Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 929178, 8 pages
http://dx.doi.org/10.1155/2013/929178

Research Article

The $L\omega$-Compactness in $L\omega$-Spaces

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Received 15 March 2013; Accepted 5 April 2013

Academic Editor: Zhenkun Huang

The concepts of $\alpha\omega$-remote neighborhood family, $\gamma\omega$-cover, and $L\omega$-compactness are defined in $L\omega$-spaces. The characterizations of $L\omega$-compactness are systematically discussed. Some important properties of $L\omega$-compactness such as ω-closed heredity, arbitrarily multiplicative property, and preserving invariance under ω-continuous mappings are obtained. Finally, the Alexander ω-subbase lemma and the Tychonoff product theorem with respect to $L\omega$-compactness are given.

1. Introduction

Compactness is one of the most important notions in general topology, fuzzy topology, and L-topology. Many research workers have presented various kinds of compactness [1–19] by means of introducing various operators, such as closure operator, θ-closure operator, δ-closure operator, R-closure operator, S-closure operator, SR-closure operator, and PS-closure operator; because the above operators are all order preserving. That is, they satisfy the following conditions: (i) if $A, B \in L^X$ and $A \leqslant B$, then $\omega(A) \leqslant \omega(B)$; (ii) for any $A \in L^X$, $A \leqslant \omega(A)$, where $\omega : L^X \to L^X$ can take and of the above operators, L^X is the family of all L-sets defined on X and with value in L, L is a fuzzy lattice, and 1_X is the greatest L-set of L^X. We introduced a kind of generalized fuzzy space called $L\omega$-space in [20] in order to unify various elementary concepts in L-topological spaces. In the present paper, we will propose and study a generalized compactness which will be called $L\omega$-compactness in $L\omega$-spaces. The $L\omega$-compactness is a unified form of N-compactness [16, 19], near N-compactness [5], almost N-compactness [6], S-compactness [13], SR-compactness [1], PS-compactness [2], δ-compactness [9], θ-compactness [18], and so forth.

2. Preliminaries

Throughout this paper, L denotes a fuzzy lattice, that is, a completely distributive lattice with order-reserving involution $'$, 0 and 1 denote the least and greatest elements of L, respectively, and M denotes the set that consisting of all nonzero $\vee$-irreducible elements of L. Let X be a nonempty crisp set, L^X the set of all L-fuzzy sets (briefly, L-sets) on X, and $M^*(L^X) = \{x_\alpha : \alpha \in M,\ x \in X\}$ the set of all nonzero $\vee$-irreducible elements (i.e., so-called molecules [17] or points for short) of L^X. The least and the greatest elements of L^X will be denoted by 0_X and 1_X, respectively. For any $\alpha \in M$, $\beta(\alpha)$ is called the greatest minimal set of α [12], and $\beta^*(\alpha) = \beta(\alpha) \cap M$ is said to be the standard minimal set of α [17].

Definition 1 (Chen and Cheng [20]). Let X be a nonempty crisp set.

(i) An operator ω: $L^X \to L^X$ is said to be an ω-operator if (1) for all $A, B \in L^X$ and $A \leqslant B$, $\omega(A) \leqslant \omega(B)$; (2) for all $A \in L^X$, $A \leqslant \omega(A)$.

(ii) An L-set $A \in L^X$ is called an ω-set if $\omega(A) = A$.

(iii) Put $\Omega = \{A \in L^X \mid \omega(A) = A\}$, and call the pair (L^X, ω) an $L\omega$-space.

Definition 2 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $x_\alpha \in M^*(L^X)$. If there exists a $Q \in \Omega$ such that $x_\alpha \nleq Q$ and $P \leqslant Q$, then call P an ω-remote neighborhood (briefly, ωR-neighborhood) of x_α. The collection of all ωR-neighborhoods of x_α is denoted by $\omega\eta(x_\alpha)$. If $A \nleq P$ for each $P \in \omega\eta(x_\alpha)$, then x_α is said to be an ω-adherence point of A and the union of all ω-adherence points of A is called the ω-closure of A and denoted by $\omega\,\mathrm{cl}(A)$. If $A = \omega\,\mathrm{cl}(A)$, then call A an ω-closed set and call A' an ω-open set. If P is an ω-closed set and $x_\alpha \nleq P$, then P is said to be an ω-closed remote neighborhood (briefly, ωCR-neighborhood) of x_α and the collection of all ωCR-neighborhoods of x_α is denoted by $\omega\eta^-(x_\alpha)$. Note that $\omega C(L^X)$ and $\omega O(L^X)$ are the family of all ω-closed sets and all ω-open sets in L^X, respectively.

Definition 3 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and $\omega\operatorname{int}(A) = \vee\{B \in L^X \mid B \leqslant A$ and B is an ω-open set in $L^X\}$. We call $\omega\operatorname{int}(A)$ the ω-interior of A. Obviously, A is ω-open if and only if $A = \omega\operatorname{int}(A)$.

Definition 4 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let N be a molecular net in L^X, and let $x_\alpha \in M^*(L^X)$. If N is eventually not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-limit point of N (or N ω-converges to x_α). If N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$, then x_α is said to be an ω-cluster point of N (or N ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of N is written by ω-lim N (ω-*adN*).

Definition 5 (Huang and Chen [11]). Let (L^X, Ω) be an $L\omega$-space, let I be an ideal in L^X, and let $x_\alpha \in M^*(L^X)$. If $\omega\eta^-(x_\alpha) \subseteq I$, then x_α is called an ω-limit point of I (or $I\omega$-converges to x_α). If $P \vee B \neq 1_X$ for each $P \in \omega\eta^-(x_\alpha)$ and each $B \in I$, then x_α is called an ω-cluster point of I (or I ω-accumulates to x_α). The union of all ω-limit points (ω-cluster points) of I is denoted by ω-lim I (ω-*adI*).

Definition 6 (Chen and Cheng [20]). Let (L^X, Ω) be an $L\omega$-space, $x_\alpha \in M^*(L^X)$, and $\beta, \gamma \in \omega O(L^X)$. Then,

(i) β is said to be an ω-base in (L^X, Ω) if for each $G \in \omega O(L^X)$, there exists a subfamily φ of β such that $G = \vee_{B\in\varphi} B$;

(ii) γ is said to be an ω-subbase in (L^X, Ω) if the collection consisting of all intersections of any finite elements in γ is an ω-base in (L^X, Ω).

Definition 7 (Chen and Cheng [20]). Assume (L^X, Ω_i) to be an $L\omega_i$-space ($i = 1, 2$) and $f : (L^X, \Omega_1) \to (L^Y, \Omega_2)$ an L-valued Zadeh's type function [17]. If $f^{\leftarrow}(B) \in \omega_1 O(L^X)$ for each $B \in \omega_2 O(L^Y)$, then call $f(\omega_1, \omega_2)$-continuous.

3. $L\omega$-Compact Set and Its Characteristics

In this section, we will introduce the concepts of $\alpha\omega$-remote neighborhood family and $\gamma\omega$-cover in an $L\omega$-space first, propose the notion of $L\omega$-compactness by making use of $\alpha\omega$-remote neighborhood family next, and then discuss the characteristics of $L\omega$-compactness.

Definition 8. Suppose (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\alpha \in M$, and $\Phi \subseteq \omega C(L^X)$. If there exists a $P \in \Phi$ such that $P \in \omega\eta^-(x_\alpha)$ for each molecule x_α in A, then Φ is called an $\alpha\omega$-remote neighborhood family (briefly, $\alpha\omega$-RF) of A, in symbol $\wedge\Phi < A(\alpha\omega)$. If there exists a nonzero $\vee$-irreducible element $\lambda \in \beta^*(\alpha)$ with $\wedge\Phi < A(\lambda\omega)$, then Φ is said to be an $(\alpha\omega)^-$-RF, in symbol $\wedge\Phi \ll A(\alpha\omega)$.

Definition 9. Assume (L^X, Ω) be an $L\omega$-space, $A \in L^X$, $\gamma' \in M$, and $\Gamma \subseteq \omega O(L^X)$. If there is a $B \in \Gamma$ such that $B(x) \nleq \gamma$ for each $x \in \tau_{\gamma'}(A) = \{x \in X \mid A(x) \geqslant \gamma'\}$, then Γ is known as a $\gamma\omega$-cover. If there exists a prime element $t \in \alpha^*(\gamma)$ such that Γ is a $t\omega$-cover of A, then Γ is said to be a $(\gamma\omega)^+$-cover of A, where $\alpha^*(\gamma)$ is the standard maximal set of γ [17].

Definition 10. Assume (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ such that Ψ is an $(\alpha\omega)^-$-RF, where $\alpha \in M$, then call A an $\alpha L\omega$-compact set. If A is an $\alpha L\omega$-compact set for any $\alpha \in M$, then call A an $L\omega$-compact set. Specially, when 1_X is $\alpha L\omega$-compact, we call (L^X, Ω) an $\alpha L\omega$-compact space, and if (L^X, Ω) is $\alpha L\omega$-compact for each $\alpha \in M$, we say that (L^X, Ω) is an $L\omega$-compact space.

Obviously, when ω is the L-closure operator on L^X, the $L\omega$-compactness is just the N-compactness in [19], and while ω takes the θ-closure operator (resp., δ-closure operator, R-closure operator, S-closure operator, PS-closure operator, and SR-closure operator) on L^X, the $L\omega$-compactness is just the θ-compactness (resp., δ-compactness, near N-compactness, S-compactness, PS-compactness, and SR-compactness). Therefore, the $L\omega$-compactness is of the universal significance.

Example 11. Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. If the support $\sigma_0(A) = \{x \in X \mid A(x) > 0\}$ of A is a finite set, then A is an $L\omega$-compact set.

Proof. Assume that $\sigma_0(A) = \{x_1, x_2, \ldots, x_n\}$ and Φ is an $\alpha\omega$-RF of A. For each $i \in \{1, 2, \ldots, n\}$ we choose an ω-closed set $P_i \in \Phi$ with $\alpha \nleq P_i(x_i)$. Being $\alpha = \sup \beta^*(\alpha)$, there is a $\lambda_i \in \beta^*(\alpha)$ such that $\lambda \nleq P_i(x_i)$. Since $\beta^*(\alpha)$ is an upper directed set, there is a $\lambda \in \beta^*(\alpha)$ with $\lambda \geqslant \lambda_i$ for each $i \in \{1, 2, \ldots, n\}$, and thus $\lambda_i \nleq P_i(x_i)$. Therefore Φ has a finite subfamily $\Psi = \{P_1, P_2, \ldots, P_n\}$ which is an $(\alpha\omega)^-$-RF of A. By Definition 10, A is an $L\omega$-compact set. □

Now we give some characteristics of $L\omega$-compactness as follows.

Theorem 12. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if the following conditions hold:*

(1) *for each $\alpha \in M$, every $\alpha\omega$-RF Φ of A has a finite subfamily Ψ with $\wedge\Psi < A(\alpha\omega)$;*

(2) *for each $\alpha \in M$, if $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ is also an $(\alpha\omega)^-$-RF of A.*

Proof. Necessity. Assume that A is $L\omega$-compact and Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. According to Definition 10, Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$ and so it certainly holds that $\wedge\Psi < A(\alpha\omega)$. Thus (1) is satisfied. If $\Phi = \{P\}$ is an $\alpha\omega$-RF of A, then Φ has a finite Ψ with $\wedge\Psi \ll A(\alpha\omega)$ by the

$L\omega$-compactness of A. Obviously, $\Psi = \Phi$, and hence Φ is an $(\alpha\omega)^{-}$-RF of A. Therefore (2) holds.

Sufficiency. Suppose that conditions (1) and (2) are satisfied, and Φ is an $\alpha\omega$-RF of A ($\alpha \in M$). By (1), there is a finite subfamily Ψ of Φ such that Ψ is an $\alpha\omega$-RF of A. Let $P = \wedge\Psi$. Then $\{P\}$ is an $\alpha\omega$-RF of A. According to (2), $\{P\}$ is also an $\alpha\omega$-RF of A; that is, there exists a $\lambda \in \beta^*(\alpha)$ with $\lambda \nleqslant P(x) = \wedge\{Q(x) \mid Q \in \Psi\}$ for each molecule $x_\lambda \leqslant A$. Since Ψ is finite, we can choose an ω-closed set $Q \in \Psi$ with $\lambda \nleqslant Q(x)$; that is, $Q \in \omega\eta^{-}(x_\lambda)$. This shows that Ψ is an $(\alpha\omega)^{-}$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 13. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is an $L\omega$-compact set if and only if for each $\gamma' \in M$, every $\gamma\omega$-cover Γ of A has a finite subfamily Ξ such that Ξ is a $(\gamma\omega)^{+}$-cover of A.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and Γ is any $\gamma\omega$-cover of A ($\gamma' \in M$). Put $\Phi = \Gamma'$. Then $\Phi \subseteq \omega C(L^X)$, and there is an ω-closed set $B' \in \Phi$ with $B(x) \nleqslant \gamma$ for each $x \in \tau_{\gamma'}(A)$; that is, $\gamma' \nleqslant B'(x)$; equivalently, $B' \in \omega\eta^{-}(x_{\gamma'})$. This implies that Φ is a $\gamma'\omega$-RF of A. Thus Φ has a finite subfamily Ψ which is a $(\gamma'\omega)^{-}$-RF of A; that is, there exists $t' \in \beta^*(\gamma')$ such that for each $x \in \tau_{\gamma'}(A)$ we can take an ω-open set $B \in \Psi'$ with $t' \nleqslant B'(x)$. In other words, there are $t \in \alpha^*(\gamma)$ and $B \in \Psi' = \Xi$ with $B(x) \nleqslant t$ for each $x \in \tau_{\gamma'}(A)$. This means that Ξ is a finite subfamily of Γ and a $(\gamma\omega)^{+}$-cover of A.

Sufficiency. Assume that every $\gamma\omega$-cover of A has a finite subfamily which is a $(\gamma\omega)^{+}$-cover of A ($\gamma' \in M$). If Φ is an $\alpha\omega$-RF of A ($\alpha \in M$), then $\Gamma = \Phi'$ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Hence Γ has a finite subfamily Ξ which is a $(\gamma\omega)^{+}$-cover of A by the hypothesis. Write $\Psi = \Xi'$. One can easily see that Ψ is a finite subfamily of Φ and is an $(\alpha\omega)^{-}$-RF of A. Therefore A is $L\omega$-compact. □

Theorem 14. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$ and each $\Phi \subseteq \omega C(L^X)$ having α-finite intersection property for A (i.e., for each finite subfamily Ψ of Φ and each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$), there exists a molecule $x_\alpha \leqslant A$ with $x_\alpha \leqslant \wedge\Phi$.*

Proof. Necessity. Grant that A is an $L\omega$-compact set, $\Phi \subseteq \omega C(L^X)$, and Φ has α-finite intersection property for A ($\alpha \in M$). If $x_\alpha \nleqslant \wedge\Phi$ for each $x_\alpha \leqslant A$, then Φ is an $\alpha\omega$-RF of A by the hypothesis of Φ. Hence Φ has a finite subfamily Ψ which is an $(\alpha\omega)^{-}$-RF of A; that is, there is a $\lambda \in \beta^*(\alpha)$ satisfying $x_\lambda \nleqslant \wedge\Psi$ for each $x_\lambda \leqslant A$; in other words, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \ngeqslant \lambda$. It contradicts the fact that Φ has α-finite intersection property for A. Hence the necessity is proved.

Sufficiency. Assume that the condition holds and that Φ is an $\alpha\omega$-RF of A. If for any finite subfamily Ψ of Φ, Ψ is not an $(\alpha\omega)^{-}$-RF of A, then for each $\lambda \in \beta^*(\alpha)$ there exists a molecule $x_\lambda \leqslant A$ with $x_\lambda \leqslant \wedge\Psi$; that is, $\vee_{x\in X}(A \wedge (\wedge\Psi))(x) \geqslant \lambda$. This shows that Φ has α-finite intersection property for A. By the assumption we have $x_\alpha \leqslant A$ satisfying $x_\alpha \leqslant \wedge\Psi$. It contradicts that Φ is an $\alpha\omega$-RF of A. Therefore Φ has a finite subfamily Ψ which is an $(\alpha\omega)^{-}$-RF of A, and hence A is $L\omega$-compact. □

Theorem 15. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-net in A has an ω-cluster point in A with height α.*

Proof. Necessity. Suppose that A is an $L\omega$-compact set and that $N = \{N(n) \mid n \in D\}$ is an α-net [16] in A. If N does not have any ω-cluster point in A with height α, then there exists a $P[x] \in \omega\eta^{-}(x_\alpha)$ such that N is eventually in $P[X]$ for each $x_\alpha \leqslant A$; that is, there is a $n(x) \in D$ with $N(n) \leqslant P[x]$ whenever $n \geqslant n(x)$. Write $\Phi = \{P[x] \mid x_\alpha \leqslant A\}$. Obviously, Φ is $\alpha\omega$-RF of A. By the $L\omega$-compactness of A, Φ has a finite subfamily $\Psi = \{P[x_i] \mid i = 1, 2, \ldots, m\}$ which is an $(\alpha\omega)^{-}$-RF of A; that is, there is an $i \in \{1, 2, \ldots, m\}$ with $y_r \nleqslant P[X_i]$ for some $r \in \beta^*(\alpha)$ and each $y_r \leqslant A$. Take $P = \wedge_{i=1}^{m} P[x_i]$. Then $y_r \nleqslant P$ for each $y_r \leqslant A$. Since D is a directed set, there is an $n_0 \in D$, such that $n_0 \geqslant n(x_i)$ and $N(n) \leqslant P[x_i]$ ($i = 1, 2, \ldots, m$) whenever $n \geqslant n_o$, and so $N(n) \leqslant P$. This shows that for each $y_r \leqslant A$, $\vee(N(n)) \ngeqslant r$ as long as $n \geqslant n_o$. It contradicts the fact that N is an α-net. Therefore N has at least an ω-cluster point in A with height α.

Sufficiency. Assume that every α-net in A has at least an ω-cluster point with hight α for each $\alpha \in M$, Φ is an $\alpha\omega$-RF of A, and $2^{(\Phi)}$ is the set of all finite subfamilies of Φ. If for each $r \in \beta^*(\alpha)$ and each $\Psi \in 2^{(\Phi)}$, Ψ is not an $r\omega$-RF of A; that is, $x_r \leqslant \wedge\Psi$ for each $x_r \leqslant A$, and hence there exists a molecule $N(r, \Psi) \leqslant A$ satisfying $N(r, \Psi) \leqslant \wedge\Psi$. In $\beta^*(\alpha) \times 2^{(\Phi)}$, we define the relation as follows: $(r_1, \Psi_1) \geqslant (r_2, \Psi_2)$ if and only if $r_1 \geqslant r_2$ and $\Psi_1 \supseteq \Psi_2$, then $\beta^*(\alpha) \times 2^{(\Phi)}$ is a directed set with the relation "$\geqslant$". Let $N = \{N(r, \Psi) \mid (r, \Psi) \in \beta^*(\alpha) \times 2^{(\Phi)}\}$. One can easily see that N is an α-net in A. We assert that N does not have any ω-cluster point in A with hight α. In fact, for each $x_\alpha \leqslant A$, we can choose an ω-closed set $P \in \Phi$ with $P \in \omega\eta^{-}(x_\alpha)$ by the definition of Φ. Taking $r_1 \in \beta^*(\alpha)$ and $\Psi \in 2^{(\Phi)}$, we have $P \in \Psi$ according to $(r, \Psi) \geqslant (r_1, \{P\})$, and hence $N(r, \Psi) \leqslant \wedge\Psi \subseteq P$. This implies that N is eventually in P, and thus x_α is not an ω-cluster point of N. It is in contradiction with the hypothesis of sufficiency. Consequently, A is $L\omega$-compact. □

Definition 16. Let (L^X, Ω) be an $L\omega$-space, let $\mathscr{F}$ be an α-filter in L^X; that is, $\vee_{x\in X}(F \wedge A)(x) \geqslant \alpha$ for each $F \in \mathscr{F}$ and $x_\alpha \in M^*(L^X)$. If $F \nleqslant P$ and for each $P \in \omega\eta^{-}(x_\alpha)$ and each $F \in \mathscr{F}$, then x_α is called an ω-cluster point of $\mathscr{F}$.

Theorem 17. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if for each $\alpha \in M$, every α-filter containing A as an element has an ω-cluster point in A with hight α.*

Proof. Necessity. Grant that A is an $L\omega$-compact set and that $\mathscr{F}$ is an α-filter containing A as an element. Then $F \wedge A \in \mathscr{F}$

for each $F \in \mathscr{F}$ and $\vee_{x\in X}(F \wedge A)(x) \geqslant \alpha$, and thus there exists a molecule $N(F, r) \leqslant A$ with hight r for each $r \in \beta^*(\alpha)$. Define $N = \{N(F, r) \leqslant F \wedge A \mid (F, r) \in \mathscr{F} \times \beta^*(\alpha)\}$ and define a relation in $\mathscr{F} \times \beta^*(\alpha)$ as follows:

$$(F_1, r_1) \geqslant (F_2, r_2) \quad \text{iff } F_1 \leqslant F_2,\ r_1 \geqslant r_2. \tag{1}$$

Evidently, $\mathscr{F} \times \beta^*(\alpha)$ is a directed set with the relation "$\geqslant$", and then N is an α-net in A. By the $L\omega$-compactness of A and Theorem 15, N has an ω-cluster point in A with hight α, say x_α. We assert that x_α is also an ω-cluster point of $\mathscr{F}$. In reality, N is frequently not in P for each $P \in \omega\eta^-(x_\alpha)$; that is, for each $F \in \mathscr{F}$ there exist $F_1 \in \mathscr{F}$ with $F_1 \leqslant F$ and some $r \in \beta^*(\alpha)$ satisfying $N(F_1, r) \nleqslant P$. Hence we have $F \nleqslant P$ by virtue of the fact that $N(F_1, r) \leqslant F_1 \leqslant F$. This means that x_α is an ω-cluster point of $\mathscr{F}$. Therefore the necessity is proved.

Sufficiency. Suppose that every α-filter containing A as an element has an ω-cluster point in A with hight α for each $\alpha \in M$ and that Φ is an $\alpha\omega$-RF of A. If for each $\Psi \in 2^{(\Phi)}$, Ψ is not an $(\alpha\omega)^-$-RF of A, then there exists a molecule $x_r \leqslant A$ and $x_r \leqslant \wedge\Psi$ for each $r \in \beta^*(\alpha)$. Put $\mathscr{F} = \{F \in L^X \mid \exists \Psi \in 2^{(\Phi)}$ with $(\wedge\Psi) \wedge A \leqslant F\}$. One can easily verify that $\mathscr{F}$ is an α-filter containing A as an element, and hence $\mathscr{F}$ has an ω-cluster point in A with hight α by the supposition, say x_α. In accordance with Definition 16, we have $F \nleqslant P$ for each $P \in \omega\eta^-(x_\alpha)$ and each $F \in \mathscr{F}$, specially, $\wedge\Psi \nleqslant P$. Since Φ is an $\alpha\omega$-RF of A, there exists an ω-closed set $Q \in \Phi$ with $Q \in \omega\eta^-(x_\alpha)$ for each $x_\alpha \leqslant A$. Obviously, $\{Q\} \in 2^{(\Phi)}$, so $Q \nleqslant Q$, and this is impossible. Hence there must be a $\Psi \in 2^{(\Phi)}$ which is an $(\alpha\omega)^-$-RF of A. This shows that A is $L\omega$-compact. □

Definition 18. Let I be an ideal in L^X. If $\vee_{x\in X} B'(x) \geqslant \alpha$ for each $B \in I$, then I is called an α-ideal ($\alpha \in M$).

Theorem 19. *Let (L^X, Ω) be an $L\omega$-space and $A \in L^X$. Then A is $L\omega$-compact if and only if every α-ideal I whose A is not in I has an ω-cluster point in A with hight α for each $\alpha \in M$.*

Proof. Necessity. Assume that A is an $L\omega$-compact set, I is an α-ideal whose A is not in I, and $N(I) = \{N(I)((b, B)) = b \leqslant A \mid (b, B) \in D(I)\}$ where $D(I) = \{(b, B) \mid b \in M^*(L^X), B \in I$ and $b \nleqslant B\}$. Then $N(I)$ is an α-net in A. Hence $N(I)$ has an ω-cluster point in A with hight α by Theorem 15, say x_α. Obviously, x_α is also an ω-cluster point of I. Consequently, the necessity is proved.

Sufficiency. Grant that every α-ideal whose A is not in it has an ω-cluster point in A with hight α for each $\alpha \in M$ and $\mathscr{F}$ is an α-filter containing A as an element. Let $I = \{F' \in L^X \mid F \in \mathscr{F}\}$. Evidently, I is an α-ideal whose A is not in I. Now we will prove that $\mathscr{F}$ has an ω-cluster point in A with hight α. Actually, by the hypothesis we know that I has an ω-cluster point in A with hight α, say x_α; that is, $F' \vee P \neq 1_X$; equivalently, $F \nleqslant P$, for each $F \in \mathscr{F}$ and each $P \in \omega\eta^-(x_\alpha)$. Therefore x_α is an ω-cluster point of $\mathscr{F}$ in line with Definition 16, and hence A is an $L\omega$-compact set by Theorem 17. This implies that the sufficiency holds. □

4. Some Important Properties of $L\omega$-Compactness

In this section, we still further deliberate the properties of $L\omega$-compactness in an $L\omega$-space.

Theorem 20. *Let (L^X, Ω) be an $L\omega$-space and $A, B \in L^X$. If A is $L\omega$-compact and B is ω-closed, then $A \wedge B$ is $L\omega$-compact.*

Proof. Assume that N is an α-net in $A \wedge B$ ($\alpha \in M$). Then N is also an α-net in A. Since A is ω-compact, N has an ω-cluster point in A with hight α, say x_α. We assert that $x_\alpha \leqslant B$. Actually, since N is an α-net in B and N ω-accumulates x_α, N has an α-subnet T which ω-converges to x_α and so $x_\alpha \leqslant \omega\operatorname{cl}(B) = B$. Hence $x_\alpha \leqslant A \wedge B$, and thus $A \wedge B$ is $L\omega$-compact in accordance with Theorem 15. □

This theorem shows that the $L\omega$-compactness is hereditary with respect to ω-closed sets.

Theorem 21. *Let A and B be both $L\omega$-compact sets in (L^X, Ω). Then $A \vee B$ is also an $L\omega$-compact set in (L^X, Ω).*

Proof. Suppose that Φ is an $\alpha\omega$-RF of $A \vee B$ ($\alpha \in M$). Then Φ is an $\alpha\omega$-RF of both A and B. Owing to the $L\omega$-compactness of A, there are $\lambda_1 \in \beta^*(\alpha)$ and $\Psi_1 \in 2^{(\Phi)}$ with $\wedge\Psi_1 < A(\lambda_1\omega)$. Similarly, there exist $\lambda_2 \in \beta^*(\alpha)$ and $\Psi_2 \in 2^{(\Phi)}$ satisfying $\wedge\Psi_2 < A(\lambda_2\omega)$. Take $\lambda = \lambda_1 \wedge \lambda_2$ and $\Psi = \Psi_1 \cup \Psi_2$; then $\lambda \in \beta^*(\alpha)$, $\Psi \in 2^{(\Phi)}$, and $\wedge\Psi < A(\lambda\omega)$; that is, Ψ is an $(\alpha\omega)^-$-RF of $A \vee B$. Consequently, $A \vee B$ is $L\omega$-compact. □

This theorem indicates that the $L\omega$-compactness is finitely additive.

Theorem 22. *Let $L = [0, 1]$, (L^X, Ω) be an $L\omega$-space and let $A \in L^X$ be an $L\omega$-compact set. Then there exists a crisp point $x \in X$ such that $A(x) = \sup\{A(t) \mid t \in X\}$.*

Proof. Let $\alpha = \sup\{A(t) \mid t \in X\}$; then $\alpha \in [0, 1]$. If $\alpha = 0$, then $A = 0_X$ and hence $A(x) = \sup\{A(t) \mid t \in X\}$ holds for each $x \in X$. If $\alpha > 0$, and D is the set of all natural numbers, then we choose $x^n \in X$ with $A(x^n) > \alpha - (1/n)$ and $N = \{x^n_{A(x^n)} \mid n \in D\}$. Obviously, N is an α-net in A, and N has an ω-cluster point x_α in A by virtue of the $L\omega$-compactness of A. Hence $A(x) \geqslant \alpha$ by $x_\alpha \leqslant A$. On the other hand, $A(x) \leqslant \alpha$ by the definition of α. Therefore $A(x) = \alpha = \sup\{A(t) \mid t \in X\}$. □

This theorem implies that an $L\omega$-compact set can reach the maximum at some point in X as a function.

Theorem 23. *Let (L^X, Ω_1) and (L^Y, Ω_2) be an $L\omega_1$-space and an $L\omega_2$-space, respectively, and let $f : L^X \to L^Y$ be an (ω_1, ω_2)-continuous L-valued Zadeh's type function. If A is an $L\omega_1$-compact set in (L^X, Ω_1), then $f^{\to}(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2).*

Proof. Assume that Φ is an $\alpha\omega_2$-RF of $f^{\to}(A)$ and $y_\alpha \in M^*(L^Y)$ with $y_\alpha \leqslant f^{\to}(A)(\alpha \in M)$. According to the definition of f, there is a molecule $x_\alpha \in M^*(L^X)$ such that $x_\alpha \leqslant A$ and $f^\rightarrow(x_\alpha) = y_\alpha$. Thus there is an ω-closed set $Q \in \Phi$ with $f^\rightarrow(x_\alpha) \nleqslant Q$; that is, $x_\alpha \nleqslant f^\leftarrow(Q)$. Since f is (ω_1, ω_2)-continuous, $f^\leftarrow(Q)$ is ω-closed in (L^X, ω_1), and hence $f^\leftarrow(Q) \in \omega_1\eta^-(x_\alpha)$. This means that $f^\leftarrow(\Phi) = \{f^\leftarrow(Q) \mid Q \in \Phi\}$ is an $\alpha\omega_1$-RF of A. Therefore Φ has a finite subfamily $\Psi = \{Q_1, Q_2, \ldots, Q_n\}$ such that $f^\leftarrow(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. We assert that Ψ is an $(\alpha\omega_2)^-$-RF of $f^\rightarrow(A)$. In reality, there exists a $\lambda \in \beta^*(\alpha)$ with $\wedge f^\leftarrow(\Psi) < A(\lambda\omega_1)$ by virtue of the fact that $f^\leftarrow(\Psi)$ is an $(\alpha\omega_1)^-$-RF of A. Since for each $y_\lambda \leqslant f^\rightarrow(A)$ there exists a $x_\lambda \leqslant A$ satisfying $f^\rightarrow(x_\lambda) = y_\lambda$, and there exists a $Q \in \Psi$ with $f^\leftarrow(Q) \in \omega_1\eta^-(x_\lambda)$, that is, $x_\lambda \nleqslant f^\leftarrow(Q)$. Hence $y_\lambda = f^\rightarrow(x_\lambda) \nleq Q$ by Lemma 3.1 in [19], and so Ψ is an $(\alpha\omega_2)^-$-RF of $f^\rightarrow(A)$. Consequently, $f^\rightarrow(A)$ is an $L\omega_2$-compact set in (L^Y, Ω_2). □

This theorem means that the $L\omega$-compactness is topological variant under (ω_1, ω_2)-continuous L-valued Zadeh's type functions.

Definition 24. Let (X, Ω) be a crisp ω-space, and let $\mathscr{P}(X)$ be the set of all subsets of X, that is, all crisp sets on X and $A \in L^X$, where $\omega : \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is a crisp ω-operator which satisfies the following conditions: (1)$\omega(U) \subseteq \omega(V)$ for each $U, V \in \mathscr{P}(X)$ and $U \subseteq V$; (2)$U \subseteq \omega(U)$ for each $U \in \mathscr{P}(X)$.

(i) If $\xi_\alpha(A) = \{x \in X \mid A(x) \leqslant \alpha\} \in \omega C(X)$, where $\omega C(X)$ denotes the set of all crisp ω-closed sets on X and $\alpha \in M$, then A is said to be an L-valued lower semicontinuous function on X.

(ii) Let $\Delta_L(\Omega)$ be the set of all L-valued lower semicontinuous functions on X, and call the pair $(L^X, \Delta_L(\Omega))$ the $L\omega$-space topologically generated by (X, Ω).

Theorem 25. *Let (X, Ω) be a crisp ω-space and let $(L^X, \Delta_L(\Omega))$ be the $L\omega$-space topologically generated by (X, Ω). Then $A \in L^X$ is $L\omega$-compact if and only if $\tau_\alpha(A) = \{x \in X \mid A(x) \geqslant \alpha\}$ is ω-compact for each $\alpha \in M$.*

Proof. Necessity. Provided that $A \in L^X$ is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ and Φ is an ω-open cover of $\tau_\alpha(A)(\alpha \in M)$, let $\Gamma = \{\chi_G \mid G \in \Phi\}$ and $\gamma = \alpha'$, where χ_G is the characteristic function of G. We assert that Γ is a $\gamma\omega$-cover of A. In fact, for each $x \in \tau_{\gamma'}(A)$, there is an ω-open set $G \in \Phi$ with $x \in G$; that is, $\chi_G(x) = 1$. Hence $\chi_G(x) \nleqslant \gamma$ by virtue of the fact that γ is a prime element in L with $\gamma \neq 1$. Thus Φ has a finite subfamily $\{G_1, G_2, \ldots, G_m\}$ such that $\mu = \{\chi_{G_i} \mid i = 1, 2, \ldots, m\} \in 2^{(\Gamma)}$ which is a $(\gamma\omega)^+$-cover of A in line with Theorem 13; that is, there is an $i \in \{1, 2, \ldots, m\}$ such that $\chi_{G_i} \in \mu$ with $\chi_{G_i}(x) \nleqslant \lambda$ for some $\lambda \in \alpha^*(\gamma)$ and each $x \in \tau_\alpha(A)$, and so $x \in G_i$. This implies that $\tau_\alpha(A) \subseteq \cup_{i=1}^m G_i$. Hence $\tau_\alpha(A)$ is an ω-compact set in (X, Ω).

Sufficiency. Grant that $\tau_\alpha(A)$ is an ω-compact set in (X, Ω) for each $\alpha \in M$ and that Γ is a $\gamma\omega$-cover of A where $\gamma = \alpha'$. Then there is an ω-open set $B_x \in \Gamma$ with $B_x(x) \nleqslant \gamma$ for each $x \in \tau_\alpha(A)$, and hence there exists a prime element $t(x) \in \alpha^*(\gamma)$ satisfying $B_x(x) \nleqslant t(x)$. Put $l_{t(x)}(B_x) = \{y \in X \mid B_x(y) \nleqslant t(x)\}$ and $\Phi = \{l_{t(x)}(B_x) \mid x \in \tau_\alpha(A)\}$; then Φ is an ω-open cover of $\tau_\alpha(A)$ according to $x \in l_{t(x)}(B_x)$ and $B_x \in \Delta_L(\Omega)$. Because of the ω-compactness of $\tau_\alpha(A)$, Φ has a finite subfamily $\Psi = \{l_{t(x_i)}(B_{x_i}) \mid i = 1, 2, \ldots, m\}$ which is an ω-open cover of $\tau_\alpha(A)$; that is, there exists an $i \in \{1, 2, \ldots, m\}$ with $x \in l_{t(x_i)}(B_{x_i})$; in other words, $B_{x_i}(x) \nleqslant t(x_i)$ for each $x \in \tau_\alpha(A)$. Take $t = \wedge_{i=1}^m t(x_i)$; evidently, $t \in \alpha^*(\gamma)$ and $B_{x_i}(x) \nleqslant t$. Hence $\mu = \{B_{x_i} \mid i = 1, 2, \ldots, m\}$ is a $(\gamma\omega)^+$-cover of A, and thus A is an $L\omega$-compact set in $(L^X, \Delta_L(\Omega))$ by Theorem 13. □

This theorem indicates that the $L\omega$-compactness is a good extension in the sense of R. Lowen.

Theorem 26. *Let (L^X, Ω) be a stratified ωT_2 and $A \in L^X$. If A is $L\omega$-compact, then A is ω-closed.*

Proof. We only prove that $x_\alpha \leqslant A$ for each $x_\alpha \in M^*(L^X)$ with $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$ by the definition of ω-operator. Actually, if $x_\alpha \leqslant \omega\,\mathrm{cl}(A)$, then there exists a molecular net $N = \{x_{t(n)}^{(n)} \in M^*(L^X) \mid n \in D\}$ in A which ω-converges to x_α in accordance with Theorem 2 in [11]. Write $\lambda = \wedge_{m\in D}\vee_{n\geqslant m} t(n)$; we assert that $\lambda \geqslant \alpha$. In fact, if $\lambda \ngeqslant \alpha$, then there is a $m \in D$ with $\vee_{n\geqslant m} t(n) \ngeqslant \alpha$, and let $d = \vee_{n\geqslant m} t(n)$. Since (L^X, Ω) is stratified, the constant L-set $[d]$ on X is ω-closed and $x_\alpha \nleqslant [d]$, that is, $[d] \in \omega\eta^-(x_\alpha)$. Obviously, N is eventually in $[d]$, and it contradicts the fact that N ω-converges to x_α. Hence $\lambda \geqslant \alpha$; that is, $\vee_{n\geqslant m} t(n) \geqslant \alpha$ for each $m \in D$. For each $r \in \beta^*(\alpha)$ and each $m \in D$ we choose $n(r, m) \in D$ such that $n(r, m) \geqslant m$ and $t(n(r, m)) \geqslant r$, and define the relation "$\geqslant$" in $\beta^*(\alpha) \times D$ as follows:

$$(r_1, m_1) \geqslant (r_2, m_2) \quad \text{iff } r_1 \geqslant r_2,\ m_1 \geqslant m_2. \tag{2}$$

Then $\beta^*(\alpha) \times D$ is a directed set with the relation. Write $S = \{x_{t(n(r,m))}^{n(r,m)} \mid (r, m) \in \beta^*(\alpha) \times D\}$; then $S = N \circ R$, where $R : \beta^*(\alpha) \times D \rightarrow D$ is defined as $R(n(r, m)) = n(r, m)$. Evidently S is a subnet of N and ω-converges to x_α, and S is an α-net in A. Being the $L\omega$-compactness of A, S has an ω-cluster point in A with hight α, say z_α. Since (L^X, Ω) is an ωT_2 space, S ω-converges to x_α and ω-accumulates to z_α, $z = x$ by Theorem 2.7 in [11], and hence $x_\alpha = z_\alpha \leqslant A$. This implies that $\omega\,\mathrm{cl}(A) \leqslant A$; that is, A is an ω-closed set. □

The following example shows that the stratified condition in Theorem 26 can not be omitted.

Example 27. Let $X = \{x\}$ be a single set, $L = [0, 1]$, and let $\omega : L^X \rightarrow L^X$ be the fuzzy closure operator. Define $\omega O(L^X) = \{0_X, x_{1/3}, 1_X\}$, where $A : x \rightarrow [0, 1]$ is defined as $A(x) = x_\alpha$, $\alpha \in [0, 1]$ for $x \in X$. Obviously, (L^X, Ω) is both an $L\omega$-compact space and an N-compact space. According to Example 11 we know that $A = x_{1/3}$ is an $L\omega$-compact set in (L^X, Ω), but A is not ω-closed.

The following theorems imply that the $L\omega$-compactness can strengthen ω-seperation properties.

Theorem 28. *If (L^X, Ω) is both ωT_2 and $L\omega$-compact $L\omega$-space, then (L^X, Ω) is an ω-regular space [11].*

Proof. Let $G \in L^X$ be an ω-closed pseudocrisp set and let x_λ be a molecule which x is not in supp G. By Definition 7.1 in [19], there is an $\alpha \in M$ such that $G(x) > 0$ implies $G(x) \geqslant \alpha$. For each $y_\alpha \in M^*(L^X)$, there are $P_y \in \omega\eta^-(x_\lambda)$ and $Q_y \in \omega\eta^-(y_\alpha)$ satisfying $P_y \vee P_y = 1_X$ by virtue of $x \neq y$ and the ωT_2 separation of (L^X, Ω). Put $\Phi = \{Q_y \mid y_\alpha \leqslant G\}$; then Φ is an $\alpha\omega$-RF of G. Since (L^X, Ω) is an $L\omega$-compact space, G is an $L\omega$-compact set in accordance with Theorem 20, and thus Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \ldots, Q_{y_n}\}$ which is an $(\alpha\omega)^-$-RF of G; that is, there is an $r \in \beta^*(\alpha)$ such that for each molecule $z_r \leqslant G$ we have $i \leqslant n$ with $z_r \nleqslant Q_{y_i}$. Let $Q = \wedge_{i=1}^n Q_{y_i}$; then $z_r \nleqslant Q$; that is, $r \nleqslant Q(z)$ for each $z_r \leqslant G$. Since $G(z) > 0$ implies that $G(z) \geqslant \alpha \geqslant r$, $G(z) \nleqslant Q(z)$ for each $z \in \operatorname{supp} G$, and hence $Q \in \omega\eta^-(G)$. Write $P = \vee_{i=1}^n Py_i$; then $P \in \omega\eta^-(x_\lambda)$ and

$$P \vee Q = \left(\vee_{i=1}^n Py_i\right) \vee \left(\wedge_{i=1}^n Qy_i\right) \geqslant \vee_{i=1}^n \left(Py_i \vee Qy_i\right) = 1. \quad (3)$$

Consequently, (L^X, Ω) is an ω-regular space. □

Theorem 29. *Let (L^X, Ω) be an $L\omega$-compact ωT_2 space. Then (L^X, Ω) is an ω-normal space [11].*

Proof. Let both G, H be ω-closed pseudocrisp sets in (L^X, Ω) with $(\operatorname{supp} G) \cap (\operatorname{supp} H) = \phi$. Then there are $\lambda, \mu \in M$ such that $G(x) > 0$ if and only if $G(x) \geqslant \lambda$, and $H(x) > 0$ if and only if $H(x) \geqslant \mu$. According to the proof of Theorem 28, for each molecule $y_\mu \leqslant G$, there is an ω-closed set $P_y \in \omega\eta^-(G)$ satisfying $\lambda \nleqslant P_y(z)$ for each $z \in \operatorname{supp} G$, and there is a $Q_y \in \omega\eta^-(y_\mu)$ such that $P_y \vee Q_y = 1$. One can easily see that $\Phi = \{Q_y \mid y_\mu \leqslant B\}$ is a $\mu\omega$-RF of H. In line with Theorem 20 we know that H is an $L\omega$-compact set, and so Φ has a finite subfamily $\Psi = \{Q_{y_1}, Q_{y_2}, \ldots, Q_{y_n}\}$ such that Ψ is a $(\mu\omega)^-$-RF of H. Put $P = \vee_{i=1}^n Py_i$; $Q = \wedge_{i=1}^n Q_{y_i}$; then $P \in \omega\eta^-(G)$, $Q \in \omega\eta^-(H)$ and $P \vee Q = 1$. Therefore (L^X, Ω) is an ω-normal space. □

5. The Tychonoff Product Theorem

In this section, we will first extend Alexandar's subbase Lemma in general topology and give the Alexandar's ω-subbase lemma and next prove that the Tychonoff product theorem holds in $L\omega$-spaces.

Theorem 30 (Alexandar ω-subbase lemma). *Let (L^X, Ω) be an $L\omega$-space, $A \in L^X$, and let γ be an ω-subbase [20] in L^X. If for each $\alpha\omega$-RF Φ of A where $\Phi \subseteq \gamma' \subseteq \omega C(L^X)$, there is a finite subfamily Ψ of Φ with $\wedge\Psi \ll A(\alpha\omega)(\alpha \in M)$, then A is $L\omega$-compact.*

Proof. Suppose that Φ is an arbitrary $\alpha\omega$-RF of A. We will prove that Φ has a finite subfamily Ψ which is an $(\alpha\omega)^+$-RF of A. In fact, if for each $\Psi \in 2^{(\Phi)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold, then $H = \{\Delta \mid \Phi \subseteq \Delta \subseteq \omega C(L^X)$, for all $\Psi \in 2^{(\Delta)}$, $\wedge\Psi \ll A(\alpha\omega)$ does not hold$\} \neq \emptyset$, and H is a partial-ordered set with respect to the upper bound and hence there exists a maximal element Δ_0 in H by Zorn's Lemma. We assert that Δ_0 satisfies the following conditions:

(1) $\wedge\Delta_0 < A(\alpha \geqslant \omega)$;

(2) if $P \in \Delta_0$, then $Q \in \Delta_0$ for each $Q \in \omega C(L^X)$ with $Q \geqslant P$;

(3) if $P, Q \in \omega C(L^X)$ and $P \vee Q \in \Delta_0$, then $P \in \Delta_0$ or $Q \in \Delta_0$.

Actually, since $\wedge\Phi < A(\alpha\omega)$ and $\Phi \subseteq \Delta_0$, condition (1) holds. If $P \in \Delta_0, Q \in \omega C(L^X), Q \geqslant P$, and Q is not in Δ_0, then $\Delta^* = \Delta_0 \cup \{Q\} \in H$ and $\Delta_0 \subset \Delta^*$. It contradicts the fact that Δ_0 is the maximal element in H thus condition (2) holds. Let $P, Q \in \omega C(L^X)$. If P and Q are both not in Δ_0, then $\Delta_0 \cup \{P\}$ and $\Delta_0 \cup \{Q\}$ are both not in H by the maximality of Δ_0, and thus there are $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$ such that $\wedge(\Psi_1 \cup \{P\}) \ll A(\alpha\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) \ll A(\alpha\omega)$ according to the definition of H; that is, there are $s, t \in \beta^*(\alpha)$ with $\wedge(\Psi_1 \cup \{P\}) < A(s\omega)$ and $\wedge(\Psi_2 \cup \{Q\}) < A(t\omega)$. Since $\beta^*(\alpha)$ is upper directed, we can choose $r \in \beta^*(\alpha)$ with $r \geqslant s \vee t$. Now we prove $\wedge\{\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}\} < A(r\omega)$. In reality, if $\Psi_2 \cup \Psi_2$ does not have any ωR-neighborhood of x_r for each $x_r \leqslant A$, then $\Psi_2 \cup \Psi$ does not have any ωR-neighborhood of x_s and x_t, respectively, and hence $P \in \omega\eta^-(x_s)$ and $Q \in \omega\eta^-(x_t)$. Particularly, $P, Q \in \omega\eta^-(x_r)$ and so $P \vee Q \in \omega\eta^-(x_r)$. This shows that $\wedge(\Psi_2 \cup \Psi_2 \cup \{P \vee Q\}) < A(r\omega)$. Therefore $P \vee Q$ is not in Δ_0 by virtue of the definition of Δ_0 and $\Psi_1, \Psi_2 \in 2^{(\Delta_0)}$. So, condition (3) holds.

From (2) and (3) we have the following result:

(4) If $R \in \Delta_0$, $P_i \in \omega C(L^X)\,(i = 1, 2, \ldots, n)$ and $R \leqslant \vee_{i=1}^n P_i$, then there is an $i \in \{1, 2, \ldots, n\}$ satisfying $P_i \in \Delta_0$.

Consider now $\gamma' \cap \Delta_0$. If $\gamma' \cap \Delta_0$ is an $\alpha\omega$-RF of A, then there is a finite subfamily δ of $\gamma' \cap \Delta_0$ which is an $(\alpha\omega)^-$-RF of A. Evidently, $\delta \in 2^{(\Delta_0)}$; it is in contradiction with $\Delta_0 \in H$. Hence $\gamma' \cap \Delta_0$ is not an $\alpha\omega$-RF of A; that is, there is a molecule x_α in A meeting $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0)$. We now verify that $x_\alpha \leqslant \wedge\Delta_0$. In fact, if there is $Q \in \Delta_0$ with $x_\alpha \nleqslant Q$, then by Definition 5 in [17] we can take a finite subfamily $\{P_{ij} \mid j \in J_i, i \in I\}$ of γ' satisfying $Q = \wedge_{i\in I} \vee_{j\in J_i} P_{ij}$, where J_i is a finite set for each $i \in I$. Because of $x_\alpha \nleqslant Q$, we can choose $i \in I$ with $x_\alpha \nleqslant \vee_{j\in J_i} P_{ij}$. Since $Q \leqslant \vee_{j\in J_i} P_{ij}$, there is a $j \in J_i$ such that $P_{ij} \in \Delta_0$ by (4). Hence $P_{ij} \in \gamma' \cap \Delta_0$ and $x_\alpha \nleqslant P_{ij}$; it contradicts the fact that $x_\alpha \leqslant \wedge(\gamma' \cap \Delta_0) \leqslant P_{ij}$; thus $x_\alpha \leqslant \wedge\Delta_0$. However, this is in contradiction with (1) again. This implies that Φ has a finite subfamily Ψ with $\wedge\Psi \ll A(\alpha\omega)$. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 31. *Let $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ be a collection of $L\omega$-spaces and let (L^X, Ω) be the product space of them. If A_t is an $L\omega$-compact set in (L^{X_t}, Ω_t) for each $t \in T$, then the product $A = \Pi_{t\in T} A_t$ of all $L\omega$-compact sets $A_t (t \in T)$ is an $L\omega$-compact set in (L^X, Ω).*

Proof. Assume that Φ is an $\alpha\omega$-RF of $A(\alpha \in M)$. By Theorem 30 we can grant that every ω-closed set in Φ is of the form $\rho_t^{\leftarrow}(B_t)$ where $B_t \in \omega C(L^{X_t})$ and $\rho_t : L^X \to L^{X_t}$ is a protection because $\{\rho_t^{\leftarrow}(U_t) \mid U_t \in \omega O(L^X), t \in T\}$ is an ω-subbase in (L^X, Ω) [20]. Now we consider the following two cases.

(i) If there exists a $t_0 \in T$ such that no molecule with hight α is contained in A_{t_0}, then by the $L\omega$-compactness of A_{t_0}, there is an $r \in \beta^*(\alpha)$ such that no molecule with hight r is contained in A_{t_0}. In reality, if there exists a molecule with hight r in A_{t_0} for each $r \in \beta^*(\alpha)$, say $N(r)$, then $N = \{N(r) \mid r \in \beta^*(\alpha)\}$ is an α-net in A_{t_0} by the directivity of $\beta^*(\alpha)$. Since A_{t_0} is $L\omega$-compact, N has an ω-cluster point in A_{t_0} with hight α according to Theorem 15. It is in contradiction with the hypothesis of A_{t_0}. Thus it can be seen that there exists an $r \in \beta^*(\alpha)$ with $A_{t_0}(x^{t_0}) \ngeq r$ for each $x^{t_0} \in X_{t_0}$. Hence for each $x \in X$, we have

$$\begin{aligned} A(x) &= \left(\Pi_{t\in T} A_t\right)(x) \\ &= \wedge_{t\in T} A_t\left(\rho_t(x)\right) \leqslant A_{t_0}\left(\rho_{t_0}(x)\right) = A_{t_0}\left(x^{t_0}\right), \end{aligned} \tag{4}$$

and hence $A(x) \ngeq r$ for each $x \in X$; that is, no molecule with hight r is contained in A. This shows that for each $\Psi \in 2^{(\Phi)}$, Ψ is an $(\alpha\omega)^-$-RF of A.

(ii) Suppose that for each $t \in T$, A_t contains a molecule with hight α, say x_α^t. Since $\Phi \subseteq \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \omega C(L^X), t \in T\}$, we can take $R \subseteq T$ such that $\Phi = \cup_{t\in R}\Phi_t$, where $\Phi_t = \{\rho_t^{\leftarrow}(B_t) \mid B_t \in \mathscr{B}_t \subseteq \omega C(L^X)\}$. Now we prove that there must be $s \in R$ with $\wedge\mathscr{B}_s < A(\alpha\omega)$. In fact, if there is a crisp point $y^t \in X_t$ such that $y^t \leqslant A_t \wedge (\wedge\mathscr{B}_t)$ for each $t \in R$, then we choose a crisp point z in X as follows: if $t \in R$, $z^t = y^t$; if t is not in R, $z^t = x^t$. Taking any ω-closed set $\rho_t^{\leftarrow}(B_t)$ in Φ, where $t \in R$ and $B_t \in \mathscr{B}_t$, we have

$$\rho_t^{\leftarrow}(B_t)(z) = B_t\left(z^t\right) = B_t\left(y^t\right) \geqslant \left(A_t \wedge \left(\wedge\mathscr{B}_t\right)\right)\left(y^t\right) \geqslant \alpha, \tag{5}$$

that is, $z_\alpha \leqslant \rho_t^{\leftarrow}(B_t)$, and hence $z_\alpha \leqslant \wedge\Phi$ by the arbitrariness of $\rho_t^{\leftarrow}(B_t) \in \Phi$. On the other hand,

$$A(z) = \wedge_{t\in R} A_t\left(z^t\right) = \left(\wedge_{t\in R} A_t\left(y^t\right)\right) \wedge \left(\wedge_{t\in R} A_t\left(x^t\right)\right) \geqslant \alpha. \tag{6}$$

This implies that z_α is a molecule in A; it contradicts the fact that Φ is an $\alpha\omega$-RF of A. Consequently, there is $s \in R$ with $\wedge\mathscr{B}_s < A_s(\alpha)$; thus there is a finite subfamily Γ_s of $\mathscr{B}_s$ with $\Gamma_s < A_s(r\omega)$ for some $r \in \beta^*(\alpha)$. Put $\Psi = \{\rho_s^{\leftarrow}(B_s) \mid B_s \in \Gamma_s\}$; then $\Psi \in 2^{(\Phi)}$. We assert that $\wedge\Psi < A(r\omega)$. Actually, for any molecule e_r in A with hight r we have $A_s(e^s) \geqslant A(e) \geqslant r$; that is, e_r^s is a molecule in A_s, where $e = \{e^t\}_{t\in T} \in X$. Hence there exists an ω-closed set $B_s \in \Gamma_s$ meeting $B_s \in \omega\eta^-(e_r^s)$ by virtue of the fact that Γ_s is an $r\omega$-RF of A_s; thus $\rho_s^{\leftarrow}(B_s)(e) = B_s(e^s) \ngeq r$; that is, $\rho_s^{\leftarrow}(B_s) \in \omega\eta^-(e_r)$. This shows that Ψ is an $r\omega$-RF of A. Therefore A is an $L\omega$-compact set in (L^X, Ω). □

Theorem 32 (Tychonoff product theorem). *Let (L^X, Ω) be the product space of a collection of $L\omega$-spaces $\{(L^{X_t}, \Omega_t) \mid t \in T\}$. Then (L^X, Ω) is $L\omega$-compact if and only if for each $t \in T$, (L^{X_t}, Ω_t) is $L\omega$-compact.*

Proof. Necessity. Assume that (L^X, Ω) is an $L\omega$-compact space. Since $\rho_t : (L^X, \Omega) \to (L^{X_t}, \Omega_t)$ is an ω-continuous L-valued Zadeh's type function for each $t \in T$, (L^{X_t}, Ω_t) is an $L\omega$-compact space by Theorem 23. Therefore the necessity holds.

Sufficiency. It follows from Theorem 31. □

The following example shows that the inverse theorem of Theorem 31 does not hold.

Example 33. Let $E = \{e_1, e_2, \ldots\}$ be a countably infinite set, $X_t = E$ for each $t \in T = \{1, 2, \ldots\}$, $L = [0, 1]$, $\Omega_t = [0, 1]^E$ and let ω be a fuzzy closure operator. Then (L^{X_t}, Ω_t) is a discrete $L\omega$-space for each $t \in T$. Define $A_t \in L^{X_t}$ $(t \in T)$ as follows:

$$\text{if } j = 1,\ A_t(e_j) = 1;\ \text{if } j \geqslant 2,\ A_t(e_j) = 1/t.$$

Suppose that (L^X, Ω) is the product space of $\{(L^{X_t}, \Omega_t) \mid t \in T\}$ and $A = \Pi_{t\in T} A_t$. Now we prove that A is an $L\omega$-compact set in (L^X, Ω), but A_t is not an ω-compact set in (L^{X_t}, Ω_t) for each $t \in T$. In reality, for each $x = (x_1, x_2, \ldots) \in X$ we put $x_t = e_{j(t)}^t$, where x_t is a crisp point e_j in X_t; then from the definitions of A_t and fuzzy product set A we know

$$\begin{aligned} A(x) &= \left(\Pi_{t\in T} A_t\right)(x) = \wedge_{t\in T} A_t\left(x_t\right) = \wedge_{t\in T} A_t\left(e_{j(t)}^t\right) \\ &= \begin{cases} 0, & \text{if there are infinite elements } t \\ & \text{such that } j(t) \geqslant 2. \\ \dfrac{1}{t_R}, & \text{if there is a } t_R \in T \text{ such that } j\left(t_R\right) \geqslant 2 \\ & \text{and } j(t) = 1 \text{ whenever } t > t_R. \end{cases} \end{aligned} \tag{7}$$

Thus it can be seen that $A \neq 0_X$ and if $A(x) \geqslant 1/t_R$, then the coordinates $x_t = e_{j(t)}^t = e_1$ of x whenever $t > t_R$. Obviously, points in X satisfying the condition are only finite. Let $\alpha \in M$, that is, $\alpha > 0$, and let Φ be an $\alpha\omega$-RF of A. Choose $t_R \in T$ with $1/t_R < \alpha$. Since there are only finite molecule in A with hight α, denote the finite crisp points as $x^1, x^2, \ldots, x^n$. If $(x^i)_\alpha \leqslant A$ for each $i \in \{1, 2, \ldots, n\}$, then there is $P_i \in \Phi$ with $P_i(x^i) < \alpha$. Put $s = \max\{P_i(x^i) \mid P_i(x^i) < \alpha, i \leqslant n\}$; then $s < \alpha$. Taking $s_1 \in (s, \alpha)$ and $r = \max(s_1, 1/t_R)$, we know that A has at most n molecules with hight r, say $(x^i)_r$ $(i \leqslant n)$. By the definition of Φ, there is a $P_i \in \Phi$ such that $P_i \in \omega\eta^-((x^i)_r)$ for each $(x^i)_r$ in A. Denote $\Psi = \{P_i \in \Phi \mid P_i \in \omega\eta^-((x^i)_r), i \leqslant n\}$; then $\Psi \in 2^{(\Phi)}$ and Ψ is an $r\omega$-RF of A. This implies that Ψ is an $(\alpha\omega)^-$-RF of A by $r \in \beta^*(\alpha)$. Hence A is $L\omega$-compact in (L^X, Ω). On the other hand, take $D = T$ and $N = \{N(m) \mid m \in D\}$ where $N(m) = (e_m)_{1/t}$ for each $m \in D$ and each $t \in T$; then N is a $(1/t)$-net in A_t. Since (L^{X_t}, Ω_t) is discrete, N does not have any ω-cluster point in A_t with hight $1/t$. Therefore A_t is not $L\omega$-compact in (L^{X_t}, Ω_t) for each $t \in T$ according to Theorem 15.

Acknowledgments

The project was supported by the Natural Science Foundation of Fujian Province of China (no. 2011J01013) and the

Programs of Science and Technology Department of Xiamen City (nos. 3502Z20110010 and 3502Z20123022).

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