

Research Article

A Linear Functional Equation of Third Order Associated with the Fibonacci Numbers

Soon-Mo Jung¹ and Michael Th. Rassias²

¹ Mathematics Section, College of Science and Technology, Hongik University, Sejong 339-701, Republic of Korea

² Department of Mathematics, ETH-Zürich, Ramistrasse 101, 8092 Zürich, Switzerland

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

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Given a vector space X , we investigate the solutions $f : \mathbb{R} \rightarrow X$ of the linear functional equation of third order $f(x) = pf(x-1) + qf(x-2) + rf(x-3)$, which is strongly associated with a well-known identity for the Fibonacci numbers. Moreover, we prove the Hyers-Ulam stability of that equation.

1. Introduction

The problem of stability of functional equations was motivated by a question of Ulam [1] and a solution to it by Hyers [2]. Since then, numerous papers have been published on that subject and we refer to [3–6] for more details, some discussions, and further references; for examples of very recent results, see, for example, [7].

In this paper, as usual, \mathbb{C} , \mathbb{R} , \mathbb{Z} , and \mathbb{N} stand for the sets of complex numbers, real numbers, integers, and positive integers, respectively. For a nonempty subset S of a vector space, let $\xi : S \rightarrow S$ be a function. Moreover, $\xi^0(x) = x$, $\xi^{n+1}(x) = \xi(\xi^n(x))$, and (only for bijective ξ) $\xi^{-n-1}(x) = \xi^{-1}(\xi^{-n}(x))$ for $x \in S$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Jung has proved in [3] (see also [8]) some results on solutions and Hyers-Ulam stability of the functional equation

$$f(x) = pf(\xi(x)) - qf(\xi^2(x)), \quad (1)$$

in the case where $S = \mathbb{R}$ and $\xi(x) = x - 1$ for $x \in \mathbb{R}$.

If $S := \mathbb{N}_0$ and $p, q \in \mathbb{Z}$, then solutions $x : \mathbb{N}_0 \rightarrow \mathbb{Z}$ of the difference equation $f(x) = pf(x-1) - qf(x-2)$ are called the Lucas sequences (see, e.g., [9]). In some special cases they are called with specific names, for example, the Fibonacci numbers ($p = 1$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the Lucas numbers ($p = 1$, $q = -1$, $x(0) = 2$, and $x(1) = 1$), the Pell numbers ($p = 2$, $q = -1$, $x(0) = 0$, and $x(1) = 1$), the

Pell-Lucas (or companion Lucas) numbers ($p = 2$, $q = -1$, $x(0) = 2$, and $x(1) = 2$), and the Jacobsthal numbers ($p = 1$, $q = -2$, $x(0) = 0$, and $x(1) = 1$).

For some information and further references concerning the functional equations in a single variable, we refer to [10–12]. Let us mention yet that the problem of Hyers-Ulam stability of functional equations is connected to the notions of controlled chaos and shadowing (see [13]).

We remark that if $\xi : S \rightarrow S$ is bijective, then (1) can be written in the following equivalent form:

$$f(\eta^2(x)) = pf(\eta(x)) - qf(x), \quad (2)$$

where $\eta := \xi^{-1}$.

In view of the last remark, the following Hyers-Ulam stability result concerning (1) can be derived from [14, Theorem 2] (see also [15]).

Theorem 1. Let $p, q \in \mathbb{R}$ be given with $q \neq 0$ and let S be a nonempty subset of a vector space. Assume that a_1, a_2 are the complex roots of the quadratic equation $x^2 - px + q = 0$ with $|a_i| \neq 1$ for $i \in \{1, 2\}$. Moreover, assume that X is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$.

Let $\xi : S \rightarrow S$ be bijective. If a function $f : S \rightarrow X$ satisfies the inequality

$$\|f(x) - pf(\xi(x)) + qf(\xi^2(x))\| \leq \varepsilon \quad (3)$$

for all $x \in S$ and for some $\varepsilon \geq 0$, then there exists a unique solution $F : S \rightarrow X$ of (1) with

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{|(|a_1| - 1)(|a_2| - 1)|} \quad (4)$$

for all $x \in S$.

In [16, Theorem 1.4], the method presented in [3] was modified so as to prove a theorem which is a complement of Theorem 1. Note that, for bijective ξ , the following theorem improves the estimation (4) in some cases (e.g., $a_1 = 3/2$, $a_2 = -3/2$, or $a_1 = 1/2$, $a_2 = -1/2$). However, in some other situations (e.g., $a_1 = 3$, $a_2 = -3$), the estimation (4) is better than (5). The following theorem also complements Theorem 1, because ξ can be quite arbitrary in the case of (α) .

Theorem 2. Given $p, q \in \mathbb{R}$ with $q \neq 0$, assume that the distinct complex roots a_1, a_2 of the quadratic equation $x^2 - px + q = 0$ satisfy one of the following two conditions:

- (α) $|a_i| < 1$ for $i \in \{1, 2\}$;
- (β) $|a_i| \neq 1$ for $i \in \{1, 2\}$ and $\xi : S \rightarrow S$ is bijective.

Moreover, assume that X is either a real vector space if $p^2 - 4q > 0$ or a complex vector space if $p^2 - 4q < 0$. If a function $f : S \rightarrow X$ satisfies inequality (3), then there exists a solution $F : S \rightarrow X$ of (1) such that

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{|a_1 - a_2|} \left(\frac{|a_1|}{||a_1| - 1|} + \frac{|a_2|}{||a_2| - 1|} \right) \quad (5)$$

for all $x \in S$. Moreover, if the condition (β) is true, then the F is the unique solution of (1) satisfying (5).

In this paper, we investigate the solutions of the functional equation

$$f(x) = pf(x-1) + qf(x-2) + rf(x-3), \quad (6)$$

where p, q, r are real constants. Moreover, we also prove the Hyers-Ulam stability of that equation. Equation (6) is a kind of linear functional equations of third order because it is of the form

$$f(x) = a_1(x)f(\xi(x)) + a_2(x)f(\xi^2(x)) + a_3(x)f(\xi^3(x)) \quad (7)$$

for the case of $a_1(x) = p$, $a_2(x) = q$, $a_3(x) = r$, and $\xi(x) = x - 1$.

2. General Solution

In the following theorem, we apply [16, Theorem 1.1] for the investigation of general solutions of the functional equation (6).

Theorem 3. Let p, q, r be real constants such that the cubic equation

$$x^3 + px^2 - qx + r = 0 \quad (8)$$

has the following properties:

- (i) α_1 and α_2 are two distinct nonzero real roots of the cubic equation (8);
- (ii) it holds true that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$.

Let X be either a real vector space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for $i \in \{1, 2\}$ or a complex vector space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for $i \in \{1, 2\}$. Then, a function $f : \mathbb{R} \rightarrow X$ is a solution of the functional equation (6) if and only if there exist functions $h_1, h_2 : [-1, 1) \rightarrow X$ such that

$$\begin{aligned} f(x) = & \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1} h_2(x - [x]) \\ & + \frac{\alpha_1 r}{\alpha_2 (\alpha_1 - \alpha_2)} V_{[x]} h_2(x - [x] - 1) \\ & - \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1} h_1(x - [x]) \\ & - \frac{\alpha_2 r}{\alpha_1 (\alpha_1 - \alpha_2)} U_{[x]} h_1(x - [x] - 1), \end{aligned} \quad (9)$$

where $[x]$ denotes the largest integer not exceeding x , and U_n, V_n are defined in (13) and (23).

Proof. Assume that $f : \mathbb{R} \rightarrow X$ is a solution of (6). If we define an auxiliary function $g_1 : \mathbb{R} \rightarrow X$ by

$$g_1(x) := f(x) + \alpha_1 f(x-1), \quad (10)$$

then it follows from (6) that g_1 satisfies

$$g_1(x) = (\alpha_1 + p)g_1(x-1) + \frac{r}{\alpha_1}g_1(x-2) \quad (11)$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_1 : [-1, 1) \rightarrow X$ such that

$$\begin{aligned} g_1(x) = & f(x) + \alpha_1 f(x-1) \\ = & U_{[x]+1} h_1(x - [x]) + \frac{r}{\alpha_1} U_{[x]} h_1(x - [x] - 1) \end{aligned} \quad (12)$$

for all $x \in \mathbb{R}$, where

$$U_n = \frac{a^n - b^n}{a - b} \quad (n \in \mathbb{Z}) \quad (13)$$

and a, b are the distinct roots of the quadratic equation

$$x^2 - (\alpha_1 + p)x - \frac{r}{\alpha_1} = 0, \quad (14)$$

that is,

$$\begin{aligned} a = & \frac{\alpha_1 + p}{2} + \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}, \\ b = & \frac{\alpha_1 + p}{2} - \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}. \end{aligned} \quad (15)$$

Since a is a root of the quadratic equation (14), we have

$$a^2 = (\alpha_1 + p)a + \frac{r}{\alpha_1}. \quad (16)$$

We multiply both sides of (16) with a and make use of (16) and (i) to get

$$\begin{aligned} a^3 &= pa^2 + \alpha_1 a^2 + \frac{r}{\alpha_1} a \\ &= pa^2 + \alpha_1 \left((\alpha_1 + p)a + \frac{r}{\alpha_1} \right) + \frac{r}{\alpha_1} a \\ &= pa^2 + \frac{a}{\alpha_1} (\alpha_1^3 + p\alpha_1^2 + r) + r \\ &= pa^2 + qa + r. \end{aligned} \quad (17)$$

Similarly, we also obtain

$$b^3 = pb^2 + qb + r. \quad (18)$$

Using (13), (17), and (18), we have

$$\begin{aligned} &pU_{n-1} + qU_{n-2} + rU_{n-3} \\ &= \frac{(pa^2 + qa + r)a^{n-3} - (pb^2 + qb + r)b^{n-3}}{a - b} \\ &= \frac{a^n - b^n}{a - b} = U_n \end{aligned} \quad (19)$$

for all $n \in \mathbb{Z}$.

If we define an auxiliary function $g_2 : \mathbb{R} \rightarrow X$ by

$$g_2(x) := f(x) + \alpha_2 f(x-1), \quad (20)$$

then it follows from (6) that g_2 satisfies

$$g_2(x) = (\alpha_2 + p)g_2(x-1) + \frac{r}{\alpha_2}g_2(x-2) \quad (21)$$

for any $x \in \mathbb{R}$. According to [16, Theorem 1.1] or [3, Theorem 2.1], there exists a function $h_2 : [-1, 1) \rightarrow X$ such that

$$\begin{aligned} g_2(x) &= f(x) + \alpha_2 f(x-1) \\ &= V_{[x]+1}h_2(x - [x]) + \frac{r}{\alpha_2}V_{[x]}h_2(x - [x] - 1) \end{aligned} \quad (22)$$

for all $x \in \mathbb{R}$, where

$$V_n = \frac{c^n - d^n}{c - d} \quad (n \in \mathbb{Z}) \quad (23)$$

and c, d are the distinct roots of the quadratic equation

$$x^2 - (\alpha_2 + p)x - \frac{r}{\alpha_2} = 0, \quad (24)$$

that is,

$$\begin{aligned} c &= \frac{\alpha_2 + p}{2} + \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}, \\ d &= \frac{\alpha_2 + p}{2} - \sqrt{\left(\frac{\alpha_2 + p}{2}\right)^2 + \frac{r}{\alpha_2}}. \end{aligned} \quad (25)$$

As in the first part, we verify that

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3} \quad (26)$$

for all $n \in \mathbb{Z}$.

We now multiply (12) with α_2 and (22) with α_1 , we subtract the former from the latter, and we then divide the resulting equation by $(\alpha_1 - \alpha_2)$ to get (9).

We assume that a function $f : \mathbb{R} \rightarrow X$ is given by (9), where $h_1, h_2 : [-1, 1) \rightarrow X$ are arbitrarily given functions and U_n, V_n are given by (13) and (23), respectively. Then, by (9), (19), and (26), we have

$$\begin{aligned} &pf(x-1) + qf(x-2) + rf(x-3) \\ &= \frac{\alpha_1}{\alpha_1 - \alpha_2} (pV_{[x]} + qV_{[x]-1} + rV_{[x]-2})h_2(x - [x]) \\ &\quad + \frac{\alpha_1 r}{\alpha_2(\alpha_1 - \alpha_2)} (pV_{[x]-1} + qV_{[x]-2} + rV_{[x]-3}) \\ &\quad \times h_2(x - [x] - 1) \\ &\quad - \frac{\alpha_2}{\alpha_1 - \alpha_2} (pU_{[x]} + qU_{[x]-1} + rU_{[x]-2})h_1(x - [x]) \\ &\quad - \frac{\alpha_2 r}{\alpha_1(\alpha_1 - \alpha_2)} (pU_{[x]-1} + qU_{[x]-2} + rU_{[x]-3}) \\ &\quad \times h_1(x - [x] - 1) \\ &= \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1}h_2(x - [x]) \\ &\quad + \frac{\alpha_1 r}{\alpha_2(\alpha_1 - \alpha_2)} V_{[x]}h_2(x - [x] - 1) \\ &\quad - \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1}h_1(x - [x]) \\ &\quad - \frac{\alpha_2 r}{\alpha_1(\alpha_1 - \alpha_2)} U_{[x]}h_1(x - [x] - 1) = f(x) \end{aligned} \quad (27)$$

for all $x \in \mathbb{R}$, which implies that f is a solution of (6). \square

According to [17, p. 92], the Fibonacci numbers F_n satisfy the identity

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2 \quad (28)$$

for all integers $n > 3$. We can easily notice that the linear equation of third order

$$f(x) = 2f(x-1) + 2f(x-2) - f(x-3) \quad (29)$$

is strongly related to identity (28).

Corollary 4. Let X be a real vector space. A function $f : \mathbb{R} \rightarrow X$ is a solution of the functional equation (29) if and only if there exist functions $h_1, h_2 : [-1, 1) \rightarrow X$ such that

$$\begin{aligned} f(x) = & \frac{5+3\sqrt{5}}{10} U_{[x]+1} h_1(x-[x]) \\ & + \frac{15+7\sqrt{5}}{10} U_{[x]} h_1(x-[x]-1) \\ & + \frac{5-3\sqrt{5}}{10} V_{[x]+1} h_2(x-[x]) \\ & + \frac{15-7\sqrt{5}}{10} V_{[x]} h_2(x-[x]-1), \end{aligned} \quad (30)$$

where U_n and V_n are defined in (33).

Proof. If we set $p = 2$, $q = 2$, and $r = -1$ in (8), then the cubic equation

$$x^3 + 2x^2 - 2x - 1 = 0 \quad (31)$$

has three distinct nonzero roots including

$$\alpha_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}, \quad \alpha_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}. \quad (32)$$

Moreover, it holds that $(\alpha_1 + p)^2 + 4r/\alpha_1 > 0$ and $(\alpha_2 + p)^2 + 4r/\alpha_2 > 0$. By (13), (15), (23), and (25), we have

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = \frac{c^n - d^n}{c - d}, \quad (33)$$

where we make use of (15) and (25) to calculate

$$a = \frac{3+\sqrt{5}}{2}, \quad b = -1, \quad c = \frac{3-\sqrt{5}}{2}, \quad d = -1. \quad (34)$$

Finally, in view of Theorem 3, we conclude that the assertion of our corollary is true. \square

Corollary 5. If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution of functional equation (29), then there exist real constants μ_1, μ_2, ν_1 , and ν_2 such that

$$\begin{aligned} f(n) = & \frac{5+3\sqrt{5}}{10} \mu_1 U_{n+1} + \frac{15+7\sqrt{5}}{10} \mu_2 U_n \\ & + \frac{5-3\sqrt{5}}{10} \nu_1 V_{n+1} + \frac{15-7\sqrt{5}}{10} \nu_2 V_n \end{aligned} \quad (35)$$

for all $n \in \mathbb{Z}$, where U_n and V_n are defined in (33).

3. Hyers-Ulam Stability

We apply the classical direct method to the proof of the following theorem. The classical direct method was first proposed by Hyers [2].

Theorem 6. Let p, q, r be real constants with $r \neq 0$, let α be a nonzero root of the cubic equation (8), and let a, b be the roots of

the quadratic equation $x^2 - (\alpha + p)x - r/\alpha = 0$ with $|a| > 1$ and $0 < |b| < 1$. Let X be either a real Banach space if $(\alpha + p)^2 + 4r/\alpha > 0$ or a complex Banach space if $(\alpha + p)^2 + 4r/\alpha < 0$. If a function $f : \mathbb{R} \rightarrow X$ satisfies the inequality

$$\|f(x) - pf(x-1) - qf(x-2) - rf(x-3)\| \leq \varepsilon \quad (36)$$

for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a solution $G : \mathbb{R} \rightarrow X$ of (6) such that

$$\|f(x) + \alpha f(x-1) - G(x)\| \leq \frac{|a| - |b|}{|a - b|} \frac{\varepsilon}{(|a| - 1)(1 - |b|)} \quad (37)$$

for all $x \in \mathbb{R}$.

Proof. If we define an auxiliary function $g : \mathbb{R} \rightarrow X$ by

$$g(x) := f(x) + \alpha f(x-1), \quad (38)$$

then, as we did in (11), it follows from (36) that g satisfies the inequality

$$\|g(x) - (\alpha + p)g(x-1) - \frac{r}{\alpha}g(x-2)\| \leq \varepsilon \quad (39)$$

or

$$\|g(x) - ag(x-1) - b[g(x-1) - ag(x-2)]\| \leq \varepsilon \quad (40)$$

for any $x \in \mathbb{R}$.

If we replace x with $x - k$ in the last inequality, then we have

$$\begin{aligned} & \|g(x-k) - ag(x-k-1) \\ & - b[g(x-k-1) - ag(x-k-2)]\| \leq \varepsilon \end{aligned} \quad (41)$$

for all $x \in \mathbb{R}$. Furthermore, we get

$$\begin{aligned} & \|b^k [g(x-k) - ag(x-k-1)] \\ & - b^{k+1} [g(x-k-1) - ag(x-k-2)]\| \leq |b|^k \varepsilon \end{aligned} \quad (42)$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (42), we obviously have

$$\begin{aligned} & \|g(x) - ag(x-1) - b^n [g(x-n) - ag(x-n-1)]\| \\ & \leq \sum_{k=0}^{n-1} \|b^k [g(x-k) - ag(x-k-1)] \\ & - b^{k+1} [g(x-k-1) - ag(x-k-2)]\| \\ & \leq \sum_{k=0}^{n-1} |b|^k \varepsilon \end{aligned} \quad (43)$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (42) implies that the sequence $\{b^n [g(x-n) - ag(x-n-1)]\}$ is a Cauchy sequence (note that $0 < |b| < 1$). Therefore, we can define a function $G_1 : \mathbb{R} \rightarrow X$ by

$$G_1(x) := \lim_{n \rightarrow \infty} b^n [g(x-n) - ag(x-n-1)], \quad (44)$$

since X is complete. In view of the definition of G_1 and using the relations, $a + b = \alpha + p$ and $ab = -r/\alpha$, we obtain

$$\begin{aligned} & (\alpha + p)G_1(x-1) + \frac{r}{\alpha}G_1(x-2) \\ &= (a+b)G_1(x-1) - abG_1(x-2) \\ &= \frac{a+b}{b} \lim_{n \rightarrow \infty} b^{n+1} [g(x-(n+1)) - ag(x-(n+1)-1)] \\ &\quad - \frac{ab}{b^2} \lim_{n \rightarrow \infty} b^{n+2} [g(x-(n+2)) - ag(x-(n+2)-1)] \\ &= \frac{a+b}{b}G_1(x) - \frac{a}{b}G_1(x) = G_1(x) \end{aligned} \quad (45)$$

for all $x \in \mathbb{R}$. Since α is a nonzero root of the cubic equation (8), it follows from (45) that

$$\begin{aligned} & G_1(x) - pG_1(x-1) - qG_1(x-2) - rG_1(x-3) \\ &= (\alpha + p)G_1(x-1) + \frac{r}{\alpha}G_1(x-2) - pG_1(x-1) \\ &\quad - qG_1(x-2) - rG_1(x-3) \\ &= \alpha G_1(x-1) + \left(-q + \frac{r}{\alpha}\right)G_1(x-2) - rG_1(x-3) \\ &= \alpha G_1(x-1) + (-\alpha^2 - p\alpha)G_1(x-2) - rG_1(x-3) \\ &= \alpha \left((\alpha + p)G_1(x-2) + \frac{r}{\alpha}G_1(x-3) \right) \\ &\quad - \alpha(\alpha + p)G_1(x-2) - rG_1(x-3) = 0 \end{aligned} \quad (46)$$

for all $x \in \mathbb{R}$. Hence, we conclude that G_1 is a solution of (6). If n tends to infinity, then (43) yields that

$$\|g(x) - ag(x-1) - G_1(x)\| \leq \frac{\varepsilon}{1-|b|} \quad (47)$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (36) that

$$\|g(x) - bg(x-1) - a[g(x-1) - bg(x-2)]\| \leq \varepsilon \quad (48)$$

for all $x \in \mathbb{R}$. Analogously to (42), replacing x by $x+k$ in the last inequality and then dividing by $|a|^k$ both sides of the resulting inequality, then we have

$$\begin{aligned} & \|a^{-k}[g(x+k) - bg(x+k-1)] \\ & - a^{-k+1}[g(x+k-1) - bg(x+k-2)]\| \leq |a|^{-k}\varepsilon \end{aligned} \quad (49)$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By using (49), we further obtain

$$\begin{aligned} & \|a^{-n}[g(x+n) - bg(x+n-1)] - [g(x) - bg(x-1)]\| \\ & \leq \sum_{k=1}^n \|a^{-k}[g(x+k) - bg(x+k-1)] \\ & \quad - a^{-k+1}[g(x+k-1) - bg(x+k-2)]\| \\ & \leq \sum_{k=1}^n |a|^{-k}\varepsilon \end{aligned} \quad (50)$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

On account of (49), we see that the sequence $\{a^{-n}[g(x+n) - bg(x+n-1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbb{R}$ (note that $|a| > 1$). Hence, we can define a function $G_2 : \mathbb{R} \rightarrow X$ by

$$G_2(x) := \lim_{n \rightarrow \infty} a^{-n}[g(x+n) - bg(x+n-1)]. \quad (51)$$

Due to the definition of G_2 and the relations, $a + b = \alpha + p$ and $ab = -r/\alpha$, we get

$$\begin{aligned} & (\alpha + p)G_2(x-1) + \frac{r}{\alpha}G_2(x-2) \\ &= (a+b)G_2(x-1) - abG_2(x-2) \\ &= \frac{a+b}{a} \lim_{n \rightarrow \infty} a^{-(n-1)} [g(x+n-1) - bg(x+n-2)] \\ &\quad - \frac{ab}{a^2} \lim_{n \rightarrow \infty} a^{-(n-2)} [g(x+n-2) - bg(x+n-3)] \\ &= \frac{a+b}{a}G_2(x) - \frac{b}{a}G_2(x) = G_2(x) \end{aligned} \quad (52)$$

for any $x \in \mathbb{R}$. Similarly as in the first part, we can show that G_2 is a solution of (6).

If we let n tend to infinity, then it follows from (50) that

$$\|G_2(x) - g(x) + bg(x-1)\| \leq \frac{\varepsilon}{|a|-1} \quad (53)$$

for $x \in \mathbb{R}$.

It follows from (47) and (53) that

$$\begin{aligned} & \left\| g(x-1) - \frac{1}{a-b}G_2(x) + \frac{1}{a-b}G_1(x) \right\| \\ & \leq \left\| \frac{1}{a-b}G_1(x) - \frac{1}{a-b}g(x) + \frac{a}{a-b}g(x-1) \right\| \\ & \quad + \left\| \frac{1}{a-b}g(x) - \frac{b}{a-b}g(x-1) - \frac{1}{a-b}G_2(x) \right\| \\ & \leq \frac{|a|-|b|}{|a-b|} \frac{\varepsilon}{(|a|-1)(1-|b|)} \end{aligned} \quad (54)$$

for any $x \in \mathbb{R}$.

Finally, if we define a function $G : \mathbb{R} \rightarrow X$ by

$$G(x) := \frac{1}{a-b}G_2(x+1) - \frac{1}{a-b}G_1(x+1) \quad (55)$$

for all $x \in \mathbb{R}$, then G is also a solution of (6). Moreover, the validity of (37) follows from the last inequality. \square

The following theorem is the main theorem of this paper.

Theorem 7. *Given real constants p, q, r with $r \neq 0$, let α_1 and α_2 be distinct nonzero roots of cubic equation (8) and let a_i, b_i be the roots of the quadratic equation $x^2 - (\alpha_i + p)x - r/\alpha_i = 0$ with $|a_i| > 1$ and $0 < |b_i| < 1$ for $i \in \{1, 2\}$. Assume that either $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ for all $i \in \{1, 2\}$ or $(\alpha_i + p)^2 + 4r/\alpha_i < 0$ for all $i \in \{1, 2\}$. Let X be either a real Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i > 0$ or a complex Banach space if $(\alpha_i + p)^2 + 4r/\alpha_i < 0$. If a function $f: \mathbb{R} \rightarrow X$ satisfies inequality (36) for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a solution $F: \mathbb{R} \rightarrow X$ of (6) such that*

$$\begin{aligned} \|f(x) - F(x)\| \leq & \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} \\ & + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \end{aligned} \quad (56)$$

for all $x \in \mathbb{R}$.

Proof. According to Theorem 6, there exists a solution $F_i: \mathbb{R} \rightarrow X$ of (6) such that

$$\|f(x) + \alpha_i f(x-1) - F_i(x)\| \leq \frac{|a_i| - |b_i|}{|a_i - b_i|} \frac{\varepsilon}{(|a_i| - 1)(1 - |b_i|)} \quad (57)$$

for any $x \in \mathbb{R}$ and $i \in \{1, 2\}$. In view of the last inequalities, we have

$$\begin{aligned} & \left\| f(x) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) + \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \right\| \\ & \leq \left\| \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} f(x) - \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) \right\| \\ & \quad + \left\| \frac{\alpha_1}{\alpha_1 - \alpha_2} f(x) + \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x-1) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) \right\| \\ & \leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_2|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_1| - 1)(1 - |b_1|)} \\ & \quad + \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} \frac{\varepsilon}{(|a_2| - 1)(1 - |b_2|)} \end{aligned} \quad (58)$$

for all $x \in \mathbb{R}$.

If we define a function $F: \mathbb{R} \rightarrow X$ by

$$F(x) := \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \quad (59)$$

for each $x \in \mathbb{R}$, then F is also a solution of (6), and inequality (56) follows from the last inequality. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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