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Research Article

Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative

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We propose the local fractional function decomposition method, which is derived from the coupling method of local fractional Fourier series and Yang-Laplace transform. The forms of solutions for local fractional differential equations are established. Some examples for inhomogeneous wave equations are given to show the accuracy and efficiency of the presented technique.

1. Introduction

Fractional differential equations with arbitrary orders [1] have attracted more and more attention to their extensive applications in various areas, such as physics, applied mathematics, and biology [2–8]. As a result, great deal of methods for solving the fractional differential equations are developed [9–21], such as the heat balance integral method [9, 10], the homotopy analysis method [11], the variational iteration method [12], the homotopy decomposition method [13, 14], and the Adomian decomposition method [15, 16].

The fractional differential equations were considered in sense of the Caputo derivative, the Riemann-Liouville derivative, and the Grünwald-Letnikov derivative [17]. However, they do not deal with the nondifferentiable functions defined on Cantor sets. Local fractional derivative [18, 19] is the best method for describing the nondifferential problems defined on Cantor sets. For example, the heat equations arising in fractal transient conduction were investigated in [19–22]. The Helmholtz and diffusion equations on the Cantor sets within local fractional derivative were discussed [23]. The Navier-Stokes equations on Cantor sets were suggested in [24]. There are some methods for solving the local fractional differential equations, such as the local fractional variational iteration method [20], the Yang-Fourier transform [21], the Yang-Laplace transform [22], the local fractional Fourier series method [25], and the local fractional Adomian decomposition method [26].

In this paper, our aims are to present the coupling method of local fractional series method and Yang-Laplace transform, which is called as the local fractional function decomposition method, and to use it to solve the differential equations with local fractional derivative. The organization of the manuscript is as follows. In Section 2, the basic mathematical tools are introduced. In Section 3, the local fractional function decomposition method for solving the differential equations with local fractional derivative is investigated. In Section 4, several examples are considered. Finally, in Section 5 the conclusions are given.

2. Mathematical Fundamentals

In this section, we introduce the basic notions of local fractional continuity, local fractional derivative, local fractional Fourier series, and special function in fractal space [18, 19], which are used in the paper.

Definition 1. Suppose that there is [19]

$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{1}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$; then $f(x)$ is called local fractional continuous at $x = x_0$ and it is denoted by $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 2. Suppose that the function $f(x)$ satisfies condition (1), for $x \in (a, b)$; it is so-called local fractional continuous on the interval (a, b), denoted by

$$f(x) \in C_\alpha(a, b). \tag{2}$$

Definition 3. In fractal space, let $f(x) \in C_\alpha(a, b)$, local fractional derivative of $f(x)$ of order α at $x = x_0$ is given by [19]

$$\begin{aligned} D_x^{(\alpha)} f(x_0) &= f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} \\ &= \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \end{aligned} \tag{3}$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\Delta(f(x) - f(x_0))$.

Local fractional derivative of high order and local fractional partial derivative are defined, respectively, in the following forms [18, 19]:

$$\begin{aligned} f^{(k\alpha)}(x) &= \overbrace{D_x^{(\alpha)} \cdots D_x^{(\alpha)}}^{k \text{ times}} f(x), \\ \frac{\partial^{k\alpha}}{\partial x^{k\alpha}} f(x, y) &= \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x, y). \end{aligned} \tag{4}$$

Definition 4. In fractal space, the Mittage Leffler function, sine function, cosine function, hyperbolic sine function, and hyperbolic cosine function are, respectively, defined by [18, 19]

$$\begin{aligned} E_\alpha(x^\alpha) &:= \sum_{K=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1 + k\alpha)}, \quad x \in R,\ 0 < \alpha \le 1, \\ \sin_\alpha x^\alpha &:= \sum_{K=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1 + (2k+1)\alpha]}, \quad x \in R,\ 0 < \alpha \le 1, \\ \cos_\alpha x^\alpha &:= \sum_{K=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1 + 2k\alpha)}, \quad x \in R,\ 0 < \alpha \le 1, \\ \sin h_\alpha x^\alpha &:= \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2}, \quad x \in R,\ 0 < \alpha \le 1, \\ \cos h_\alpha x^\alpha &:= \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2}, \quad x \in R,\ 0 < \alpha \le 1. \end{aligned} \tag{5}$$

Definition 5. Let $f(x)$ be $2l$-periodic. For $k \in Z$ and $f(x) \in C_\alpha(-\infty, +\infty)$, the local fraction Fourier series of $f(x)$ is defined as (see [18, 25])

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} + b_k \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} \right), \tag{6}$$

where

$$\begin{aligned} a_k &= \frac{1}{l^\alpha} \int_{-l}^{+l} f(x) \cos_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha, \\ b_k &= \frac{1}{l^\alpha} \int_{-l}^{+l} f(x) \sin_\alpha \frac{\pi^\alpha (kx)^\alpha}{l^\alpha} (dx)^\alpha \end{aligned} \tag{7}$$

are the local fraction Fourier coefficients.

Definition 6. Let $(1/\Gamma(1 + \alpha)) \int_0^\infty |f(x)|(dx)^\alpha < k < \infty$. The Yang-Laplace transforms of $f(x)$ is given by [18, 22]

$$\begin{aligned} L_\alpha\{f(x)\} &= f_s^{L,\alpha}(s) \\ &:= \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha, \\ &\qquad 0 < \alpha \le 1, \end{aligned} \tag{8}$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 7. The inverse formula of the Yang-Laplace transforms of $f(x)$ is given by [18, 22]:

$$\begin{aligned} L_\alpha^{-1}\left\{f_s^{L,\alpha}(s)\right\} &= f(t) \\ &:= \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha(s^\alpha t^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha, \\ &\qquad 0 < \alpha \le 1, \end{aligned} \tag{9}$$

where $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$; fractal imaginary unit i^α and $\mathrm{Re}(s) = \beta > 0$.

3. Local Fractional Function Decomposition Method

In this section we will present the local fractional function decomposition method.

At first, we present the local fractional differential equation

$$\begin{aligned} \frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + k_1 \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + k_2 \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} \\ + k_3 \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = f(x,t) \end{aligned} \tag{10}$$

with constants k_1, k_2, k_3, $0 < \alpha \le 1/2$ and with boundary and initial conditions

$$\begin{aligned} u(0,t) &= u(l,t) = 0, \\ u(x,0) &= \varphi(x), \\ \frac{\partial^\alpha u(x,o)}{\partial t^\alpha} &= \psi(x). \end{aligned} \tag{11}$$

Now we discuss the solution of (10).

According to the decomposition of the local fractional function, with respect to the system $\{\sin_\alpha n^\alpha (\pi x/l)^\alpha\}$, the following functions coefficients can be given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} v_n(t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \\ f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \\ \varphi(x) &= \sum_{n=1}^{\infty} C_n \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \\ \psi(x) &= \sum_{n=1}^{\infty} D_n \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \end{aligned} \tag{12}$$

where

$$\begin{aligned} f_n(t) &= \frac{2}{l^\alpha} \int_0^l f(x,t) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha (dx)^\alpha, \\ C_n &= \frac{2}{l^\alpha} \int_0^l \varphi(x) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha (dx)^\alpha, \\ D_n &= \frac{2}{l^\alpha} \int_0^l \varphi(x) \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha (dx)^\alpha. \end{aligned} \tag{13}$$

Substituting (12) into (10) implies that

$$\begin{aligned} &\frac{\partial^\alpha v_n(t)}{\partial t^\alpha} + k_1 \frac{\partial^\alpha v_n(t)}{\partial t^\alpha} + k_2 \left(\frac{n\pi}{l}\right)^{2\alpha} v_n(t) \\ &\quad + k_3 \left(\frac{n\pi}{l}\right)^{\alpha} v_n(t) = f_n(t), \\ &v_n(0) = C_n, \qquad v_n'(0) = D_n. \end{aligned} \tag{14}$$

Suppose that the Yang-Laplace transforms of functions $v_n(t)$ and $f_n(t)$ are $V_n(s)$ and $F_n(s)$, respectively. Then we obtain

$$\begin{aligned} &s^{2\alpha} V_n(s) - C_n s^\alpha - D_n + k_1 \left(s^\alpha V_n(s) - D_n\right) \\ &\quad + k_2 \left(\frac{n\pi}{l}\right)^{2\alpha} V_n(s) + k_3 \left(\frac{n\pi}{l}\right)^{\alpha} V_n(s) = F_n(s). \end{aligned} \tag{15}$$

That is

$$\begin{aligned} V_n(s) &= \frac{D_n + k_1 D_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} \\ &\quad + \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha}. \end{aligned} \tag{16}$$

Hence, we have

$$\begin{aligned} v_n(t) &= L_\alpha^{-1}\left[V_n(s)\right] \\ &= \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha\left(s^\alpha t^\alpha\right) V_n(s) (ds)^\alpha \\ &= \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha\left(s^\alpha t^\alpha\right) \\ &\qquad \times \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha \\ &\quad + \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha\left(s^\alpha t^\alpha\right) \\ &\qquad \times \frac{D_n + k_1 D_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha. \end{aligned} \tag{17}$$

Let

$$v_n(t) = v_{1,n}(t) + v_{2,n}(t),$$

$$\begin{aligned} v_{1,n}(t) &= \frac{1}{(2\pi)^\alpha} \\ &\quad \times \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha\left(s^\alpha t^\alpha\right) \\ &\qquad \times \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha, \\ v_{2,n}(t) &= \frac{1}{(2\pi)^\alpha} \\ &\quad \times \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha\left(s^\alpha t^\alpha\right) \\ &\qquad \times \frac{D_n + k_1 D_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha} (ds)^\alpha. \end{aligned} \tag{18}$$

Hence, we get

$$\begin{aligned} V_{1,n}(s) &= \frac{F_n(s)}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha}, \\ V_{2,n}(s) &= \frac{D_n + k_1 D_n + C_n s^\alpha}{s^{2\alpha} + k_1 s^\alpha + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^\alpha}. \end{aligned} \tag{19}$$

Then, making use of (8) and (9) and rearranging integration sequence, we have the following several formulas about $v_{1,n}(t)$ and $v_{2,n}(t)$.

(I) Suppose that

$$-\frac{1}{4}k_1^2 + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} > 0,$$
$$s^{2\alpha} + k_1 s^{\alpha} + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} = \left(s^{\beta} + \frac{k_1}{2}\right)^2 + D_n'^2, \tag{20}$$

where $D_n' = \sqrt{-(1/4)k_1^2 + k_2(n\pi/l)^{2\alpha} + k_3(n\pi/l)^{\alpha}}$.

Then, we get

$$v_{1,n}(t) = \frac{1}{D_n'\Gamma(1+\alpha)} \times \int_0^t E_\alpha\left(\frac{-k_1\tau^\alpha}{2^\alpha}\right)\sin_\alpha D_n'\tau f_n(t-\tau)(d\tau)^\alpha. \tag{21}$$

When

$$V_{2,n}(s) = \frac{C_n(s^\alpha + (k_1/2)) + (D_n + k_1 D_n - (k_1/2)C_n)}{(s^\alpha + (k_1/2))^2 + D_n'^2}, \tag{22}$$

we get

$$v_{2,n}(t) = C_n E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right)\cos_\alpha\sqrt{D'}t^\alpha + \left(D_n + k_1 D_n - \frac{k_1}{2}C_n\right)E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right)\sin_\alpha\sqrt{D'}t^\alpha. \tag{23}$$

(II) If

$$-\frac{1}{4}k_1^2 + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} = 0,$$
$$s^{2\alpha} + k_1 s^{\alpha} + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} = \left(s^{\alpha} + \frac{k_1}{2}\right)^2, \tag{24}$$

then we have

$$v_{1,n}(t) = \frac{1}{\Gamma(1+\beta)}\int_0^t E_\alpha\left(\frac{-k_1\tau^\alpha}{2^\alpha}\right)\tau^\alpha f_n(t-\tau)(d\tau)^\alpha. \tag{25}$$

When

$$V_{2,n}(s) = \frac{C_n(s^\alpha + (k_1/2)) + (D_n + k_1 D_n - (k_1/2)C_n)}{(s^\alpha + (k_1/2))^2}, \tag{26}$$

we arrive at

$$v_{2,n}(t) = \frac{C_n}{\Gamma(1+\alpha)}E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right) + \frac{D_n + k_1 D_n - (k_1/2)C_n}{\Gamma(1+\alpha)}t^\alpha E_\alpha\left(\frac{-k_1 t^\alpha}{2^\alpha}\right). \tag{27}$$

(III) Let

$$-\frac{1}{4}k_1^2 + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} < 0, \tag{28}$$
$$s^{2\alpha} + k_1 s^{\alpha} + k_2\left(\frac{n\pi}{l}\right)^{2\alpha} + k_3\left(\frac{n\pi}{l}\right)^{\alpha} = \left(s^{\alpha} + \frac{k_1}{2}\right)^2 - D_n'^2, \tag{29}$$

where $D_n' = \sqrt{(1/4)k_1^2 - k_2(n\pi/l)^{2\alpha} - k_3(n\pi/l)^{\alpha}}$.

Then, we have

$$v_{1,n}(t) = \frac{1}{\Gamma(1+\alpha)D_n'} \times \int_0^t E_\alpha\left(\frac{-k_1\tau^\alpha}{2^\alpha}\right)sh_\alpha D_n'\tau^\alpha f_n(t-\tau)(d\tau)^\beta. \tag{30}$$

When

$$\begin{aligned} V_{2,n}(s) &= \frac{C_n(s^\alpha + (k_1/2)) + (D_n + k_1 D_n - (k_1/2)C_n)}{(s^\alpha + (k_1/2))^2 - D_n'^2} \\ &= \frac{(k_1+1)D_n + (D_n' - (k_1/2))C_n}{2D_n'(s^\alpha + (k_1/2) - D_n')} \\ &\quad + \frac{(D_n' + (k_1/2))C_n - (k_1+1)D_n}{2D_n'(s^\alpha + (k_1/2) + D_n')}, \end{aligned} \tag{31}$$

we obtain

$$\begin{aligned} v_{2,n}(t) &= \frac{(k_1+1)D_n + (D_n' - (k_1/2))C_n}{2D_n'} \times E_\alpha\left(\left(D_n' - \frac{k_1}{2^\alpha}\right)t^\alpha\right) \\ &\quad + \frac{(D_n' + (k_1/2))C_n - (k_1+1)D_n}{2D_n'} \times E_\alpha\left(\left(-D_n' - \frac{k_1}{2^\alpha}\right)t^\alpha\right). \end{aligned} \tag{32}$$

The above results are the desired solutions.

4. Illustrative Examples

In order to illustrate the above results in Section 3, we give the following several examples.

Example 1. The local fractional Laplace differential equation is written in the following form [18, 19]:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} = 0 \tag{33}$$

subjected to the boundary and initial conditions described by

$$u(x,0)=\sin_\alpha(x^\alpha),$$
$$\frac{\partial^\alpha u(x,o)}{\partial t^\alpha}=\sin_\alpha(x^\alpha), \tag{34}$$
$$u(0,t)=u(\pi,t)=0.$$

From (33), the final solution can be easily deduced as follows:

$$u(x,t)=\sin_\alpha(x^\alpha)E_\alpha(t^\alpha). \tag{35}$$

Example 2. We consider the following inhomogeneous wave equation with local fractional derivative:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-\sin_\alpha x^\alpha=0 \tag{36}$$

subjected to the boundary and initial conditions

$$u(x,0)=\sin_\alpha(x^\alpha),$$
$$\frac{\partial^\alpha u(x,o)}{\partial t^\alpha}=0, \tag{37}$$
$$u(0,t)=u(\pi,t)=0.$$

In order to find its solution, we suppose that

$$u(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin_\alpha n^\alpha x^\alpha,$$
$$f(x,t)=\sin_\alpha x^\alpha=\sum_{n=1}^{\infty}f_n(t)\sin_\alpha n^\alpha x^\alpha,$$
$$\varphi(x)=\sin_\alpha x^\alpha=\sum_{n=1}^{\infty}C_n\sin_\alpha n^\alpha x^\alpha, \tag{38}$$
$$\psi(x)=0=\sum_{n=1}^{\infty}D_n\sin_\alpha n^\alpha x^\alpha,$$

which leads to

$$f_n(t)=0,\quad (n\neq 1),$$
$$f_1(t)=1,$$
$$C_n=0,\quad (n\neq 1), \tag{39}$$
$$C_1=1,$$
$$D_n=0.$$

Contrasting (37) with (35), we directly get

$$k_1=k_3=0,$$
$$k_2=-1,$$
$$D_n'=\sqrt{\frac{1}{4}k_1^2-k_2\left(\frac{n\pi}{l}\right)^{2\alpha}-k_3\left(\frac{n\pi}{l}\right)^{\alpha}}=1,\quad (n=1), \tag{40}$$
$$D_n'=0,\quad (n\neq 1).$$

According to (30) and (32), we can derive

$$v_n(t)=0,\quad (n\neq 1), \tag{41}$$

$$\begin{aligned}v_{1,1}(t)&=\frac{1}{\Gamma(1+\alpha)D_1'}\int_0^t E_\alpha\left(\frac{-k_1\tau^\alpha}{2^\alpha}\right)sh_\alpha D_1'\tau^\alpha f_1(t-\tau)(d\tau)^\alpha\\&=\frac{1}{\Gamma(1+\alpha)}\int_0^t sh_\alpha\tau^\alpha(d\tau)^\alpha\\&=\frac{E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)}{2}-1,\end{aligned} \tag{42}$$

$$v_{1,2}(t)=\frac{E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)}{2}. \tag{43}$$

Conclusively, we get

$$v_1(t)=E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)-1,\qquad v_n(t)=0,\quad (n\neq 1). \tag{44}$$

Thus, we obtain

$$u(x,t)=\left[E_\alpha(t^\alpha)+E_\alpha(-t^\alpha)-1\right]\sin_\alpha(x^\alpha). \tag{45}$$

Example 3. The inhomogeneous wave equation with local fractional differential operator is written in the following form:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}=1. \tag{46}$$

The boundary and initial conditions are described by

$$u(x,0)=\sin_\alpha x^\alpha,$$
$$\frac{\partial^\alpha u(x,o)}{\partial t^\alpha}=0. \tag{47}$$

In order to find the solution of (46), we set

$$u(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin_\alpha n^\alpha x^\alpha,$$
$$f(x,t)=1=\sum_{n=1}^{\infty}f_n(t)\sin_\alpha n^\alpha x^\alpha,$$
$$\varphi(x)=\sin_\alpha x^\alpha=\sum_{n=1}^{\infty}C_n\sin_\alpha n^\alpha x^\alpha, \tag{48}$$
$$\psi(x)=0=\sum_{n=1}^{\infty}D_n\sin_\alpha n^\alpha x^\alpha.$$

Hence, we get

$$f_n(t)=\frac{2\left[1-(-1)^n\right]}{n},$$
$$C_n=0\quad (n\neq 1), \tag{49}$$
$$C_1=1,$$
$$D_n=0.$$

Let

$$s^{2\alpha} + n^{2\alpha} = \left(s^{\alpha}\right)^2 + D_n'^2,$$
$$D_n' = n^{\alpha}. \tag{50}$$

Making use of (21) and (23),we can write

$$\begin{aligned} v_{1,n}(t) &= \frac{1}{D_n'\Gamma(1+\alpha)} \frac{2\left[1-(-1)^n\right]}{n^{\alpha}} \int_0^t \sin_{\alpha}(n\tau)^{\alpha}(d\tau)^{\alpha} \\ &= \left(1-\cos_{\alpha}(n\tau)^{\alpha}\right) \frac{2\left[1-(-1)^n\right]}{n^{\alpha}}. \end{aligned} \tag{51}$$

When

$$V_{2,n}(t) = \frac{D_n + k_1 D_n + C_n s^{\alpha}}{s^{2\alpha} + k_1 s^{\alpha} + k_2 (n\pi/l)^{2\alpha} + k_3 (n\pi/l)^{\alpha}} = \frac{s^{\alpha}}{s^{2\alpha} + n^{2\alpha}}, \tag{52}$$

we obtain

$$v_{2,1}(t) = \cos_{\alpha} t^{\alpha},$$
$$v_{2,n}(t) = 0, \quad (n \neq 1). \tag{53}$$

Conclusively, we arrive at

$$v_1(t) = v_{1,1}(t) + v_{2,1}(t) = 4 - 3\cos_{\alpha} t^{\alpha},$$
$$v_n(t) = \left(1-\cos_{\alpha}(nt)^{\alpha}\right) \frac{2\left[1-(-1)^n\right]}{n^{\alpha}}, \quad (n \neq 1). \tag{54}$$

Hence, we obtain the solution of (46) in the following form:

$$\begin{aligned} u(x,t) = {} & \left(4 - 3\cos_{\alpha} t^{\alpha}\right) \sin_{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha} \\ & + \sum_{n=2}^{\infty} \left(1-\cos_{\alpha}(nt)^{\alpha}\right) \frac{2\left[1-(-1)^n\right]}{n^{\alpha}} \sin_{\alpha} n^{\alpha} \left(\frac{\pi x}{l}\right)^{\alpha}. \end{aligned} \tag{55}$$

Example 4. The inhomogeneous wave equation with local fractional differential operator is written in the form

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} - \cos_{\alpha} t^{\alpha} \cos_{\alpha} x^{\alpha} = 0. \tag{56}$$

The boundary and initial conditions are presented as follows:

$$u(x,0) = \cos_{\alpha}\left(x^{\alpha}\right),$$
$$\frac{\partial^{\alpha} u(x,o)}{\partial t^{\alpha}} = 0, \tag{57}$$
$$u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = 0.$$

Let

$$u(x,t) = \sum_{n=1}^{\infty} v_n(t) \cos_{\alpha} n^{\alpha} x^{\alpha},$$
$$f(x,t) = \cos_{\alpha} t^{\alpha} \cos_{\alpha} x^{\alpha} = \sum_{n=1}^{\infty} f_n(t) \cos_{\alpha} n^{\alpha} x^{\alpha},$$
$$\varphi(x) = \cos_{\alpha} x^{\alpha} = \sum_{n=1}^{\infty} C_n \cos_{\alpha} n^{\alpha} x^{\alpha}, \tag{58}$$
$$\psi(x) = 0 = \sum_{n=1}^{\infty} D_n \cos_{\alpha} n^{\alpha} x^{\alpha}.$$

We can write

$$f_n(t) = 0, \quad (n \neq 1),$$
$$f_1(t) = \cos_{\alpha} t^{\alpha},$$
$$C_n = 0, \quad (n \neq 1), \tag{59}$$
$$C_1 = 0,$$
$$D_n = 0.$$

Obviously, we have

$$k_1 = 0, \qquad k_2 = -1, \qquad k_3 = 1,$$
$$D_n' = 0. \tag{60}$$

From (25) and (27) we obtain

$$v_{1,1} = v_n(t) = 0, \quad (n \neq 1),$$
$$v_{1,1}(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t \cos_{\alpha}(t-\tau)^{\alpha}(d\tau)^{\alpha} = \sin_{\alpha} t^{\alpha}. \tag{61}$$

Hence, the nondifferentiable solution of (56) reads as

$$u(x,t) = \sin_{\alpha} t^{\alpha} \cos_{\alpha}\left(x^{\alpha}\right). \tag{62}$$

5. Conclusions

In this work we proposed the local fractional function decomposition method. The applications of the methods for solving the inhomogeneous wave equations with local fractional derivative are discussed in detail. The new technique is an efficient mathematical tool for the scientists to deal with local fractional differential equations.

Conflict of Interests

The authors declare that have no conflict of interests regarding this paper.

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