Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 947487, 7 pages
http://dx.doi.org/10.1155/2013/947487

Research Article

On Complete Convergence for Weighted Sums of ρ^*-Mixing Random Variables

Aiting Shen, Xinghui Wang, and Huayan Zhu

School of Mathematical Science, Anhui University, Hefei 230039, China

Correspondence should be addressed to Aiting Shen; empress201010@126.com

Received 29 July 2013; Revised 15 September 2013; Accepted 17 September 2013

Academic Editor: Ciprian A. Tudor

We prove the strong law of large numbers for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$, which generalizes and improves the corresponding one for independent and identically distributed random variables and φ-mixing random variables. In addition, we present some results on complete convergence for weighted sums of ρ^*-mixing random variables under some suitable conditions, which generalize the corresponding ones for independent random variables.

1. Introduction

Throughout the paper, we let $I(A)$ be the indicator function of the set A. We assume that $\phi(x)$ is a positive increasing function on $(0,\infty)$ satisfying $\phi(x) \uparrow \infty$ as $x \to \infty$ and $\psi(x)$ is the inverse function of $\phi(x)$. Since $\phi(x) \uparrow \infty$, it follows that $\psi(x) \uparrow \infty$. For easy notation, we let $\phi(0) = 0$ and let $\psi(0) = 0$. $a_n = O(b_n)$ denotes that there exists a positive constant C such that $|a_n/b_n| \le C$. C denotes a positive constant.

Let $\{X_i, i \ge 1\}$ be a sequence of independent observations from a population distribution. A common expression for these linear statistics is $T_n \doteq \sum_{i=1}^{n} a_{ni}X_i$, where the weights a_{ni} are either real constants or random variables independent of X_i. Many authors have studied the strong convergence properties for linear statistics T_n and obtained some interesting results. For the details, one can refer to Bai and Cheng [1], Sung [2], Cai [3], Jing and Liang [4], Zhou et al. [5], Wang et al. [6–8] and Wu and Chen [9], Tang [10], and so forth.

Recently, Cai [11] proved the following strong law of large numbers for weighted sums of independent and identically distributed random variables.

Theorem A. *Let $\{X, X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with $EX = 0$, $EX^2 < \infty$, and $E[\phi(|X|)] < \infty$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n). \tag{1}$$

Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants such that

(*i*) $\max_{1\le i\le n}|a_{ni}| = O(1/(\psi(n)))$;

(*ii*) $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ *for some* $\alpha > 0$.

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{2}$$

Wang et al. [12] generalized the result of Theorem A for independent sequences to the case of φ-mixing sequences under conditions of (1), (*i*), and (*ii*). Conditions (1) and (*i*) in Theorem A look ugly, since the conclusion (2) does not contain any information on $\psi(n)$. The main purpose of the paper is to show (2) for ρ^*-mixing random variables without conditions of (1) and (*i*). So our result generalizes and improves the corresponding one of Cai [11] and Wang et al. [12]. In addition, we will present some results on complete convergence for weighted sums of ρ^*-mixing random variables under some suitable conditions, which generalize the corresponding ones for independent random variables.

Firstly, let us recall the concept of ρ^*-mixing random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. Write $\mathscr{F}_S = \sigma(X_i, i \in S \subset \mathbf{N})$. Given σ-algebras $\mathscr{B}, \mathscr{R}$ in $\mathscr{F}$, let

$$\rho(\mathscr{B}, \mathscr{R}) = \sup_{X \in L_2(\mathscr{B}), Y \in L_2(\mathscr{R})} \frac{|EXY - EXEY|}{(\operatorname{Var} X \ \operatorname{Var} Y)^{1/2}}. \quad (3)$$

Define the ρ^*-mixing coefficients by

$$\rho^*(k) = \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T) : \text{finite subsets } S, T \subset \mathbf{N}, \text{ such that } \operatorname{dist}(S, T) \geq k\}, \quad k \geq 0. \quad (4)$$

Obviously, $0 \leq \rho^*(k+1) \leq \rho^*(k) \leq 1$, and $\rho^*(0) = 1$.

Definition 1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be ρ^*-mixing if there exists $k \in \mathbf{N}$ such that $\rho^*(k) < 1$.

It is easily seen that ρ^*-mixing (i.e., $\tilde{\rho}$-mixing) sequence contains independent sequence as a special case. ρ^*-mixing random variables were introduced by Bradley [13] and many applications have been found. $\tilde{\rho}$-mixing is similar to ρ-mixing, but both are quite different. Many authors have studied this concept providing interesting results and applications. See, for example, Bradley [13] for the central limit theorem, Bryc and Smoleński [14], Peligrad and Gut [15], and Utev and Peligrad [16] for moment inequalities, Gan [17], Kuczmaszewska [18], Wu and Jiang [19] and Wang et al. [20, 21] for almost sure convergence, Peligrad and Gut [15], Cai [22], Kuczmaszewska [23], Zhu [24], An and Yuan [25], Wang et al. [26], and Sung [27] for complete convergence, Peligrad [28] for invariance principle, Wu and Jiang [29] for strong limit theorems for weighted product sums of ρ^*-mixing sequences of random variables, Wu and Jiang [30] for Chover-type laws of the k-iterated logarithm, Wu [31] for strong consistency of estimator in linear model, Wang et al. [32] for complete consistency of the estimator of nonparametric regression models, Wu et al. [33] and Guo and Zhu [34] for complete moment convergence, and so forth. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired. So studying the limit behavior of ρ^*-mixing random variables is of interest.

The following concepts of slowly varying function and stochastic domination will be used in this work.

Definition 2. A real-valued function $l(x)$, positive and measurable on $(0, \infty)$, is said to be slowly varying if

$$\lim_{x \to \infty} \frac{l(x\lambda)}{l(x)} = 1 \quad (5)$$

for each $\lambda > 0$.

Definition 3. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x) \quad (6)$$

for all $x \geq 0$ and $n \geq 1$.

This work is organized as follows. Some important lemmas are presented in Section 2. Main results and their proofs are provided in Section 3.

2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results of the paper. The first one is the Rosenthal type maximal inequality for ρ^*-mixing random variables, which was obtained by Utev and Peligrad [16].

Lemma 4 (cf. Utev and Peligrad [16]). *For a positive integer $N \geq 1$ and positive real numbers $q \geq 2$ and $0 \leq r < 1$, there exists a positive constant $D = D(q, N, r)$ such that if $\{X_n, n \geq 1\}$ is a sequence of random variables with $\rho^*(N) < r$, $EX_i = 0$, and $E|X_i|^q < \infty$ for every $i \geq 1$, then for all $n \geq 1$,*

$$E\left(\max_{1 \leq i \leq n}\left|\sum_{j=1}^{i} X_j\right|^q\right) \leq D\left\{\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right\}. \quad (7)$$

The next one is a basic property for stochastic domination. For the details of the proof, one can refer to Wu [35] or Tang [36].

Lemma 5. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 [E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)], \quad (8)$$

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$, where C is a positive constant.

The last one is the basic properties for slowly varying function, which was obtained by Bai and Su [37].

Lemma 6 (cf. Bai and Su [37]). *If $l(x) > 0$ is a slowly varying function as $x \to \infty$, then*

(*i*) $\lim_{x \to \infty}(l(xu)/l(x)) = 1$ *for each* $u > 0$;

(*ii*) $\lim_{k \to \infty} \sup_{2^k \leq x < 2^{k+1}}(l(x)/l(2^k)) = 1$;

(*iii*) $\lim_{x \to \infty} x^\delta l(x) = \infty$, $\lim_{x \to \infty} x^{-\delta} l(x) = 0$ *for each* $\delta > 0$;

(*iv*) $c_1 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr} l(\varepsilon 2^j) \leq c_2 2^{kr} l(\varepsilon 2^k)$ *for every* $r > 0$, $\varepsilon > 0$, *positive integer* k *and some* $c_1 > 0$, $c_2 > 0$;

(*v*) $c_3 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=k}^{\infty} 2^{jr} l(\varepsilon 2^j) \leq c_4 2^{kr} l(\varepsilon 2^k)$ *for every* $r < 0$, $\varepsilon > 0$, *positive integer* k *and some* $c_3 > 0$, $c_4 > 0$.

3. Main Results and Their Proofs

In this section, we will generalize and improve the result of Theorem A for independent and identically distributed

random variables to the case of ρ^*-mixing random variables. In addition, we will present some results on complete convergence for weighted sums of ρ^*-mixing random variables.

Our main results are as follows.

Theorem 7. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables, which is stochastically dominated by a random variable X with $EX_n = 0$, and $EX^2 < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants such that*

$$\sum_{i=1}^{n} a_{ni}^2 = O\left(\log^{-1-\alpha} n\right) \quad \text{for some } \alpha > 0. \tag{9}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{10}$$

Proof. By Markov's inequality, Lemmas 4 and 5, $EX^2 < \infty$ and condition (9), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) \\
&\quad \leq C\sum_{n=1}^{\infty} n^{-1} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right|^2\right) \\
&\quad \leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} a_{ni}^2 EX_i^2 \\
&\quad \leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} a_{ni}^2 EX^2 \\
&\quad \leq C\sum_{n=1}^{\infty} n^{-1}\log^{-1-\alpha} n < \infty,
\end{aligned} \tag{11}$$

which implies (10). This completes the proof of the theorem. □

Remark 8. The key to the proof of Theorem 7 is the Rosenthal type maximal inequality for ρ^*-mixing sequences (i.e., Lemma 4). Similar to the proof of Theorem 7, we have the following result.

Theorem 9. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X with $EX_n = 0$ and $EX^2 < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants such that $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ for some $\alpha > 0$. Suppose that there exists a positive constant C such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right|^2\right) \leq C\sum_{i=1}^{n} a_{ni}^2 EX_i^2. \tag{12}$$

Then (10) *holds for any $\varepsilon > 0$.*

If the array of constants $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is replaced by the sequence of constants $\{a_n, n \geq 1\}$, then we can get the following strong law of large numbers for weighted sums $\sum_{i=1}^{n} a_i X_i$. The proof is standard, so we omit the details.

Theorem 10. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X with $EX_n = 0$ and $EX^2 < \infty$. Let $\{a_n, n \geq 1\}$ be a sequence of constants such that $\sum_{i=1}^{n} a_i^2 = O(\log^{-1-\alpha} n)$ for some $\alpha > 0$. Suppose that there exists a positive constant C such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_i X_i\right|^2\right) \leq C\sum_{i=1}^{n} a_i^2 EX_i^2. \tag{13}$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon\right) < \infty, \tag{14}$$

and in consequence, $\sum_{i=1}^{n} a_i X_i \to 0$ a.s.

Remark 11. There are many sequences of random variables satisfying (12), such as negatively associated (NA, in short) sequence (see Shao [38]), negatively superadditive-dependent (NSD, in short) sequence (see Wang et al. [39]), asymptotically almost negatively associated (AANA, in short) sequence (see Yuan and An [40]), φ-mixing sequence (see Wang et al. [12]), and ρ^*-mixing sequence (see Utev and Peligrad [16]). Comparing Theorems 7 and 9 with Theorem A, conditions (1) and (*i*) in Theorem A can be removed. In addition, the condition "identical distribution" in Theorem A can be weakened by "stochastic domination." Hence, the results of Theorem 7 and Theorem 9 generalize and improve the corresponding one of Theorem A.

In the following, we will present some results on complete convergence for weighted sums of ρ^*-mixing random variables. The main ideas are inspired by Kuczmaszewska [41]. The first one is a very general result of complete convergence for weighted sums of ρ^*-mixing random variables, which can be applied to obtain other result's of complete convergence, such as Baum-Katz type complete convergence and Hsu-Robbins type complete convergence.

Theorem 12. *Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables and let $\{a_{ni}, n \geq 1, i \geq 1\}$ be an array of real numbers. Let $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers and let $\{c_n, n \geq 1\}$ be a sequence of positive real numbers. If for some $q \geq 2$, $0 < t < 2$ and for any $\varepsilon > 0$, the following conditions are satisfied:*

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}\right) < \infty, \tag{15}$$

$$\sum_{n=1}^{\infty} c_n b_n^{-(q/t)} \sum_{i=1}^{b_n} |a_{ni}|^q \, E|X_i|^q I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) < \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} c_n b_n^{-(q/2)} \left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{q/2} < \infty, \tag{17}$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_jI\left(\left|a_{nj}X_j\right|<\varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} < \infty. \tag{18}$$

Proof. Let

$$\begin{aligned} Y_i^{(n)} &= a_{ni}X_iI\left(\left|a_{ni}X_i\right|<\varepsilon b_n^{1/t}\right), \quad S'_{ni} = \sum_{j=1}^{i} Y_j^{(n)}, \quad n\ge 1, \quad i\ge 1,\\ A &= \bigcap_{i=1}^{b_n}\left\{Y_i^{(n)} = a_{ni}X_i\right\},\\ B &= \overline{A} = \bigcup_{i=1}^{b_n}\left\{Y_i^{(n)} \ne a_{ni}X_i\right\} = \bigcup_{i=1}^{b_n}\left(\left|a_{ni}X_i\right|\ge \varepsilon b_n^{1/t}\right),\\ E_n &= \left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_jI\left(\left|a_{nj}X_j\right|<\varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\}. \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned} P &= \left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_{nj} - a_{nj}EX_jI\left(\left|a_{nj}X_j\right|<\varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\}\\ &= P\left(E_n\right) = P\left(E_nA\right) + P\left(E_nB\right)\\ &\le P\left(E_nA\right) + P\left(B\right)\\ &\le \sum_{i=1}^{b_n} P\left(\left|a_{ni}X_i\right|\ge \varepsilon b_n^{1/t}\right)\\ &\quad + \varepsilon^{-q} b_n^{-(q/t)} E\left(\max_{1\le i\le b_n}\left|S'_{ni} - ES'_{ni}\right|\right)^q. \end{aligned} \tag{20}$$

Using the C_r-inequality and Jensen's inequality, we can estimate $E|Y_i^{(n)} - EY_i^{(n)}|^q$ in the following way:

$$E\left|Y_i^{(n)} - EY_i^{(n)}\right|^q \le C\left|a_{ni}\right|^q E\left|X_i\right|^q I\left(\left|a_{ni}X_i\right|<\varepsilon b_n^{1/t}\right). \tag{21}$$

The desired result (18) follows from (15), (16), (17), (20), (21), and Lemma 4 immediately. The proof is completed. □

In what follows, we will give some applications for Theorem 12.

Corollary 13. *Let $\{X_n, n \ge 1\}$ be a sequence of ρ^*-mixing random variables and let $\{a_{ni}, n\ge 1, i\ge 1\}$ be an array of real numbers. Let $l(x) > 0$ be a slowly varying function as $x \to \infty$, $\alpha > (1/2)$, and $\alpha r > 1$. If for some $q \ge 2$ and $0 < t < 2$, the following conditions are satisfied for any $\varepsilon > 0$:*

$$\sum_{n=1}^{\infty} n^{\alpha r-2} l(n) \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right|\ge \varepsilon n^{1/t}\right) < \infty, \tag{22}$$

$$\sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)} l(n) \sum_{i=1}^{n}\left|a_{ni}\right|^q E\left|X_i\right|^q I\left(\left|a_{ni}X_i\right|<\varepsilon n^{1/t}\right) < \infty, \tag{23}$$

$$\sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)} l(n) \left[\sum_{i=1}^{n} a_{ni}^2 EX_i^2 I\left(\left|a_{ni}X_i\right|<\varepsilon n^{1/t}\right)\right]^{q/2} < \infty, \tag{24}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r-2} l(n) P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_jI\left(\left|a_{nj}X_j\right|<\varepsilon n^{1/t}\right)\right]\right| \ge \varepsilon n^{1/t}\right\} < \infty. \tag{25}$$

Proof. Let $c_n = n^{\alpha r-2}l(n)$ and let $b_n = n$. The desired result (25) follows from conditions (22)–(24) and Theorem 12 immediately. The proof is completed. □

If $a_{ni} \equiv 1$ for $n \ge 1$ and $i \ge 1$ in Corollary 13, then we can get the following result for ρ^*-mixing random variables.

Corollary 14. *Let $\{X_n, n\ge 1\}$ be a sequence of mean zero ρ^*-mixing random variables, which is stochastically dominated by a random variable X and, let $l(x) > 0$ be a slowly varying function as $x \to \infty$. If for some $\alpha > (1/2)$, $\alpha r > 1$ and $0 < t < 2$,*

$$E|X|^{\alpha rt} l\left(|X|^t\right) < \infty, \tag{26}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r-2} l(n) P\left(\max_{1\le i\le n}\left|\sum_{j=1}^{i} X_j\right| \ge \varepsilon n^{1/t}\right) < \infty. \tag{27}$$

Proof. The proof is similar to that of Corollary 2.8 in Kuczmaszewska [41]. Take $a_{ni} = 1$, $n \ge 1$, $i \ge 1$, and $q > \max(2, (2t(\alpha r-1))/2 - t)$. In order to prove (27), it is enough

to prove that under, the conditions of Corollary 14, conditions (22) and (24) hold.

In fact, by Lemma 6 and similar to the proof of Corollary 2.8 in Kuczmaszewska [41], we can obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-2} l(n) \sum_{i=1}^{n} P\left(\left|X_i\right| \geq \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\alpha r-1} l(n) P\left(|X| \geq \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}} n^{\alpha r-1} l(n) P\left(|X| \geq \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{k=1}^{\infty} \left(2^k\right)^{\alpha r} l\left(2^k\right) P\left[|X| \geq \varepsilon\left(2^k\right)^{1/t}\right] \\
&\quad \leq C E|X|^{\alpha r t} l\left(|X|^t\right) < \infty,
\end{aligned} \tag{28}$$

which implies that (22) holds.

Let $F(x)$ be the distribution function of X. It follows, by Lemmas 5 and 6 and the inequality above, that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)} l(n) \sum_{i=1}^{n} E\left|X_i\right|^q I\left(\left|X_i\right| < \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\alpha r-1} l(n) P\left(|X| \geq \varepsilon n^{1/t}\right) \\
&\qquad + C \sum_{n=1}^{\infty} n^{(\alpha r-1)-(q/t)} l(n) E|X|^q I\left(|X| < \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{(\alpha r-1)-(q/t)} l(n) E|X|^q I\left(|X| < \varepsilon n^{1/t}\right) \\
&\quad \leq C \sum_{k=1}^{\infty} \left(2^k\right)^{\alpha r-(q/t)} l\left(2^k\right) \int_0^{\left(2^k\right)^{1/t}} |x|^q dF(x) \\
&\quad \leq C E|X|^{\alpha r t} l\left(|X|^t\right) < \infty,
\end{aligned} \tag{29}$$

where the fourth inequality above is followed by the proof of Corollary 2.8 in Kuczmaszewska [41]. This shows that (23) holds.

By Lemma 5, C_r-inequality, and Markov's inequality, we can see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)} l(n) \left[\sum_{i=1}^{n} E X_i^2 I\left(\left|X_i\right| < \varepsilon n^{1/t}\right)\right]^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)+(q/2)} l(n) \\
&\qquad \times \left[\varepsilon^2 n^{2/t} P\left(|X| \geq \varepsilon n^{1/t}\right) + E X^2 I\left(|X| < \varepsilon n^{1/t}\right)\right]^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{(\alpha r-2)+(q/2)} l(n) \left[P\left(|X| \geq \varepsilon n^{1/t}\right)\right]^{q/2} \\
&\qquad + C \sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)+(q/2)} l(n) \left[E X^2 I\left(|X| < \varepsilon n^{1/t}\right)\right]^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{(\alpha r-1)(1-(q/2))-1} l(n) \left[E|X|^{\alpha r t}\right]^{q/2} \\
&\qquad + C \sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)+(q/2)} l(n) \left[E X^2 I\left(|X| < \varepsilon n^{1/t}\right)\right]^{q/2} \\
&\quad \leq C + C \sum_{n=1}^{\infty} n^{(\alpha r-2)-(q/t)+(q/2)} l(n) \\
&\qquad \times \left[E X^2 I\left(|X| < \varepsilon n^{1/t}\right)\right]^{q/2} < \infty,
\end{aligned} \tag{30}$$

where the last inequality is followed by the proof of Corollary 2.8 in Kuczmaszewska [41]. Hence, condition (24) is satisfied.

The proof will be completed if we show that

$$n^{-(1/t)} \max_{1 \leq i \leq n} \left|\sum_{j=1}^{i} E X_j I\left(\left|X_j\right| < \varepsilon n^{1/t}\right)\right| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{31}$$

If $\alpha r t < 1$, then we have by Lemma 5 that

$$\begin{aligned}
&n^{-(1/t)} \max_{1 \leq i \leq n} \left|\sum_{j=1}^{i} E X_j I\left(\left|X_j\right| < \varepsilon n^{1/t}\right)\right| \\
&\quad \leq n^{-(1/t)} \sum_{j=1}^{n} E\left|X_j\right| I\left(\left|X_j\right| < \varepsilon n^{1/t}\right) \\
&\quad \leq C n P\left(|X| \geq \varepsilon n^{1/t}\right) + C n^{1-(1/t)} E|X| I\left(|X| < \varepsilon n^{1/t}\right) \\
&\quad \leq C n P\left(\frac{|X|^{\alpha r t}}{\varepsilon^{\alpha r t}} \geq n\right) \\
&\qquad + C n^{1-\alpha r} E|X|^{\alpha r t} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{32}$$

If $\alpha r t \geq 1$, then it follows, by $E X_j = 0$ and Lemma 5, that

$$\begin{aligned}
&n^{-(1/t)} \max_{1 \leq i \leq n} \left|\sum_{j=1}^{i} E X_j I\left(\left|X_j\right| < \varepsilon n^{1/t}\right)\right| \\
&\quad \leq n^{-(1/t)} \sum_{j=1}^{n} \left|E X_j I\left(\left|X_j\right| < \varepsilon n^{1/t}\right)\right| \\
&\quad \leq n^{-(1/t)} \sum_{j=1}^{n} E\left|X_j\right| I\left(\left|X_j\right| \geq \varepsilon n^{1/t}\right) \\
&\quad \leq C n^{1-(1/t)} E|X| I\left(|X| \geq \varepsilon n^{1/t}\right) \\
&\quad \leq C n^{1-\alpha r} E|X|^{\alpha r t} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{33}$$

This completes the proof of the theorem. □

Remark 15. Noting that, for typical slowly varying functions $l(x) = 1$ and $l(x) = \log x$, we can get the simpler formulas in the above theorems.

Corollary 16. *Let $1 \leq p \leq 2$ and let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $E X_n = 0$ and $E|X_n|^p < \infty$*

for $n \geq 1$. Let $\{a_{ni}, n \geq 1, i \geq 1\}$ be an array of real numbers satisfying the condition

$$\sum_{i=1}^{n} \left|a_{ni}\right|^p E\left|X_i\right|^p = O\left(n^{\delta}\right), \quad \text{as } n \longrightarrow \infty \tag{34}$$

for some $0 < \delta \leq (2/q)$ and $q > 2$. Then for any $\varepsilon > 0$ and $\alpha p \geq 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} a_{nj}X_j\right| \geq \varepsilon n^{\alpha}\right) < \infty. \tag{35}$$

Proof. Take $c_n = n^{\alpha p-2}$, $b_n = n$, and $(1/t) = \alpha$ in Theorem 12. Similar to the proof of Corollary 2.2 in Kuczmaszewska [41], we can see that conditions (15)–(17) in Theorem 12 are satisfied by (34).

Noting that $\delta < 1$ and $\alpha p \geq 1$, it follows, by $EX_n = 0$ for $n \geq 1$ and (34), that

$$\begin{aligned}
&\frac{1}{n^{\alpha}}\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} a_{nj}EX_jI\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| \\
&\quad \leq \frac{1}{n^{\alpha}}\sum_{j=1}^{n}\left|a_{nj}EX_jI\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| \\
&\quad = \frac{1}{n^{\alpha}}\sum_{j=1}^{n}\left|a_{nj}EX_jI\left(\left|a_{nj}X_j\right| \geq \varepsilon n^{\alpha}\right)\right| \\
&\quad \leq \frac{1}{n^{\alpha p}}\sum_{j=1}^{n}\left|a_{nj}\right|^p E\left|X_j\right|^p \leq Cn^{\delta-\alpha p} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{36}$$

The desired result (35) follows from the statements above and Theorem 12 immediately. The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author is most grateful to the Editor Ciprian A. Tudor and anonymous referee for careful reading of the paper and valuable suggestions which helped in improving an earlier version of this paper. This work was supported by the National Natural Science Foundation of China (11201001, 11171001), the Natural Science Foundation of Anhui Province (1308085QA03, 11040606M12, and 1208085QA03), the 211 Project of Anhui University, the Youth Science Research Fund of Anhui University, Applied Teaching Model Curriculum of Anhui University (XJYYXKC04), the Students Science Research Training Program of Anhui University (kyxl2013003 and KYXL2012007), and the Students Innovative Training Project of Anhui University (201310357004).

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