &\quad + b\frac{[1 + d(x_1, u_1)]\, d(x_2, u_2)}{1 + d(x_1, x_2)} \\ &\quad + c[d(x_1, u_1) + d(x_2, u_2)] \\ &\quad + d[d(x_1, u_2) + d(x_2, u_1)]. \end{aligned} \tag{10}$$

In case $\eta(x, y) = 1$ for all $x, y \in A$, then $T : A \to B$ is said to be α-rational proximal contraction of the first kind.

Definition 10. Let A and B be nonempty closed subsets of a metric space (X, d). Then $T : A \to B$ is said to be a α-η-rational proximal contraction of the second kind if there exist nonnegative real numbers a, b, c, and d with $a + b + 2c + 2d < 1$, such that, for all $x_1, x_2, u_1, u_2 \in A$,

$$\begin{aligned} &\alpha(x_1, x_2) \geq \eta(x_1, x_2), \\ &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B) \\ &\Longrightarrow d(Tu_1, Tu_2) \\ &\leq ad(Tx_1, Tx_2) \\ &\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\ &\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\ &\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)]. \end{aligned} \tag{11}$$

In case $\eta(x, y) = 1$ for all $x, y \in A$, then $T : A \to B$ is said to be α-rational proximal contraction of the second kind.

We are ready to prove the following best proximity point result for α-η-rational proximal contraction of the first kind.

Theorem 11. *Let A and B be nonempty closed subsets of a complete metric space (X, d) and let B be approximatively compact with respect to A. Assume that $\alpha, \eta : A \times A \to [0, \infty)$ are functions, A_0 and B_0 are nonempty, and $T : A \to B$ is an α-η-rational proximal contraction of the first kind which satisfies the following assertions:*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is α-proximal admissible with respect to η,*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1), \tag{12}$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.*

Then there exists $z \in A_0$, such that

$$d(z, Tz) = d(A, B). \tag{13}$$

Moreover, if $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in B_{est}(T)$, then z is unique best proximity point of T.

Proof. By (iii) there exist elements x_0 and x_1 in A_0, such that

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1). \tag{14}$$

On the other hand, $T(A_0) \subseteq B_0$, then and there exists $x_2 \in A_0$, such that

$$d(x_2, Tx_1) = d(A, B). \tag{15}$$

Now, since, T is α-η-proximal admissible, then we have $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. That is,

$$d(x_2, Tx_1) = d(A, B), \qquad \alpha(x_1, x_2) \geq \eta(x_1, x_2). \tag{16}$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$, such that

$$d(x_3, Tx_2) = d(A, B). \tag{17}$$

Thus,

$$\begin{gathered} d(x_2, Tx_1) = d(A, B), \\ d(x_3, Tx_2) = d(A, B), \\ \alpha(x_1, x_2) \geq \eta(x_1, x_2) \end{gathered} \tag{18}$$

together with T is α-η-proximal admissible imply that $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$. Hence,

$$d(x_3, Tx_2) = d(A, B), \qquad \alpha(x_2, x_3) \geq \eta(x_2, x_3). \tag{19}$$

Continuing this process, we get

$$\begin{gathered} d(x_{n+1}, Tx_n) = d(A, B), \\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{gathered} \tag{20}$$

Since T is α-η-rational proximal contraction of the first kind, then we have

$$\begin{aligned} d(x_n, x_{n+1}) \leq\ & ad(x_{n-1}, x_n) \\ & + b\frac{[1 + d(x_{n-1}, x_n)]\, d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \\ & + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ & + d[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ \leq\ & ad(x_{n-1}, x_n) + bd(x_n, x_{n+1}) \\ & + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ & + d[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned} \tag{21}$$

which implies

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n), \tag{22}$$

where $h = (a + c + d)/(1 - b - c - d) < 1$. That is, $\{x_n\}$ is a Cauchy sequence in A and since (X, d) is a complete metric space and A is closed, so there exists an element $z \in A$ such that $x_n \to z$ as $n \to \infty$. Also, we have

$$\begin{aligned} d(z, B) &\leq d(z, Tx_n) \\ &\leq d(z, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(z, x_{n+1}) + d(A, B) \\ &\leq d(z, x_{n+1}) + d(z, B). \end{aligned} \tag{23}$$

Taking limit as $n \to \infty$ in the previous inequality, we have

$$\lim_{n \to \infty} d(z, Tx_n) = d(z, B). \tag{24}$$

As B is approximatively compact with respect to A, so the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y \in B$. Hence,

$$d(z, y) = \lim_{n \to \infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \tag{25}$$

and so $z \in A_0$. Now, since $T(A_0) \subseteq B_0$, then, $d(w, Tz) = d(A, B)$ for some $w \in A$. From (iv) and (66), we have $\alpha(x_n, z) \geq \eta(x_n, z)$ for all $n \in \mathbb{N}$. Therefore, we proved that

$$\begin{gathered} \alpha(x_n, z) \geq \eta(x_n, z), \\ d(w, Tz) = d(A, B), \\ d(x_{n+1}, Tx_n) = d(A, B) \end{gathered} \tag{26}$$

for all $n \in \mathbb{N}$. Since T is a α-η-rational proximal contraction of the first kind, so we have

$$\begin{aligned} d(w, x_{n+1}) \leq\ & ad(z, x_n) \\ & + b\frac{[1 + d(z, w)]\, d(x_n, x_{n+1})}{1 + d(z, x_n)} \\ & + c[d(z, w) + d(x_n, x_{n+1})] \\ & + d[d(z, x_{n+1}) + d(x_n, w)]. \end{aligned} \tag{27}$$

Taking limit as $n \to \infty$ in the previous inequality, we get

$$d(w, z) \leq (c + d)\, d(w, z). \tag{28}$$

As $c + d < 1$, so $w = z$. This implies that

$$d(z, Tz) = d(w, Tz) = d(A, B). \tag{29}$$

Assume that y^* is another best proximity point of T such that $\alpha(z, y^*) \geq \eta(z, y^*)$. That is,

$$\begin{gathered} \alpha(z, y^*) \geq \eta(z, y^*), \\ d(z, Tx^*) = d(A, B), \\ d(y^*, Ty^*) = d(A, B). \end{gathered} \tag{30}$$

Now, since T is α-η-rational proximal contraction of the first kind, so we have

$$\begin{aligned} d(z, y^*) \leq\ & ad(z, y^*) \\ & + b\frac{[1 + d(z, z)]\, d(y^*, y^*)}{1 + d(z, y^*)} \\ & + c[d(z, z) + d(y^*, y^*)] \\ & + d[d(z, y^*) + d(y^*, z)], \end{aligned} \tag{31}$$

which implies that $d(z, y^*) \leq (a + 2d)d(z, y^*)$. As $a + 2d < 1$, so $z = y^*$. That is, z is a unique best proximity point of T. □

By taking $\eta(x, y) = 1$ in Theorem 11, we deduce the following corollary.

Corollary 12. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that B is approximatively compact with respect to A. Assume that $\alpha : A \times A \to [0, \infty)$, A_0 and B_0 are nonempty, and $T : A \to B$ is an α-rational proximal contraction of the first kind satisfying the following assertions:*

(i) *$T(A_0) \subseteq B_0$,*

(ii) *T is α-proximal admissible,*

(iii) *there exist elements x_0 and x_1 in A_0, such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1, \tag{32}$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists $z \in A_0$, such that

$$d(z, Tz) = d(A, B). \tag{33}$$

Moreover, if $\alpha(x, y) \geq 1$ for all $x, y \in B_{est}(T)$, then z is unique best proximity point of T.

If in the previous corollary we take $\alpha(x, y) = 1$, then we obtain the following result.

Corollary 13 (see [22, Theorem 3.1]). *Let A and B be nonempty closed subsets of a complete metric space (X, d) and let B be approximatively compact with respect to A. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a rational proximal contraction of the first kind with $T(A_0) \subseteq B_0$. Then there exists a unique $z \in A_0$, such that*

$$d(z, Tz) = d(A, B). \tag{34}$$

Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to z.

Example 14. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ be metric on X. Suppose $A = (-\infty, -1]$ and $B = [5/4, +\infty)$. Define $T : A \to B$ by

$$Tx = \begin{cases} -x^3 + \dfrac{5}{4}, & \text{if } x \in (-\infty, -2) \setminus \{-3\}, \\ \dfrac{9}{4}, & \text{if } x = -3, \\ -\dfrac{1}{4}x + 1, & \text{if } x \in [-2, -1]. \end{cases} \tag{35}$$

Also, define $\alpha : X^2 \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in [-2, -1], \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Clearly, $d(A, B) = 9/4$. Now, we have

$$\begin{aligned} A_0 &= \Big\{x \in A : d(x, y) = d(A, B) \\ &= \frac{9}{4} \text{ for some } y \in B\Big\} = \{-1\}, \\ B_0 &= \Big\{y \in B : d(x, y) = d(A, B) \\ &= \frac{9}{4} \text{ for some } x \in A\Big\} = \Big\{\frac{5}{4}\Big\}. \end{aligned} \tag{37}$$

Also, $T(A_0) \subseteq B_0$. Let

$$\begin{aligned} &\alpha(x_1, x_2) \geq 1, \\ &d(u_1, Tx_1) = d(A, B) = \frac{9}{4}, \\ &d(u_2, Tx_2) = d(A, B) = \frac{9}{4}. \end{aligned} \tag{38}$$

Then,

$$\begin{aligned} &x_1, x_2 \in [-2, -1], \\ &d(u_1, Tx_1) = \frac{9}{4}, \\ &d(u_2, Tx_2) = \frac{9}{4}. \end{aligned} \tag{39}$$

Note that $Tw \in [5/4, 3/2]$ for all $w \in [-2, -1]$. Hence, $u_1 = u_2 = -1$. That is, $\alpha(u_1, u_2) \geq 1$. That is, T is a α-proximal admissible mapping. Also, assume that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Therefore, $\{x_n\} \subseteq [-2, -1]$ and then $x \in [-2, -1]$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Further, $d(-1, T(-1)) = d(A, B) = 9/4$ and $\alpha(-1, -1) \geq 1$.

Again, assume that

$$\begin{aligned} &\alpha(x_1, x_2) \geq 1, \\ &d(u_1, Tx_1) = d(A, B) = \frac{9}{4}, \\ &d(u_2, Tx_2) = d(A, B) = \frac{9}{4}. \end{aligned} \tag{40}$$

Then $x_1, x_2 \in [-2, -1]$ and $u_1 = u_2 = -1$. Hence,

$$\begin{aligned} d(u_1, u_2) = 0 \leq a d(x_1, x_2) & \\ &+ b \frac{[1 + d(x_1, u_1)] d(x_2, u_2)}{1 + d(x_1, x_2)} \\ &+ c [d(x_1, u_1) + d(x_2, u_2)] \\ &+ d [d(x_1, u_2) + d(x_2, u_1)]. \end{aligned} \tag{41}$$

Thus, all of the conditions of Corollary 12 (Theorem 11) hold and there exists a unique $z = -1 \in A_0$, such that

$$d(-1, T(-1)) = d(A, B) = \frac{9}{4}. \tag{42}$$

Example 15. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ be metric on X. Suppose $A = [0, 1] \cup \{2, 20\}$ and $B = [0, 1/4] \cup \{2, 20\}$. Define $T : A \to B$ by

$$Tx = \begin{cases} 20, & \text{if } x = 2, \\ 2, & \text{if } x = 20, \\ \dfrac{1}{4}x, & \text{if } x \in [0, 1]. \end{cases} \tag{43}$$

Also, define $\alpha : X^2 \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{44}$$

Clearly, $d(A, B) = 0$. Now, we have

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = d(A, B) = 0 \\ &\quad \text{for some } \in B\} = B, \\ B_0 &= \{y \in B : d(x, y) = d(A, B) = 0 \\ &\quad \text{for some } x \in A\} = B. \end{aligned} \tag{45}$$

Also, $T(A_0) \subseteq B_0$. Let

$$\begin{gathered}\alpha(x_1, x_2) \geq 1,\\ d(u_1, Tx_1) = d(A, B) = 0,\\ d(u_2, Tx_2) = d(A, B) = 0.\end{gathered} \tag{46}$$

Then, $x_1, x_2 \in [0, 1]$ and $u_1 = Tx_1$ and $u_2 = Tx_2$. Note that $Tw \in [0, 1]$ for all $w \in [0, 1]$; Hence, $u_1, u_2 \in [0, 1]$ that is, $\alpha(u_1, u_2) \geq 1$. That is, T is a α-proximal admissible mapping. Also, assume that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Therefore, $\{x_n\} \subseteq [0, 1]$ and then $x \in [0, 1]$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Further, $d(0, T(0)) = d(A, B) = 0$ and $\alpha(0, 0) \geq 1$

Let $a = 1/2$, $b = 1/6$, and $c = d = 1/24$.

Again, assume that

$$\begin{gathered}\alpha(x_1, x_2) \geq 1,\\ d(u_1, Tx_1) = d(A, B) = 0,\\ d(u_2, Tx_2) = d(A, B) = 0.\end{gathered} \tag{47}$$

Then $x_1, x_2 \in [0, 1]$ and $u_1 = Tx_1 = (1/4)x_1$ and $u_2 = Tx_2 = (1/4)x_2$. Hence,

$$\begin{aligned}d(u_1, u_2) &= \frac{1}{4}|x - y| \leq \frac{1}{2}|x - y|\\ &\leq \frac{1}{2} d(x_1, x_2) d(x_1, x_2)\\ &\quad + \frac{1}{6}\frac{[1 + d(x_1, u_1)] d(x_2, u_2)}{1 + d(x_1, x_2)}\\ &\quad + \frac{1}{24}[d(x_1, u_1) + d(x_2, u_2)]\\ &\quad + \frac{1}{24}[d(x_1, u_2) + d(x_2, u_1)].\end{aligned} \tag{48}$$

All of the conditions of Corollary 12 (Theorem 11) hold and there exists a unique $z = 0 \in A_0$, such that

$$d(0, T(0)) = d(A, B) = 0. \tag{49}$$

Let

$$\begin{gathered}d(0, T0) = d(A, B) = 0,\\ d(2, T20) = d(A, B) = 0,\end{gathered} \tag{50}$$

imply

$$\begin{aligned}d(0, 20) &\leq \frac{1}{2} d(0, 2)\\ &\quad + \frac{1}{6}\frac{[1 + d(0, 0)] d(2, 20)}{1 + d(0, 2)}\\ &\quad + \frac{1}{24}[d(0, 0) + d(2, 20)]\\ &\quad + \frac{1}{24}[d(0, 20) + d(2, 0)].\end{aligned} \tag{51}$$

Then,

$$20 \leq 1 + 1 + \frac{18}{24} + \frac{22}{24}, \tag{52}$$

which is a contradiction. Therefore, Corollary 32 [22, Theorem 3.1] cannot be applied here.

If in Theorem 11 we take $\alpha(x, y) = 1$, then we obtain the following result.

Corollary 16. *Let A and B be nonempty closed subsets of a complete metric space (X, d) and let B be approximatively compact with respect to A. Assume that $\eta : A \times A \to [0, \infty)$, A_0 and B_0 are nonempty, and $T : A \to B$ is a nonself-mapping satisfying the following assertions:*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is η-proximal subadmissible,*

(iii) *there exist elements x_0 and x_1 in A_0, such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \eta(x_0, x_1) \leq 1, \tag{53}$$

(iv)

$$\begin{aligned}&\eta(x_1, x_2) \leq 1,\\ &d(u_1, Tx_1) = d(A, B),\\ &d(u_2, Tx_2) = d(A, B)\\ &\qquad \Longrightarrow d(u_1, u_2)\\ &\qquad \leq ad(x_1, x_2)\\ &\qquad\quad + b\frac{[1 + d(x_1, u_1)] d(x_2, u_2)}{1 + d(x_1, x_2)}\\ &\qquad\quad + c[d(x_1, u_1) + d(x_2, u_2)]\\ &\qquad\quad + d[d(x_1, u_2) + d(x_2, u_1)],\end{aligned} \tag{54}$$

(v) *if $\{x_n\}$ is a sequence in A such that $\eta(x_n, x_{n+1}) \leq 1$ and $x_n \to x \in A$ as $n \to \infty$, then $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$,*

where $a + b + 2c + 2d < 1$. Then there exists $z \in A_0$, such that,

$$d(z, Tz) = d(A, B). \tag{55}$$

Moreover, if $\eta(x, y) \leq 1$ for all $x, y \in B_{est}(T)$, then z is unique.

The following are immediate consequences of Theorem 11.

Theorem 17. *Let X be a complete metric space and let $T : X \to X$ be a mapping satisfying the following assertions:*

(i) *T is α-admissible with respect to η,*

(ii) *there exists element x_0 in X, such that*

$$\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0), \tag{56}$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}$,

(iv)

$$\begin{aligned}\alpha(x_1, x_2) \geq \eta(x_1, x_2) &\Longrightarrow d(Tx_1, Tx_2) \\ &\leq ad(x_1, x_2) \\ &\quad + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \\ &\quad + c[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\quad + d[d(x_1, Tx_2) + d(x_2, Tx_1)],\end{aligned} \tag{57}$$

where $a + b + 2c + 2d < 1$. *Then* T *has a unique fixed point in* X.

If in Theorem 17 we take $\eta(x, y) = 1$, then we obtain the following result.

Theorem 18. *Let* X *be a complete metric space and let* $T : X \to X$ *be a mapping satisfying the following assertions:*

(i) T *is* α*-admissible,*

(ii) *there exists element* x_0 *in* X, *such that,*

$$\alpha(x_0, Tx_0) \geq 1 \tag{58}$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$,

(iv)

$$\begin{aligned}\alpha(x_1, x_2) \geq 1 &\Longrightarrow d(Tx_1, Tx_2) \\ &\leq ad(x_1, x_2) \\ &\quad + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \\ &\quad + c[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\quad + d[d(x_1, Tx_2) + d(x_2, Tx_1)],\end{aligned} \tag{59}$$

where $a + b + 2c + 2d < 1$. *Then* T *has a unique fixed point in* X.

If in Theorem 17 we take $\alpha(x, y) = 1$, then we obtain the following result.

Theorem 19. *Let* X *be a complete metric space and* $T : X \to X$ *be a mapping satisfying the following assertions:*

(i) T *is* η*-subadmissible,*

(ii) *there exists element* x_0 *in* X, *such that*

$$\eta(x_0, Tx_0) \leq 1, \tag{60}$$

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\eta(x_n, x_{n+1}) \leq 1$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\eta(x_n, x) \leq 1$ *for all* $n \in \mathbb{N}$,

(iv)

$$\begin{aligned}\eta(x_1, x_2) \leq 1 &\Longrightarrow d(Tx_1, Tx_2) \\ &\leq ad(x_1, x_2) \\ &\quad + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \\ &\quad + c[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\quad + d[d(x_1, Tx_2) + d(x_2, Tx_1)],\end{aligned} \tag{61}$$

where $a + b + 2c + 2d < 1$. *Then* T *has a unique fixed point in* X.

If in Theorem 18 we take $\alpha(x, y) = 1$, then we obtain the following fixed point result for rational contraction of first kind.

Theorem 20. *Let* X *be a complete metric space and let* $T : X \to X$ *be a mapping satisfying the following rational inequality:*

$$\begin{aligned}d(Tx_1, Tx_2) &\leq ad(x_1, x_2) \\ &\quad + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \\ &\quad + c[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ &\quad + d[d(x_1, Tx_2) + d(x_2, Tx_1)],\end{aligned} \tag{62}$$

where $a + b + 2c + 2d < 1$. *Then* T *has a unique fixed point in* X.

We now establish best proximity point result for α-η-rational proximal contraction of the second kind.

Theorem 21. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* A *is approximatively compact with respect to* B. *Assume that* $\alpha, \eta : A \times A \to [0, \infty)$, A_0 *and* B_0 *are nonempty, and* $T : A \to B$ *is a continuous* α*-*η*-rational proximal contraction of the second kind, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) T *is* α*-proximal admissible with respect to* η,

(iii) *There exist elements* x_0 *and* x_1 *in* A_0, *such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq \eta(x_0, x_1). \tag{63}$$

Then there exists $x \in B_{est}(T)$ and, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, and $Tx = Tz$ for all $x, z \in B_{est}(T)$ when $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in B_{est}(T)$.

Proof. Following the same lines of the proof of Theorem 11, there exists a sequence $\{x_n\} \in A_0$, such that

$$\begin{gathered} d(x_{n+1}, Tx_n) = d(A, B), \\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{gathered} \tag{64}$$

Since T is a α-η-rational proximal contraction of the second kind, we get

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq ad(Tx_{n-1}, Tx_n) \\ &\quad + b\frac{[1 + d(Tx_{n-1}, Tx_n)]\, d(Tx_n, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_n)} \\ &\quad + c[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad + d[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)] \\ &\leq ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) \\ &\quad + c[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad + d[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})], \end{aligned} \tag{65}$$

which implies

$$d(Tx_n, Tx_{n+1}) \leq hd(Tx_{n-1}, Tx_n), \tag{66}$$

where $h = (a + c + d)/(1 - b - c - d) < 1$. That is, $\{Tx_n\}$ is a Cauchy sequence and since (X, d) is a complete metric space and B is closed, so there exists an element $y^* \in B$ such that $Tx_n \to y^*$ as $n \to \infty$. Also, we have

$$\begin{aligned} d(y^*, A) &\leq d(y^*, x_{n+1}) \\ &\leq d(y^*, Tx_n) + d(Tx_n, x_{n+1}) \\ &= d(y^*, Tx_n) + d(A, B) \\ &\leq d(y^*, Tx_n) + d(y^*, A). \end{aligned} \tag{67}$$

Taking limit as $n \to \infty$ in the previous inequality, we have

$$\lim_{n \to \infty} d(y^*, x_n) = d(y^*, A). \tag{68}$$

Since A is approximatively compact with respect to B, so the sequence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $x^* \in A$. Now, by applying continuity of T, we get

$$d(x^*, Tx^*) = \lim_{k \to \infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B). \tag{69}$$

That is, $x^* \in B_{\text{est}}(T)$. Now, assume that z^* is a another best proximity point of T. That is, $d(z^*, Tz^*) = d(A, B)$. Now, since, T is a α-η-rational proximal contraction of the second kind and $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in B_{\text{est}}(T)$, then

$$\begin{aligned} d(Tx^*, Tz^*) &\leq ad(Tx^*, Tz^*) \\ &\quad + b\frac{[1 + d(Tx^*, Tx^*)]\, d(Tz^*, Tz^*)}{1 + d(Tx^*, Tz^*)} \\ &\quad + c[d(Tx^*, Tx^*) + d(Tz^*, Tz^*)] \\ &\quad + d[d(Tx^*, Tz^*) + d(Tz^*, Tx^*)]. \end{aligned} \tag{70}$$

This implies that

$$d(Tx^*, Tz^*) \leq (c + d)\, d(Tx^*, Tz^*). \tag{71}$$

And, hence, $d(Tx^*, Tz^*) = 0$ gives us $Tx^* = Tz^*$. □

Corollary 22. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B. Assume that $\alpha : A \times A \to [0, \infty)$, A_0 and B_0 are nonempty, and $T : A \to B$ is a continuous mapping, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is α-proximal admissible,*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1, \tag{72}$$

(iv)

$$\begin{aligned} &\alpha(x_1, x_2) \geq 1, \\ &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B) \\ &\qquad \Longrightarrow d(Tu_1, Tu_2) \\ &\qquad \leq ad(Tx_1, Tx_2) \\ &\qquad\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\ &\qquad\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\ &\qquad\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)], \end{aligned} \tag{73}$$

where $a+b+2c+2d < 1$. Then there exists $x \in B_{est}(T)$ and, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, and $Tx = Tz$ for all $x, z \in B_{est}(T)$ when $\alpha(x, y) \geq 1$ for all $x, y \in B_{est}(T)$.

If in the previous corollary we take $\alpha(x, y) = 1$, then we have the following result.

Corollary 23 (see [22, Theorem 3.2]). *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a continuous* *rational proximal contraction of the second kind, such that $T(A_0) \subseteq B_0$. Then there exists $x \in B_{est}(T)$ and, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, and $Tx = Tz$ for all $x, z \in B_{est}(T)$.*

Corollary 24. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A is approximatively compact with respect to B. Assume that $\eta : A \times A \to [0, \infty)$, A_0 and B_0 are nonempty and $T : A \to B$ is a continuous mapping, such that $T(A_0) \subseteq B_0$ and T is η-proximal subadmissible, such that*

$$\begin{aligned} &\eta(x_1, x_2) \le 1 \\ &d(u_1, Tx_1) = d(A, B) \\ &d(u_2, Tx_2) = d(A, B) \\ &\qquad \Longrightarrow d(Tu_1, Tu_2) \\ &\qquad \le ad(Tx_1, Tx_2) \\ &\qquad\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\ &\qquad\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\ &\qquad\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)] \end{aligned} \tag{74}$$

for all $x_1, x_2, u_1, u_2 \in A$, where $a + b + 2c + 2d < 1$. Then there exists $x \in B_{est}(T)$ and, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, and $Tx = Tz$ for all $x, z \in B_{est}(T)$ when $\eta(x, y) \le 1$ for all $x, y \in B_{est}(T)$.

The following are immediate consequences of Theorem 21.

Theorem 25. *Let X be a complete metric space and let $T : X \to X$ be a continuous mapping satisfying the following assertions:*

(i) *T is α-admissible with respect to η,*

(ii) *there exists element x_0 in X, such that*

$$\alpha(x_0, Tx_0) \ge \eta(x_0, Tx_0), \tag{75}$$

(iii)

$$\begin{aligned} &\alpha(x_1, x_2) \ge \eta(x_1, x_2) \\ &\qquad \Longrightarrow d(T^2x_1, T^2x_2) \\ &\qquad \le ad(Tx_1, Tx_2) \\ &\qquad\quad + b\frac{[1 + d(Tx_1, T^2x_1)]\, d(Tx_2, T^2x_2)}{1 + d(Tx_1, Tx_2)} \\ &\qquad\quad + c[d(Tx_1, T^2x_1) + d(Tx_2, T^2x_2)] \\ &\qquad\quad + d[d(Tx_1, T^2x_2) + d(Tx_2, T^2x_1)], \end{aligned} \tag{76}$$

where $a + b + 2c + 2d < 1$. Then T has a unique fixed point in X.

If in Theorem 25 we take $\eta(x, y) = 1$, then we obtain the following result.

Theorem 26. *Let X be a complete metric space and let $T : X \to X$ be a continuous mapping satisfying the following assertions:*

(i) *T is α-admissible,*

(ii) *there exists element x_0 in X, such that*

$$\alpha(x_0, Tx_0) \ge 1 \tag{77}$$

(iii)

$$\begin{aligned} &\alpha(x_1, x_2) \ge 1 \\ &\qquad \Longrightarrow d(T^2x_1, T^2x_2) \\ &\qquad \le ad(Tx_1, Tx_2) \\ &\qquad\quad + b\frac{[1 + d(Tx_1, T^2x_1)]\, d(Tx_2, T^2x_2)}{1 + d(Tx_1, Tx_2)} \\ &\qquad\quad + c[d(Tx_1, T^2x_1) + d(Tx_2, T^2x_2)] \\ &\qquad\quad + d[d(Tx_1, T^2x_2) + d(Tx_2, T^2x_1)], \end{aligned} \tag{78}$$

where $a + b + 2c + 2d < 1$. Then T has a unique fixed point in X.

If in Theorem 25 we take $\alpha(x, y) = 1$, then we obtain the following result.

Theorem 27. *Let X be a complete metric space and let $T : X \to X$ be a continuous mapping satisfying the following assertions:*

(i) *T is η-subadmissible*

(ii) *there exists element x_0 in X, such that*

$$\eta(x_0, Tx_0) \le 1, \tag{79}$$

(iii)

$$\begin{aligned} &\eta(x_1, x_2) \le 1 \\ &\qquad \Longrightarrow d(T^2x_1, T^2x_2) \\ &\qquad \le ad(Tx_1, Tx_2) \\ &\qquad\quad + b\frac{[1 + d(Tx_1, T^2x_1)]\, d(Tx_2, T^2x_2)}{1 + d(Tx_1, Tx_2)} \\ &\qquad\quad + c[d(Tx_1, T^2x_1) + d(Tx_2, T^2x_2)] \\ &\qquad\quad + d[d(Tx_1, T^2x_2) + d(Tx_2, T^2x_1)], \end{aligned} \tag{80}$$

where $a + b + 2c + 2d < 1$*. Then* T *has a unique fixed point in* X*.*

If in Theorem 26 we take $\alpha(x, y) = 1$, then we obtain the following result.

Theorem 28. *Let* X *be a complete metric space and let* $T : X \to X$ *be a continuous mapping satisfying the following rational inequality:*

$$\begin{aligned} d\left(T^2x_1, T^2x_2\right) &\leq ad\left(Tx_1, Tx_2\right) \\ &+ b\frac{\left[1 + d\left(Tx_1, T^2x_1\right)\right] d\left(Tx_2, T^2x_2\right)}{1 + d\left(Tx_1, Tx_2\right)} \\ &+ c\left[d\left(Tx_1, T^2x_1\right) + d\left(Tx_2, T^2x_2\right)\right] \\ &+ d\left[d\left(Tx_1, T^2x_2\right) + d\left(Tx_2, T^2x_1\right)\right], \end{aligned} \tag{81}$$

where $a + b + 2c + 2d < 1$*. Then* T *has a unique fixed point in* X*.*

Our next best proximity point result is about α-η-rational proximal contraction of the first and second kinds where we consider only completeness of (X, d) without assuming continuity of the mapping T and approximative compactness of A and B.

Theorem 29. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d)*. Assume that* $\alpha, \eta : A \times A \to [0, \infty)$*,* A_0 *and* B_0 *are nonempty, and* $T : A \to B$ *is* α*-*η*-rational proximal contraction of the first and second kinds, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is α-proximal admissible with respect to η,*

(iii) *there exists elements x_0 and x_1 in A_0 such that*

$$d\left(x_1, Tx_0\right) = d\left(A, B\right), \qquad \alpha\left(x_0, x_1\right) \geq \eta\left(x_0, x_1\right), \tag{82}$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.*

Then there exists unique $x \in B_{est}(T)$*. Also, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by* $d(x_{n+1}, Tx_n) = d(A, B)$*, converges to* x*, whenever* $\alpha(x, y) \geq \eta(x, y)$ *for all* $x, y \in B_{est}(T)$*.*

Proof. As in proof of Theorem 11, there exists a sequence $\{x_n\} \in A_0$, such that

$$\begin{aligned} d\left(x_{n+1}, Tx_n\right) &= d\left(A, B\right), \\ \alpha\left(x_n, x_{n+1}\right) &\geq \eta\left(x_n, x_{n+1}\right), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{83}$$

and the sequence $\{x_n\}$ is a Cauchy sequence and so converges to some $x^* \in A$. Also, by proof of Theorem 21, we obtain that the sequence $\{Tx_n\}$ is a Cauchy sequence and converges to some $y^* \in B$. Hence, we have

$$d\left(x^*, y^*\right) = \lim_{n \to \infty} d\left(x_n, Tx_n\right) = d\left(A, B\right). \tag{84}$$

That is, $x^* \in A_0$. Since $TA_0 \subseteq B_0$, so $d(u, Tx^*) = d(A, B)$ for some $u \in A$. Thus we have $d(x_{n+1}, Tx_n) = d(A, B)$ and $d(u, Tx^*) = d(A, B)$ and so by (iv) this implies that $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all $n \geq 0$. Now, since T is a α-η-rational proximal contraction of the first kind, we get

$$\begin{aligned} d\left(x_{n+1}, u\right) &\leq ad\left(x_n, x^*\right) \\ &+ b\frac{\left[1 + d\left(x_n, x_{n+1}\right)\right] d\left(x^*, u\right)}{1 + d\left(x_n, x^*\right)} \\ &+ c\left[d\left(x_n, x_{n+1}\right) + d\left(x^*, u\right)\right] \\ &+ d\left[d\left(x_n, u\right) + d\left(x^*, x_{n+1}\right)\right]. \end{aligned} \tag{85}$$

Taking limit as $n \to \infty$ in the previous inequality, we get

$$d\left(x^*, u\right) \leq (b + c + d)\, d\left(x^*, u\right), \tag{86}$$

which implies that $d(x^*, u) = 0$. That is, $x^* = u$. Hence, $d(x^*, Tx^*) = d(A, B)$. Further, following similar proof of Theorem 11 we can deduce the uniqueness of best proximity point of T. □

If $\eta(x, y) = 1$ for all $x, y \in A$ in the previous theorem, we obtain the following result.

Corollary 30. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d)*. Assume that* $\alpha : A \times A \to [0, \infty)$*,* A_0 *and* B_0 *are nonempty, and* $T : A \to B$ *is* α*-rational proximal contraction of the first and second kinds, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is α-proximal admissible,*

(iii) *there exists elements x_0 and x_1 in A_0, such that*

$$d\left(x_1, Tx_0\right) = d\left(A, B\right), \qquad \alpha\left(x_0, x_1\right) \geq 1, \tag{87}$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists unique $x \in B_{est}(T)$*. Also, for any fixed* $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by* $d(x_{n+1}, Tx_n) = d(A, B)$*, converges to* x*, whenever* $\alpha(x, y) \geq 1$ *for all* $x, y \in B_{est}(T)$*.*

Corollary 31. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* B *is approximatively compact with respect to* A*. Assume that* $\eta : A \times A \to [0, \infty)$ *and* $a + b + 2c + 2d < 1$*. Let* A_0 *and* B_0 *be nonempty and let*

$T : A \to B$ be a nonself-mapping such that $T(A_0) \subseteq B_0$ and T is η-proximal subadmissible mapping such that

$$\begin{aligned}
&\eta(x_1, x_2) \le 1, \\
&d(u_1, Tx_1) = d(A, B), \\
&d(u_2, Tx_2) = d(A, B) \\
&\qquad \Longrightarrow d(u_1, u_2) \\
&\qquad \le ad(x_1, x_2) \\
&\qquad\quad + b\frac{[1 + d(x_1, u_1)]\, d(x_2, u_2)}{1 + d(x_1, x_2)} \\
&\qquad\quad + c[d(x_1, u_1) + d(x_2, u_2)] \\
&\qquad\quad + d[d(x_1, u_2) + d(x_2, u_1)], \\
&\eta(x_1, x_2) \le 1, \\
&d(u_1, Tx_1) = d(A, B), \\
&d(u_2, Tx_2) = d(A, B) \\
&\qquad \Longrightarrow d(Tu_1, Tu_2) \\
&\qquad \le ad(Tx_1, Tx_2) \\
&\qquad\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\
&\qquad\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\
&\qquad\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)]
\end{aligned} \tag{88}$$

for all x_1, x_2, u_1, u_2. Then there exists unique $x \in B_{est}(T)$. Also, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, where $\eta(x, y) \le 1$ for all $x, y \in B_{est}(T)$.

Corollary 32 (see [22, Theorem 3.3]). *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 and B_0 are nonempty and $T : A \to B$ is rational proximal contraction of the first and second kinds, such that $T(A_0) \subseteq B_0$. Then there exists unique $x \in B_{est}(T)$. Also, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x.*

Remark 33. (1) Similarly we may obtain many results as an immediate consequence of Theorem 29.

(2) If $b = 0$ in our results (Theorem 11–Corollary 32), we get the modified and improved versions of recent results in [26].

3. Best Proximity and Fixed Point Results in Partially Ordered Metric Spaces

The aim of this section is to deduce main results (Theorems 3.1–3.3 [22]) in the context of partially ordered metric spaces. Moreover, we obtain certain recent fixed point results as corollaries in partially ordered metric spaces. Existence of best proximity and fixed points in partially ordered metric spaces has been considered recently by many authors (see, [7, 8, 11, 25, 33]).

Definition 34 (see [25]). A mapping $T : A \to B$ is said to be proximally order preserving if and only if it satisfies the condition that

$$\begin{aligned}
&x_1 \preceq x_2, \\
&d(u_1, Tx_1) = d(A, B), \\
&d(u_2, Tx_2) = d(A, B) \\
&\qquad \Longrightarrow u_1 \preceq u_2
\end{aligned} \tag{89}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $A = B$, then proximally order-preserving map T reduces to nondecreasing map.

Definition 35. Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$. Then $T : A \to B$ is said to be an ordered rational proximal contraction of the first kind if there exist nonnegative real numbers a, b, c, and d with $a + b + 2c + 2d < 1$, such that

$$\begin{aligned}
&x_1 \preceq x_2, \\
&d(u_1, Tx_1) = d(A, B), \\
&d(u_2, Tx_2) = d(A, B) \\
&\qquad \Longrightarrow d(u_1, u_2) \\
&\qquad \le ad(x_1, x_2) \\
&\qquad\quad + b\frac{[1 + d(x_1, u_1)]\, d(x_2, u_2)}{1 + d(x_1, x_2)} \\
&\qquad\quad + c[d(x_1, u_1) + d(x_2, u_2)] \\
&\qquad\quad + d[d(x_1, u_2) + d(x_2, u_1)].
\end{aligned} \tag{90}$$

Definition 36. Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$. Then $T : A \to B$ is said to be an ordered rational proximal contraction of the second kind if there exist nonnegative real numbers a, b, c, and d with $a + b + 2c + 2d < 1$, such that

$$\begin{aligned}
&x_1 \preceq x_2, \\
&d(u_1, Tx_1) = d(A, B), \\
&d(u_2, Tx_2) = d(A, B) \\
&\qquad \Longrightarrow d(Tu_1, Tu_2) \\
&\qquad \le ad(Tx_1, Tx_2) \\
&\qquad\quad + b\frac{[1 + d(Tx_1, Tu_1)]\, d(Tx_2, Tu_2)}{1 + d(Tx_1, Tx_2)} \\
&\qquad\quad + c[d(Tx_1, Tu_1) + d(Tx_2, Tu_2)] \\
&\qquad\quad + d[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)].
\end{aligned} \tag{91}$$

Theorem 37. *Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$ and let B be approximatively compact with respect to A. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is an ordered rational proximal contraction of the first kind which satisfies the following assertions:*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is proximally order-preserving,*

(iii) *there exist elements x_0 and x_1 in A_0, such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1, \tag{92}$$

(iv) *if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists $z \in A_0$, such that,

$$d(z, Tz) = d(A, B). \tag{93}$$

Proof. Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{iff } x \preceq y \\ 0, & \text{otherwise.} \end{cases} \tag{94}$$

At first, we prove that T is α-proximal admissible mapping. For this, assume that

$$\begin{gathered} \alpha(x, y) \geq 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{gathered} \tag{95}$$

So,

$$\begin{gathered} x \preceq y, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{gathered} \tag{96}$$

Now since, T is a proximally order preserving, so, $u \preceq v$. That is, $\alpha(u, v) \geq 1$ which implies that T is α-proximal admissible.

By (iii), we have

$$d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 1. \tag{97}$$

Further as T is an ordered rational proximal contraction, we have

$$\begin{aligned} &\alpha(x_1, x_2) \geq 1, \\ &d(u_1, Tx_1) = d(A, B), \\ &d(u_2, Tx_2) = d(A, B) \\ &\qquad \Longrightarrow d(u_1, u_2) \\ &\qquad \leq ad(x_1, x_2) \\ &\qquad\quad + b\frac{[1 + d(x_1, u_1)]\, d(x_2, u_2)}{1 + d(x_1, x_2)} \\ &\qquad\quad + c[d(x_1, u_1) + d(x_2, u_2)] \\ &\qquad\quad + d[d(x_1, u_2) + d(x_2, u_1)], \end{aligned} \tag{98}$$

which implies that $T : A \to B$ is α-rational proximal contraction of the first kind. Assume that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by (iv) we get $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. That is, all conditions of Corollary 12 hold and consequently there exists $z \in A_0$, such that

$$d(z, Tz) = d(A, B). \tag{99}$$

□

Similarly, we can prove following best proximity point result in partially ordered metric space.

Theorem 38. *Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$ and B be approximatively compact with respect to A. Assume that A_0 and let B_0 are nonempty and $T : A \to B$ is a continuous ordered rational proximal contraction of the second kind, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is proximally order-preserving,*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1, \tag{100}$$

where $a + b + 2c + 2d < 1$. Then there exists $x \in B_{est}(T)$ and, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to x, and $Tx = Tz$ for all $x, z \in B_{est}(T)$ when $x \preceq y$ for all $x, y \in B_{est}(T)$.

Theorem 39. *Let A and B be nonempty closed subsets of a complete partially ordered metric space $(X, d, \preceq)$. Assume that A_0 and B_0 are nonempty and $T : A \to B$ is a continuous ordered rational proximal contraction of the first and second kind, such that*

(i) $T(A_0) \subseteq B_0$,

(ii) *T is proximally order-preserving,*

(iii) *there exist elements x_0 and x_1 in A_0, such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \preceq x_1, \tag{101}$$

(iv) *if $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists $z \in A_0$, such that

$$d(z, Tz) = d(A, B). \tag{102}$$

Further, we can easily deduce the following recent fixed point results from Theorem 37.

Theorem 40. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that $T : X \to X$ is self-mapping on X which satisfies the following assertions:*

(i) *T is nondecreasing mapping,*

(ii) *there exists* $x_0 \in X$ *such that,* $x_0 \preceq Tx_0$,

(iii) *if* $\{x_n\}$ *is nondecreasing sequence in* X *such that* $x_n \to x \in X$ *as* $n \to \infty$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$,

(iv)

$$\begin{aligned} d(Tx_1, Tx_2) \leq{} & ad(x_1, x_2) \\ & + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \\ & + c[d(x_1, Tx_1) + d(x_2, Tx_2)] \\ & + d[d(x_1, Tx_2) + d(x_2, Tx_1)] \end{aligned} \tag{103}$$

for all $x_1 \preceq x_2 \in X$. *Then* T *has a fixed point.*

If we put $c = d = 0$ in the previous theorem, we obtain the following recent results.

Corollary 41 (see [11, Theorems 2 and 3]). *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space. Assume that* $T : X \to X$ *is self-mapping on* X *which satisfies the following assertions:*

(i) T *is nondecreasing mapping,*

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$,

(iii) *if* T *is continuous or* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x \in X$ *as* $n \to \infty$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$,

(iv)

$$\begin{aligned} d(Tx_1, Tx_2) \leq{} & ad(x_1, x_2) \\ & + b\frac{[1 + d(x_1, Tx_1)]\, d(x_2, Tx_2)}{1 + d(x_1, x_2)} \end{aligned} \tag{104}$$

for all $x_1 \preceq x_2$ *in* X, *where* $a + b < 1$. *Then* T *has a fixed point.*

Remark 42. (1) Similarly we may obtain many results as an immediate consequence of Theorems 38 and 39.

(2) If in Corollary 41 we put $b = 0$, then Theorems 2.1 and 2.2 of [23] are obtained. If $a = 0$ in Corollary 41, we get Theorem 5 in [11].

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