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Research Article

A Unification of G-Metric, Partial Metric, and b-Metric Spaces

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Using the concepts of G-metric, partial metric, and b-metric spaces, we define a new concept of generalized partial b-metric space. Topological and structural properties of the new space are investigated and certain fixed point theorems for contractive mappings in such spaces are obtained. Some examples are provided here to illustrate the usability of the obtained results.

1. Introduction and Mathematical Preliminaries

The concept of a b-metric space was introduced by Czerwik in [1, 2]. After that, several interesting results about the existence of fixed point for single-valued and multivalued operators in (ordered) b-metric spaces have been obtained (see, e.g., [3–13]).

Definition 1 (see [1]). Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [14].

Definition 2 (see [14]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all the three variables);

(G5) $G(x, y, z) \leq G(x, a, a)+G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Aghajani et al. in [15] introduced the class of generalized b-metric spaces (G_b-metric spaces) and then they presented some basic properties of G_b-metric spaces.

The following is their definition of G_b-metric spaces.

Definition 3 (see [15]). Let X be a nonempty set and $s \geq 1$ a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s[G(x, a, a)+G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

On the other hand, Matthews [16] has introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, self-distance of an arbitrary point need not to be equal to zero.

Definition 4 (see [16]). A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(p_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \le p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

In this case, (X, p) is called a partial metric space.

For a survey of fixed point theory, its applications, and comparison of different contractive conditions and related results in both partial metric spaces and G-metric spaces we refer the reader to, for example, [17–27] and references mentioned therein.

Recently, Zand and Nezhad [28] have introduced a new generalized metric space (G_p-metric spaces) as a generalization of both partial metric spaces and G-metric spaces.

We will use the following definition of a G_p-metric space.

Definition 5 (see [29]). Let X be a nonempty set. Suppose that a mapping $G_p : X \times X \times X \to \mathbb{R}^+$ satisfies

(G_p1) $x = y = z$ if $G_p(x,y,z) = G_p(z,z,z) = G_p(y,y,y) = G_p(x,x,x)$;

(G_p2) $G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x, y, z \in X$ with $z \ne y$;

(G_p3) $G_p(x,y,z) = G_p(p\{x,y,z\})$, where p is any permutation of x, y, and z (symmetry in all the three variables);

(G_p4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G_p is called a G_p-metric and (X, G_p) is called a G_p-metric space.

As a generalization and unification of partial metric and b-metric spaces, Shukla [30] presented the concept of a partial b-metric space as follows.

Definition 6 (see [30]). A partial b-metric on a nonempty set X is a mapping $p_b : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(p_{b1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,

(p_{b2}) $p_b(x,x) \le p_b(x,y)$,

(p_{b3}) $p_b(x,y) = p_b(y,x)$,

(p_{b4}) $p_b(x,y) \le s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A partial b-metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number $s \ge 1$ is called the coefficient of (X, p_b).

In a partial b-metric space (X, p_b), if $x, y \in X$ and $p_b(x,y) = 0$, then $x = y$, but the converse may not be true. It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of these facts needs not to be hold.

Example 7 (see [30]). Let $X = \mathbb{R}^+$, $q > 1$ a constant, and $p_b : X \times X \to \mathbb{R}^+$ defined by

$$p_b(x,y) = [\max\{x,y\}]^q + |x-y|^q \quad \forall x, y \in X. \tag{1}$$

Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Note that in a partial b-metric space the limit of a convergent sequence may not be unique (see [30, Example 2]).

In [31], Mustafa et al. introduced a modified version of ordered partial b-metric spaces in order to obtain that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 8 (see [31]). Let X be a (nonempty) set and $s \ge 1$ a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(p_{b1}) $x = y \Leftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$,

(p_{b2}) $p_b(x,x) \le p_b(x,y)$,

(p_{b3}) $p_b(x,y) = p_b(y,x)$,

($p_{b4'}$) $p_b(x,y) \le s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + ((1-s)/2)(p_b(x,x) + p_b(y,y))$.

The pair (X, p_b) is called a partial b-metric space.

Since $s \ge 1$, from ($p_{b4'}$), we have

$$\begin{aligned} p_b(x,y) &\le s(p_b(x,z) + p_b(z,y) - p_b(z,z)) \\ &\le s(p_b(x,z) + p_b(z,y)) - p_b(z,z). \end{aligned} \tag{2}$$

Hence, a partial b-metric in the sense of Definition 8 is also a partial b-metric in the sense of Definition 6.

The following example shows that a partial b-metric on X (in the sense of Definition 8) is neither a partial metric nor a b-metric on X.

Example 9 (see [31]). Let (X, d) be a metric space and $p_b(x,y) = d(x,y)^q + a$, where $q > 1$ and $a \ge 0$ are real numbers. Then p_b is a partial b-metric with $s = 2^{q-1}$.

Proposition 10 (see [31]). *Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y) \tag{3}$$

for all $x, y \in X$.

Hence, the importance of our definition of partial b-metric is that by using it we can define a dependent b-metric which we call the b-metric associated with p_b.

Now, we present some definitions and propositions in a partial b-metric space.

Definition 11 (see [31]). Let (X, p_b) be a partial b-metric space. Then, for an $x \in X$ and an $\epsilon > 0$, the p_b-ball with center x and radius $\epsilon > 0$ is

$$B_{p_b}(x, \epsilon) = \{y \in X \mid p_b(x, y) < p_b(x, x) + \epsilon\}. \quad (4)$$

Lemma 12 (see [31]). *Let (X, p_b) be a partial b-metric space. Then,*

(A) *if $p_b(x, y) = 0$, then $x = y$;*

(B) *if $x \neq y$, then $p_b(x, y) > 0$.*

Proposition 13 (see [31]). *Let (X, p_b) be a partial b-metric space, $x \in X$, and $\epsilon > 0$. If $y \in B_{p_b}(x, \epsilon)$ then there exists a $\delta > 0$ such that $B_{p_b}(y, \delta) \subseteq B_{p_b}(x, \varepsilon)$.*

Thus, from the above proposition the family of all open p_b-balls

$$\Delta = \left\{B_{p_b}(x, r) \mid x \in X, r > 0\right\} \quad (5)$$

is a base of a T_0-topology τ_{p_b} on X which we call the p_b-metric topology.

The topological space (X, p_b) is T_0 but need not be T_1.

The following lemma shows the relationship between the concepts of p_b-convergence, p_b-Cauchyness, and p_b-completeness in two spaces (X, p_b) and (X, d_{p_b}).

Lemma 14 (see [31]). *(1) A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial b-metric space (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).*

(2) A partial b-metric space (X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x, x_n) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x, x). \quad (6)$$

Now, we introduce the concept of generalized partial b-metric space, a G_{p_b}-metric space, as a generalization of both partial b-metric space and G-metric space.

Definition 15. Let X be a nonempty set. Suppose that the mapping $G_{p_b} : X \times X \times X \to \mathbb{R}^+$ satisfies the following conditions:

($G_{p_b}1$) $x = y = z$ if $G_{p_b}(x, y, z) = G_{p_b}(z, z, z) = G_{p_b}(y, y, y) = G_{p_b}(x, x, x)$;

($G_{p_b}2$) $G_{p_b}(x, x, x) \leq G_{p_b}(x, x, y) \leq G_{p_b}(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

($G_{p_b}3$) $G_{p_b}(x, y, z) = G_{p_b}(p\{x, y, z\})$, where p is any permutation of x, y, or z (symmetry in all three variables);

($G_{p_b}4$) $G_{p_b}(x, y, z) \leq s[G_{p_b}(x, a, a) + G_{p_b}(a, y, z) - G_{p_b}(a, a, a)] + ((1-s)/3)[G_{p_b}(x, x, x) + G_{p_b}(y, y, y) + G_{p_b}(z, z, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G_{p_b} is called a G_{p_b}-metric and (X, G_{p_b}) is called a G_{p_b}-metric space.

Since $s \geq 1$, so from $G_{p_b}4$ we have the following inequality:

$$G_{p_b}(x, y, z) \leq s\left[G_{p_b}(x, a, a) + G_{p_b}(a, y, z) - G_{p_b}(a, a, a)\right]. \quad (7)$$

The G_{p_b}-metric space G_{p_b} is called symmetric if

$$G_{p_b}(x, x, y) = G_{p_b}(x, y, y) \quad (8)$$

holds for all $x, y \in X$. Otherwise, G_{p_b} is an asymmetric G_{p_b}-metric.

Now we present some examples of G_{p_b}-metric space.

Example 16. Let $X = [0, +\infty)$ and let $G_{p_b} : X^3 \to \mathbb{R}^+$ be given by $G_{p_b}(x, y, z) = [\max\{x, y, z\}]^p$, where $p > 1$. Obviously, (X, G_{p_b}) is a symmetric G_{p_b}-metric space which is not a G-metric space. Indeed, if $x = y = z > 0$, then $G_{p_b}(x, y, z) = x^p > 0$. It is easy to see that $G_{p_b}1$–$G_{p_b}3$ are satisfied. Now we show that $G_{p_b}4$ holds. For each $x, y, z, a \in X$, we have

$$\frac{x^p + y^p + z^p}{3} \leq [\max\{x, y, z\}]^p, \quad (9)$$

so

$$\begin{aligned}
&[\max\{x, y, z\}]^p + \frac{s-1}{3}(x^p + y^p + z^p) + sa^p \\
&\leq [\max\{x, y, z\}]^p + (s-1)[\max\{x, y, z\}]^p + sa^p \\
&= s[\max\{x, y, z\}]^p + sa^p \\
&\leq s[\max\{x, a\}]^p + s[\max\{a, y, z\}]^p.
\end{aligned} \quad (10)$$

Thus,

$$\begin{aligned}
&[\max\{x, y, z\}]^p \\
&\leq s\left([\max\{x, a\}]^p + [\max\{a, y, z\}]^p - a^p\right) \\
&+ \frac{1-s}{3}(x^p + y^p + z^p)
\end{aligned} \quad (11)$$

which implies the required inequality

$$\begin{aligned}
&G_{p_b}(x, y, z) \\
&\leq s\left[G_{p_b}(x, a, a) + G_{p_b}(a, y, z) - G_{p_b}(a, a, a)\right] \\
&+ \frac{1-s}{3}\left[G_{p_b}(x, x, x) + G_{p_b}(y, y, y) + G_{p_b}(z, z, z)\right].
\end{aligned} \quad (12)$$

Example 17. Let $X = \{0, 1, 2, 3\}$. Let

$$\begin{aligned} A = \{&(1,0,0), (0,1,0), (0,0,1), (2,0,0), (0,2,0), \\ &(0,0,2), (3,0,0), (0,3,0), (0,0,3), (1,2,2), \\ &(2,1,2), (2,2,1), (2,3,3), (3,2,3), (3,3,2)\}, \\ B = \{&(0,1,1), (1,0,1), (1,1,0), (0,2,2), (2,0,2), \\ &(2,2,0), (0,3,3), (3,0,3), (3,3,0), (2,1,1), \\ &(1,2,1), (1,1,2), (3,2,2), (2,3,2), (2,2,3)\}. \end{aligned} \tag{13}$$

Define $G_{p_b} : X^3 \to \mathbb{R}^+$ by

$$G_{p_b}(x,y,z) = \begin{cases} \frac{3}{2}, & \text{if } x = y = z \neq 2, \\ 0, & \text{if } x = y = z = 2, \\ 2, & \text{if } (x,y,z) \in A, \\ \frac{5}{2}, & \text{if } (x,y,z) \in B, \\ 3, & \text{otherwise}. \end{cases} \tag{14}$$

It is easy to see that (X, G_{p_b}) is an asymmetric G_{p_b}-metric space with coefficient $s \geq 6/5$.

Proposition 18. *Every G_{p_b}-metric space (X, G_{p_b}) defines a b-metric space $(X, d_{G_{p_b}})$, where*

$$\begin{aligned} d_{G_{p_b}}(x,y) = G_{p_b}(x,y,y) + G_{p_b}(y,x,x) \\ - G_{p_b}(x,x,x) - G_{p_b}(y,y,y), \end{aligned} \tag{15}$$

for all $x, y \in X$.

Proof. Let $x, y, z \in X$. Then we have

$$\begin{aligned} &d_{G_{p_b}}(x,y) \\ &= G_{p_b}(x,y,y) + G_{p_b}(y,x,x) - G_{p_b}(x,x,x) \\ &\quad - G_{p_b}(y,y,y) \\ &\leq s\left(G_{p_b}(x,z,z) + G_{p_b}(z,y,y) - G_{p_b}(z,z,z)\right) \\ &\quad + \left(\frac{1-s}{3}\right)\left(G_{p_b}(x,x,x) + 2G_{p_b}(y,y,y)\right) \\ &\quad + s\left(G_{p_b}(y,z,z) + G_{p_b}(z,x,x) - G_{p_b}(z,z,z)\right) \\ &\quad + \left(\frac{1-s}{3}\right)\left(2G_{p_b}(x,x,x) + G_{p_b}(y,y,y)\right) \\ &\quad - G_{p_b}(x,x,x) - G_{p_b}(y,y,y) \\ &= s\left[G_{p_b}(x,z,z) + G_{p_b}(z,x,x) - G_{p_b}(x,x,x)\right. \\ &\quad - G_{p_b}(z,z,z) + G_{p_b}(z,y,y) \\ &\quad \left. + G_{p_b}(y,z,z) - G_{p_b}(z,z,z) - G_{p_b}(y,y,y)\right] \\ &= s\left[d_{G_{p_b}}(x,z) + d_{G_{p_b}}(z,y)\right]. \end{aligned} \tag{16}$$

□

With straightforward calculations, we have the following proposition.

Proposition 19. *Let X be a G_{p_b}-metric space. Then for each $x, y, z, a \in X$ it follows that*

(1) $G_{p_b}(x,y,z) \leq sG_{p_b}(x,a,a) + s^2G_{p_b}(y,a,a) + s^2G_{p_b}(z,a,a) - (s+s^2)G_{p_b}(a,a,a)$;

(2) $G_{p_b}(x,y,z) \leq s[G_{p_b}(x,x,y) + G_{p_b}(x,x,z) - G_{p_b}(x,x,x)] + ((1-s)/3)[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)]$;

(3) $G_{p_b}(x,y,y) \leq 2sG_{p_b}(x,x,y) + ((1-4s)/3)G_{p_b}(x,x,x) + (2/3)(1-s)G_{p_b}(y,y,y)$;

(4) $G_{p_b}(x,y,z) \leq s[G_{p_b}(x,a,z) + G_{p_b}(a,y,z) - G_{p_b}(a,a,a)] + ((1-s)/3)[G_{p_b}(x,x,x) + G_{p_b}(y,y,y) + G_{p_b}(z,z,z)]$, $a \neq z$.

Lemma 20. *Let (X, G_{p_b}) be a G_{p_b}-metric space. Then,*

(A) *if $G_{p_b}(x,y,z) = 0$, then $x = y = z$;*

(B) *if $x \neq y$, then $G_{p_b}(x,y,y) > 0$.*

Proof. If $G_{p_b}(x,y,z) = 0$, then from $G_{p_b}2$ we have $G_{p_b}(x,x,x) = G_{p_b}(y,y,y) = G_{p_b}(z,z,z) = 0$, so from $G_{p_b}1$ we obtain (A). To prove (B), on the contrary, if $G_{p_b}(x,y,y) = 0$, then from (A) we have $x = y$, a contradiction. □

Definition 21. Let (X, G_{p_b}) be a G_{p_b}-metric space. Then for an $x \in X$ and an $\epsilon > 0$, the G_{p_b}-ball with center x and radius $\epsilon > 0$ is

$$B_{G_{p_b}}(x,\epsilon) = \left\{y \in X \mid G_{p_b}(x,x,y) < G_{p_b}(x,x,x) + \epsilon\right\}. \tag{17}$$

By motivation of Proposition 4 in [31], we provide the following proposition.

Proposition 22. *Let (X, G_{p_b}) be a G_{p_b}-metric space, $x \in X$, and $\epsilon > 0$. If $y \in B_{G_{p_b}}(x,\epsilon)$, then there exists a $\delta > 0$ such that $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\varepsilon)$.*

Proof. Let $y \in B_{G_{p_b}}(x,\epsilon)$; if $y = x$, then we choose $\delta = \epsilon$. Suppose that $y \neq x$; then, by Lemma 20, we have $G_{p_b}(x,x,y) \neq 0$. Now, we consider two cases.

Case 1. If $G_{p_b}(x,x,y) = G_{p_b}(x,x,x)$, then for $s = 1$ we choose $\delta = \epsilon$. If $s > 1$, then we consider the set

$$A = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2s^{n+1}(s-1)} < G_{p_b}(x,x,x)\right\}. \tag{18}$$

By Archmedean property, A is a nonempty set; then by the well ordering principle, A has a least element m. Since $m - 1 \notin A$, we have $G_{p_b}(x,x,x) \leq (\epsilon/(2s^m(s-1)))$ and we choose $\delta = \epsilon/2s^{m+1}$. Let $z \in B_{G_{p_b}}(y,\delta)$; by property $G_{p_b}4$ we have

$$\begin{aligned} G_{p_b}(x,x,z) &\le s\left(G_{p_b}(x,x,y) + G_{p_b}(y,y,z) - G_{p_b}(y,y,y)\right) \\ &\le s\left(G_{p_b}(x,x,x) + \delta\right) \\ &\le G_{p_b}(x,x,x) + \frac{\epsilon}{2s^m} + \frac{\epsilon}{2s^m} \\ &= G_{p_b}(x,x,x) + \frac{\epsilon}{s^m} \\ &< G_{p_b}(x,x,x) + \epsilon. \end{aligned} \tag{19}$$

Hence, $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$.

Case 2. If $G_{p_b}(x,x,y) \neq G_{p_b}(x,x,x)$, then, from property $G_{p_b}2$, we have $G_{p_b}(x,x,x) < G_{p_b}(x,x,y)$, and. for $s = 1$, we consider the set

$$B = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2^{n+3}} < G_{p_b}(x,x,y) - G_{p_b}(x,x,x)\right\}. \tag{20}$$

Similarly, by the well ordering principle there exists an element m such that $G_{p_b}(x,x,y) - G_{p_b}(x,x,x) \le \epsilon/2^{m+2}$, and we choose $\delta = \epsilon/2^{m+2}$. One can easily obtain that $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$.

For $s > 1$, we consider the set

$$C = \left\{n \in \mathbb{N} \mid \frac{\epsilon}{2s^{n+2}} < G_{p_b}(x,x,y) - \frac{1}{s}G_{p_b}(x,x,x)\right\} \tag{21}$$

and by the well ordering principle there exists an element m such that $G_{p_b}(x,x,y) - (1/s)G_{p_b}(x,x,x) \le \epsilon/2s^{m+1}$ and we choose $\delta = \epsilon/2s^{m+1}$. Let $z \in B_{G_{p_b}}(y,\delta)$. By property $G_{p_b}4$ we have

$$\begin{aligned} G_{p_b}(x,x,z) &\le s\left(G_{p_b}(x,x,y) + G_{p_b}(y,y,z) - G_{p_b}(y,y,y)\right) \\ &\le s\left(G_{p_b}(x,x,y) + \delta\right) \\ &\le G_{p_b}(x,x,x) + \frac{\epsilon}{2s^m} + \frac{\epsilon}{2s^m} \\ &= G_{p_b}(x,x,x) + \frac{\epsilon}{s^m} < G_{p_b}(x,x,x) + \epsilon. \end{aligned} \tag{22}$$

Hence, $B_{G_{p_b}}(y,\delta) \subseteq B_{G_{p_b}}(x,\epsilon)$. □

Thus, from the above proposition the family of all open G_{p_b}-balls

$$F = \left\{B_{G_{p_b}}(x,\epsilon) \mid x \in X, \epsilon > 0\right\} \tag{23}$$

is a base of a T_0-topology $\tau_{G_{p_b}}$ on X which we call the G_{p_b}-metric topology.

The topological space (X, G_{p_b}) is T_0, but need not be T_1.

Definition 23. Let (X, G_{p_b}) be a G_{p_b}-metric space. Let $\{x_n\}$ be a sequence in X.

(1) A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{n,m\to\infty} G_{p_b}(x,x_n,x_m) = G_{p_b}(x,x,x)$.

(2) $\{x_n\}$ is said to be a G_{p_b}-Cauchy sequence, if $\lim_{n,m\to\infty} G_{p_b}(x_n,x_m,x_m)$ exists (and is finite).

(3) (X, G_{p_b}) is said to be G_{p_b}-complete if every G_{p_b}-Cauchy sequence in X is G_{p_b}-convergent to an $x \in X$.

Using the above definitions, one can easily prove the following proposition.

Proposition 24. *Let (X, G_{p_b}) be a G_{p_b}-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following statements are equivalent:*

(1) $\{x_n\}$ is G_{p_b}-convergent to x.

(2) $G_{p_b}(x_n,x_n,x) \to G_{p_b}(x,x,x)$, as $n \to \infty$.

(3) $G_{p_b}(x_n,x,x) \to G_{p_b}(x,x,x)$, as $n \to \infty$.

Based on Lemma 2.2 of [27], we prove the following essential lemma.

Lemma 25. *(1) A sequence $\{x_n\}$ is a G_{p_b}-Cauchy sequence in a G_{p_b}-metric space (X, G_{p_b}) if and only if it is a b-Cauchy sequence in the b-metric space $(X, d_{G_{p_b}})$.*

(2) A G_{p_b}-metric space (X, G_{p_b}) is G_{p_b}-complete if and only if the b-metric space $(X, d_{G_{p_b}})$ is b-complete. Moreover, $\lim_{n\to\infty} d_{G_{p_b}}(x,x_n) = 0$ if and only if

$$\begin{aligned} \lim_{n\to\infty} G_{p_b}(x,x_n,x_n) &= \lim_{n\to\infty} G_{p_b}(x_n,x,x) \\ &= \lim_{n,m\to\infty} G_{p_b}(x_n,x_n,x_m) = G_{p_b}(x,x,x). \end{aligned} \tag{24}$$

Proof. First, we show that every G_{p_b}-Cauchy sequence in (X, G_{p_b}) is a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Let $\{x_n\}$ be a G_{p_b}-Cauchy sequence in (X, G_{p_b}). Then, there exists $\alpha \in \mathbb{R}$ such that, for arbitrary $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with

$$\left|G_{p_b}(x_n,x_m,x_m) - \alpha\right| < \frac{\varepsilon}{4}, \tag{25}$$

for all $n,m \ge n_\varepsilon$. Hence,

$$\begin{aligned} &\left|d_{G_{p_b}}(x_n,x_m)\right| \\ &\quad = G_{p_b}(x_n,x_m,x_m) + G_{p_b}(x_m,x_n,x_n) \\ &\qquad - G_{p_b}(x_n,x_n,x_n) - G_{p_b}(x_m,x_m,x_m) \\ &\quad = \left|G_{p_b}(x_n,x_m,x_m) - \alpha + \alpha - G_{p_b}(x_n,x_n,x_n)\right. \\ &\qquad \left. + G_{p_b}(x_m,x_n,x_n) - \alpha + \alpha - G_{p_b}(x_m,x_m,x_m)\right| \\ &\quad \le \left|G_{p_b}(x_n,x_m,x_m) - \alpha\right| + \left|\alpha - G_{p_b}(x_n,x_n,x_n)\right| \\ &\qquad + \left|G_{p_b}(x_m,x_n,x_n) - \alpha\right| + \left|\alpha - G_{p_b}(x_m,x_m,x_m)\right| \\ &\quad < \varepsilon, \end{aligned} \tag{26}$$

for all $n, m \geq n_\varepsilon$. Hence, we conclude that $\{x_n\}$ is a b-Cauchy sequence in $(X, d_{G_{p_b}})$.

Next, we prove that b-completeness of $(X, d_{G_{p_b}})$ implies G_{p_b}-completeness of (X, G_{p_b}). Indeed, if $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}), then it is also a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Since the b-metric space $(X, d_{G_{p_b}})$ is b-complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} d_{G_{p_b}}(y, x_n) = 0$. Hence,

$$\lim_{n\to\infty}\left[G_{p_b}(x_n, y, y) - G_{p_b}(y, y, y) + G_{p_b}(y, x_n, x_n) - G_{p_b}(x_n, x_n, x_n)\right] = 0, \tag{27}$$

therefore; $\lim_{n\to\infty}[G_{p_b}(x_n, y, y) - G_{p_b}(y, y, y)] = 0$.

On the other hand,

$$\begin{aligned}
&\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, y) \\
&\quad\leq \lim_{n,m\to\infty} sG_{p_b}(x_n, y, y) + \lim_{n,m\to\infty} sG_{p_b}(x_m, y, y) \\
&\qquad - sG_{p_b}(y, y, y) \\
&\qquad + \frac{1-s}{3}\Big[\lim_{n,m\to\infty} G_{p_b}(x_n, x_n, x_n) \\
&\qquad\qquad + \lim_{n,m\to\infty} G_{p_b}(x_m, x_m, x_m) \\
&\qquad\qquad + G_{p_b}(y, y, y)\Big] \\
&\quad= G_{p_b}(y, y, y).
\end{aligned} \tag{28}$$

Also, from $(G_{p_b}2)$,

$$G_{p_b}(y, y, y) \leq \lim_{n,m\to\infty} G_{p_b}(x_n, x_m, y). \tag{29}$$

Hence, we obtain that $\{x_n\}$ is a G_{p_b}-convergent sequence in (X, G_{p_b}).

Now, we prove that every b-Cauchy sequence $\{x_n\}$ in $(X, d_{G_{p_b}})$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}). Let $\varepsilon = 1/2$. Then, there exists $n_0 \in \mathbb{N}$ such that $d_{G_{p_b}}(x_n, x_m) < 1/2$ for all $n, m \geq n_0$. Since

$$G_{p_b}(x_n, x_{n_0}, x_{n_0}) - G_{p_b}(x_{n_0}, x_{n_0}, x_{n_0}) \leq d_{G_{p_b}}(x_n, x_{n_0}) < \frac{1}{2}, \tag{30}$$

then

$$\begin{aligned}
G_{p_b}(x_n, x_n, x_n) &\leq d_{G_{p_b}}(x_n, x_{n_0}) + G_{p_b}(x_n, x_{n_0}, x_{n_0}) \\
&< \frac{1}{2} + G_{p_b}(x_{n_0}, x_{n_0}, x_{n_0}).
\end{aligned} \tag{31}$$

Consequently, the sequence $\{G_{p_b}(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$ and so there exists $a \in \mathbb{R}$ such that a subsequence $\{G_{p_b}(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to a; that is,

$$\lim_{k\to\infty} G_{p_b}(x_{n_k}, x_{n_k}, x_{n_k}) = a. \tag{32}$$

Now, we prove that $\{G_{p_b}(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a b-Cauchy sequence in $(X, d_{G_{p_b}})$, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_{G_{p_b}}(x_n, x_m) < \varepsilon$, for all $n, m \geq n_\varepsilon$. Thus, for all $n, m \geq n_\varepsilon$,

$$\begin{aligned}
&G_{p_b}(x_n, x_n, x_n) - G_{p_b}(x_m, x_m, x_m) \\
&\quad\leq G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x_m, x_m, x_m) \\
&\quad\leq d_{G_{p_b}}(x_m, x_n) < \varepsilon.
\end{aligned} \tag{33}$$

Therefore, $\lim_{n\to\infty} G_{p_b}(x_n, x_n, x_n) = a$.

On the other hand,

$$\begin{aligned}
&\left|G_{p_b}(x_n, x_m, x_m) - a\right| \\
&\quad= \big|G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x_n, x_n, x_n) \\
&\qquad + G_{p_b}(x_n, x_n, x_n) - a\big| \\
&\quad\leq d_{G_{p_b}}(x_m, x_n) + \left|G_{p_b}(x_n, x_n, x_n) - a\right|,
\end{aligned} \tag{34}$$

for all $n, m \geq n_\varepsilon$. Hence, $\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, x_m) = a$, and consequently $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}).

Conversely, let $\{x_n\}$ be a b-Cauchy sequence in $(X, d_{G_{p_b}})$. Then, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in (X, G_{p_b}) and so it is G_{p_b}-convergent to a point $x \in X$ with

$$\begin{aligned}
\lim_{n\to\infty} G_{p_b}(x, x_n, x_n) &= \lim_{n\to\infty} G_{p_b}(x_n, x, x) \\
&= \lim_{n,m\to\infty} G_{p_b}(x, x_m, x_n) = G_{p_b}(x, x, x).
\end{aligned} \tag{35}$$

Then, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}
G_{p_b}(x, x_n, x_n) - G_{p_b}(x, x, x) &< \frac{\varepsilon}{4}, \\
G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x) &< \frac{\varepsilon}{4}.
\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}
&\left|d_{G_{p_b}}(x_n, x)\right| \\
&\quad= \big|G_{p_b}(x_n, x, x) - G_{p_b}(x_n, x_n, x_n) \\
&\qquad + G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\big| \\
&\quad\leq \left|G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x)\right| \\
&\qquad + \left|G_{p_b}(x, x, x) - G_{p_b}(x_n, x_n, x_n)\right| \\
&\qquad + \left|G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\right| < \varepsilon,
\end{aligned} \tag{37}$$

whenever $n \geq n_\varepsilon$. Therefore, $(X, d_{G_{p_b}})$ is b-complete.

Finally, let $\lim_{n\to\infty} d_{G_{p_b}}(x_n, x) = 0$. So

$$\begin{aligned}
&\lim_{n\to\infty}\left[G_{p_b}(x_n, x, x) - G_{p_b}(x_n, x_n, x_n)\right] \\
&\quad + \lim_{n\to\infty}\left[G_{p_b}(x_n, x_n, x) - G_{p_b}(x, x, x)\right] = 0, \\
&\lim_{n\to\infty}\left[G_{p_b}(x_n, x, x) - G_{p_b}(x, x, x)\right] \\
&\quad + \lim_{n\to\infty}\left[G_{p_b}(x_n, x_n, x) - G_{p_b}(x_n, x_n, x_n)\right] = 0.
\end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned}
&\lim_{n,m\to\infty} G_{p_b}(x_n, x_m, x_m) - G_{p_b}(x, x, x) \\
&\quad \leq s\left[\lim_{n\to\infty} G_{p_b}(x_n, x, x) + \lim_{m\to\infty} G_{p_b}(x, x_m, x_m)\right. \\
&\qquad \left. - G_{p_b}(x, x, x)\right] + \frac{1-s}{3} \\
&\quad \times \left[\lim_{n\to\infty} G_{p_b}(x_n, x_n, x_n) + 2\lim_{m\to\infty} G_{p_b}(x_m, x_m, x_m)\right] \\
&\quad - G_{p_b}(x, x, x) = 0.
\end{aligned} \tag{39}$$

□

Definition 26. Let (X, G_{p_b}) and (X', G'_{p_b}) be two generalized partial b-metric spaces and let $f : (X, G_{p_b}) \to (X', G'_{p_b})$ be a mapping. Then f is said to be G_{p_b}-continuous at a point $a \in X$ if, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $G_{p_b}(a, a, x) < \delta + G_{p_b}(a, a, a)$ imply that $G'_{p_b}(f(a), f(a), f(x)) < \varepsilon + G'_{p_b}(f(a), f(a), f(a))$. The mapping f is G_{p_b}-continuous on X if it is G_{p_b}-continuous at all $a \in X$. For simplicity, we say that f is continuous.

From the above definition, with straightforward calculations, we have the following proposition.

Proposition 27. *Let (X, G_{p_b}) and (X', G'_{p_b}) be two generalized partial b-metric spaces. Then a mapping $f : X \to X'$ is G_{p_b}-continuous at a point $x \in X$ if and only if it is G_{p_b} sequentially continuous at x; that is, whenever $\{x_n\}$ is G_{p_b}-convergent to x, $\{f(x_n)\}$ is G'_{p_b}-convergent to $f(x)$.*

Definition 28. A triple $(X, \preceq, G_{p_b})$ is called an ordered generalized partial b-metric space if $(X, \preceq)$ is a partially ordered set and G_{p_b} is a generalized partial b-metric on X.

We will need the following simple lemma of the G_{p_b}-convergent sequences in the proof of our main results.

Lemma 29. *Let (X, G_{p_b}) be a G_{p_b}-metric space and suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_{p_b}-convergent to x, y, and z, respectively. Then we have*

$$\begin{aligned}
&\frac{1}{s^3} G_{p_b}(x, y, z) - \frac{1}{s^2} G_{p_b}(x, x, x) \\
&\quad - \frac{1}{s} G_{p_b}(y, y, y) - G_{p_b}(z, z, z) \\
&\quad \leq \liminf_{n\to\infty} G_{p_b}(x_n, y_n, z_n) \leq \limsup_{n\to\infty} G_{p_b}(x_n, y_n, z_n) \\
&\quad \leq s^3 G_{p_b}(x, y, z) + s G_{p_b}(x, x, x) + s^2 G_{p_b}(y, y, y) \\
&\qquad + s^3 G_{p_b}(z, z, z).
\end{aligned} \tag{40}$$

In particular, if $\{y_n\} = \{z_n\} = a$ are constant, then

$$\begin{aligned}
&\frac{1}{s} G_{p_b}(x, a, a) - G_{p_b}(x, x, x) \\
&\quad \leq \liminf_{n\to\infty} G_{p_b}(x_n, a, a) \leq \limsup_{n\to\infty} G_{p_b}(x_n, a, a) \\
&\quad \leq s G_{p_b}(x, a, a) + s G_{p_b}(x, x, x).
\end{aligned} \tag{41}$$

Proof. Using the rectangle inequality, we obtain

$$\begin{aligned}
G_{p_b}(x, y, z) &\leq s G_{p_b}(x, x_n, x_n) + s^2 G_{p_b}(y, y_n, y_n) \\
&\quad + s^3 G_{p_b}(z, z_n, z_n) + s^3 G_{p_b}(x_n, y_n, z_n), \\
G_{p_b}(x_n, y_n, z_n) &\leq s G_{p_b}(x_n, x, x) + s^2 G_{p_b}(y_n, y, y) \\
&\quad + s^3 G_{p_b}(z_n, z, z) + s^3 G_{p_b}(x, y, z).
\end{aligned} \tag{42}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

If $\{y_n\} = \{z_n\} = a$, then

$$\begin{aligned}
G_{p_b}(x, a, a) &\leq s G_{p_b}(x, x_n, x_n) + s G_{p_b}(x_n, a, a), \\
G_{p_b}(x_n, a, a) &\leq s G_{p_b}(x_n, x, x) + s G_{p_b}(x, a, a).
\end{aligned} \tag{43}$$

□

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{44}$$

In order to generalize the Banach contraction principle, Geraghty proved the following result.

Theorem 30 (see [32]). *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx, fy) \leq \beta(d(x, y))\, d(x, y) \tag{45}$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In [33], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces.

As in [33], we will consider the class of functions $\mathcal{F}$, where $\beta \in \mathcal{F}$ if $\beta : [0, \infty) \to [0, 1/s)$ and has the property

$$\beta(t_n) \longrightarrow \frac{1}{s} \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

Theorem 31 (see [33]). *Let $s > 1$ and let (X, D, s) be a complete metric type space. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$D(fx, fy) \leq \beta(D(x, y)) D(x, y) \tag{47}$$

for all $x, y \in X$ and some $\beta \in \mathcal{F}$. Then f has a unique fixed point $z \in X$ and for each $x \in X\{f^n x\}$ converges to z in (X, D, s).

The aim of this paper is to present certain new fixed point theorems for hybrid rational Geraghty-type and ψ-contractive mappings in partially ordered G_{p_b}-metric spaces. Our results improve and generalize many comparable results in literature. Some examples are established to prove the generality of our results.

2. Main Results

Recall that $\mathcal{F}$ denotes the class of all functions $\beta : [0, \infty) \to [0, 1/s)$ satisfying the following condition:

$$\beta(t_n) \longrightarrow \frac{1}{s} \text{ implies } t_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{48}$$

Theorem 32. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \leq \beta(G_{p_b}(x, y, z)) M(x, y, z) \tag{49}$$

for all comparable elements $x, y, z \in X$, where

$$\begin{aligned} M(x, y, z) &= \max\left\{G_{p_b}(x, y, z), \right. \\ &\left. \frac{G_{p_b}(x, x, fx) G_{p_b}(y, y, fy) G_{p_b}(z, z, fz)}{1 + [sG_{p_b}(fx, fy, fz)]^2}\right\}. \end{aligned} \tag{50}$$

If f is continuous, then f has a fixed point.

Proof. Put $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots. \tag{51}$$

Step 1. We will show that $\lim_{n\to\infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (49) we have

$$\begin{aligned} sG_{p_b}&(x_n, x_{n+1}, x_{n+2}) \\ &= sG_{p_b}(fx_{n-1}, fx_n, fx_{n+1}) \\ &\leq \beta(G_{p_b}(x_{n-1}, x_n, x_{n+1})) M(x_{n-1}, x_n, x_{n+1}) \\ &\leq \frac{1}{s} G_{p_b}(x_{n-1}, x_n, x_{n+1}) \\ &\leq G_{p_b}(x_{n-1}, x_n, x_{n+1}), \end{aligned} \tag{52}$$

because

$$\begin{aligned} M&(x_{n-1}, x_n, x_{n+1}) \\ &= \max\left\{G_{p_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad (G_{p_b}(x_{n-1}, x_{n-1}, fx_{n-1}) G_{p_b}(x_n, x_n, fx_n) \\ &\quad \times G_{p_b}(x_{n+1}, x_{n+1}, fx_{n+1})) \\ &\quad \left. \times \left(1 + [sG_{p_b}(fx_{n-1}, fx_n, fx_{n+1})]^2\right)^{-1}\right\} \\ &= \max\left\{G_{p_b}(x_{n-1}, x_n, x_{n+1}), \right. \\ &\quad (G_{p_b}(x_{n-1}, x_{n-1}, x_n) G_{p_b}(x_n, x_n, x_{n+1}) \\ &\quad \times G_{p_b}(x_{n+1}, x_{n+1}, x_{n+2})) \\ &\quad \left. \times \left(1 + [sG_{p_b}(x_n, x_{n+1}, x_{n+2})]^2\right)^{-1}\right\} \\ &= G_{p_b}(x_{n-1}, x_n, x_{n+1}). \end{aligned} \tag{53}$$

Therefore, $\{G_{p_b}(x_n, x_{n+1}, x_{n+2})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = r$. Letting $n \to \infty$ in (52) we have

$$sr \leq r. \tag{54}$$

Since $s > 1$, we deduce that $r = 0$; that is,

$$\lim_{n\to\infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = 0. \tag{55}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a G_{p_b}-Cauchy sequence. By rectangular inequality and (49), we have

$$\begin{aligned} G_{p_b}&(x_n, x_m, x_m) \\ &\leq sG_{p_b}(x_n, x_{n+1}, x_{n+1}) + s^2 G_{p_b}(x_{n+1}, x_{m+1}, x_{m+1}) \\ &\quad + s^2 G_{p_b}(x_{m+1}, x_m, x_m) \\ &\leq sG_{p_b}(x_n, x_{n+1}, x_{n+1}) + s\beta(G_{p_b}(x_n, x_m, x_m)) \\ &\quad \times M(x_n, x_m, x_m) + s^2 G_{p_b}(x_{m+1}, x_m, x_m). \end{aligned} \tag{56}$$

Letting $m, n \to \infty$ in the above inequality and applying (55) we have

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \leq s \lim_{n,m\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right) \lim_{n,m\to\infty} M(x_n, x_m, x_m). \tag{57}$$

Here,

$$\begin{aligned} G_{p_b}(x_n, x_m, x_m) &\leq M(x_n, x_m, x_m) \\ &= \max\left\{G_{p_b}(x_n, x_m, x_m), \frac{G_{p_b}(x_n, x_n, f(x_n))\left[G_{p_b}(x_m, x_m, f(x_m))\right]^2}{1+\left[sG_{p_b}(fx_n, fx_m, fx_m)\right]^2}\right\} \\ &= \max\left\{G_{p_b}(x_n, x_m, x_m), \frac{G_{p_b}(x_n, x_n, x_{n+1})\left[G_{p_b}(x_m, x_m, x_{m+1})\right]^2}{1+\left[sG_{p_b}(x_{n+1}, x_{m+1}, x_{m+1})\right]^2}\right\}. \end{aligned} \tag{58}$$

Letting $m, n \to \infty$ in the above inequality we get

$$\lim_{m,n\to\infty} M(x_n, x_m, x_m) = \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m). \tag{59}$$

Hence, from (57) and (59), we obtain

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \leq s \lim_{m,n\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right) \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \tag{60}$$

and so we get

$$\frac{1}{s} \leq \lim_{m,n\to\infty} \beta\left(G_{p_b}(x_n, x_m, x_m)\right). \tag{61}$$

Since $\beta \in \mathcal{F}$ we deduce that

$$\lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) = 0. \tag{62}$$

Consequently, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in X. Thus, from Lemma 25, $\{x_n\}$ is a b-Cauchy sequence in the b-metric space $(X, d_{G_{p_b}})$. Since (X, G_{p_b}) is G_{p_b}-complete, then, from Lemma 25, $(X, d_{G_{p_b}})$ is a b-complete b-metric space. Therefore, the sequence $\{x_n\}$ b-converges to some $u \in X$; that is, $\lim_{n\to\infty} d_{G_{p_b}}(x_n, u) = 0$. Again, from Lemma 25 and (62),

$$\begin{aligned} \lim_{n\to\infty} G_{p_b}(u, x_n, x_n) &= \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) \\ &= G_{p_b}(u, u, u) = 0. \end{aligned} \tag{63}$$

Step 3. Now, we show that u is a fixed point of f. Suppose to the contrary; that is, $fu \neq u$; then, from Lemma 20, we have $G_{p_b}(u, u, fu) > 0$.

Using the rectangular inequality, we get

$$G_{p_b}(u, u, fu) \leq sG_{p_b}(fu, fx_n, fx_n) + sG_{p_b}(fx_n, u, u). \tag{64}$$

Letting $n \to \infty$ and using the continuity of f and (63), we get

$$\begin{aligned} G_{p_b}(u, u, fu) &\leq s \lim_{n\to\infty} G_{p_b}(fu, fx_n, fx_n) \\ &\quad + s \lim_{n\to\infty} G_{p_b}(fx_n, u, u) = sG_{p_b}(fu, fu, fu). \end{aligned} \tag{65}$$

Note that, from (49), we have

$$sG_{p_b}(fu, fu, fu) \leq \beta\left(G_{p_b}(u, u, u)\right) M(u, u, u), \tag{66}$$

where by (65)

$$\begin{aligned} M(u, u, u) &= \max\left\{G_{p_b}(u, u, u), \frac{G_{p_b}(u, u, fu)\, G_{p_b}(u, u, fu)\, G_{p_b}(u, u, fu)}{1+\left[sG_{p_b}(fu, fu, fu)\right]^2}\right\} \\ &\leq G_{p_b}(u, u, fu). \end{aligned} \tag{67}$$

Hence, as $\beta(t) \leq 1$ for all $t \in [0, \infty)$, we have $sG_{p_b}(fu, fu, fu) \leq G_{p_b}(u, u, fu)$. Thus, by (65) we obtain that $sG_{p_b}(fu, fu, fu) = G_{p_b}(u, u, fu)$. But then, using (66), we get that $G_{p_b}(u, u, fu) = sG_{p_b}(fu, fu, fu) \leq \beta(G_{p_b}(u, u, u))M(u, u, u) < G_{p_b}(u, u, fu)$, which is a contradiction. Hence, we have $fu = u$. Thus, u is a fixed point of f. □

Now we replace the continuity of f in Theorem 32 by the regularity of the space to get the required conclusion.

Theorem 33. *Under the same hypotheses of Theorem 32, instead of the continuity assumption of f, assume that, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Repeating the proof of Theorem 32, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 29 and (63)

$$\begin{aligned} s\left[\frac{1}{s}G_{p_b}(u, fu, fu) - G_{p_b}(u,u,u)\right] &\le s\limsup_{n\to\infty} G_{p_b}(x_{n+1}, fu, fu) \\ &\le \limsup_{n\to\infty}\left(\beta\left(G_{p_b}(x_n,u,u)\right)M(x_n,u,u)\right) \\ &\le \frac{1}{s}\limsup_{n\to\infty} M(x_n,u,u), \end{aligned} \tag{68}$$

where

$$\begin{aligned} \lim_{n\to\infty} M(x_n,u,u) &= \lim_{n\to\infty}\max\left\{G_{p_b}(x_n,u,u), \frac{\left[G_{p_b}(x_n,x_n,fx_n), G_{p_b}(u,u,fu)\right]^2}{1+\left[sG_{p_b}(fx_n,fu,fu)\right]^2}\right\} \\ &= \lim_{n\to\infty}\max\left\{G_{p_b}(x_n,x_n,u), \frac{\left[G_{p_b}(x_n,x_n,x_{n+1}), G_{p_b}(u,u,fu)\right]^2}{1+\left[sG_{p_b}(x_{n+1},fu,fu)\right]^2}\right\} \\ &= \max\left\{G_{p_b}(u,u,u), 0\right\} = 0 \quad \text{(see (55) and (63))}. \end{aligned} \tag{69}$$

Therefore, we deduce that $G_{p_b}(u, fu, fu) \le sG_{p_b}(u,u,u) = 0$. Hence, we have $u = fu$. □

If in the above theorems we assume $\beta(t) = r$, where $0 \le r \le 1/s$, we obtain the following corollary.

Corollary 34. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a G_{p_b}-metric on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \le rM(x,y,z) \tag{70}$$

for all comparable elements $x, y, z \in X$, where $0 \le r < 1/s$ and

$$M(x,y,z) = \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\}. \tag{71}$$

If f is continuous or for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 35. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a G_{p_b}-metric space G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$\begin{aligned} sG_{p_b}(fx, fy, fz) &\le aG_{p_b}(x,y,z) \\ &\quad + b\frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2} \end{aligned} \tag{72}$$

for all comparable elements $x, y, z \in X$, where $a, b \ge 0$ and $a + b \le 1/s$.

If f is continuous or for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Proof. Since

$$\begin{aligned} &aG_{p_b}(x,y,z) + b\frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2} \\ &\le (a+b) \\ &\times \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\}, \end{aligned} \tag{73}$$

taking $r = a + b$, all the conditions of Corollary 34 hold and hence f has a fixed point. □

Let Ψ be the family of all continuous and nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty}\psi^n(t) = 0 \tag{74}$$

for all $t > 0$. It is known that, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Theorem 36. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \le \psi(M(x,y,z)), \tag{75}$$

where

$$M(x,y,z) = \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\} \tag{76}$$

for all comparable elements $x, y, z \in X$. If f is continuous, then f has a fixed point.

Proof. Since $x_0 \preceq f(x_0)$ and f is an increasing function we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots. \tag{77}$$

Putting $x_n = f^n(x_0)$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{78}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, for all $n \in \mathbb{N}$, we assume that $x_n \neq x_{n+1}$.

Step 1. We will prove that

$$\lim_{n\to\infty} G_{p_b}(x_n, x_{n+1}, x_{n+2}) = 0. \tag{79}$$

Using condition (75), we obtain

$$\begin{aligned} G_{p_b}(x_n,x_{n+1},x_{n+2}) &\le sG_{p_b}(x_n,x_{n+1},x_{n+2}) \\ &= sG_{p_b}(fx_{n-1},fx_n,fx_{n+1}) \\ &\le \psi(M(x_{n-1},x_n,x_{n+1})). \end{aligned} \tag{80}$$

Here,

$$\begin{aligned} &M(x_{n-1},x_n,x_{n+1}) \\ &\quad= \max\Big\{G_{p_b}(x_{n-1},x_n,x_{n+1}), \\ &\qquad \times\left(G_{p_b}(x_{n-1},x_{n-1},fx_{n-1})\,G_{p_b}(x_n,x_n,fx_n)\right. \\ &\qquad \left.\times G_{p_b}(x_{n+1},x_{n+1},fx_{n+1})\right) \\ &\qquad \times\left(1+\left[G_{p_b}(fx_{n-1},fx_n,fx_{n+1})\right]\right)^{-1}\Big\} \\ &\quad= G_{p_b}(x_{n-1},x_n,x_{n+1}). \end{aligned} \tag{81}$$

Hence,

$$\begin{aligned} G_{p_b}(x_n,x_{n+1},x_{n+2}) &\le sG_{p_b}(x_n,x_{n+1},x_{n+2}) \\ &\le \psi\left(G_{p_b}(x_{n-1},x_n,x_{n+1})\right). \end{aligned} \tag{82}$$

By induction, we get that

$$\begin{aligned} G_{p_b}(x_{n+2},x_{n+1},x_n) &\le \psi\left(G_{p_b}(x_{n+1},x_n,x_{n-1})\right) \\ &\le \psi^2\left(G_{p_b}(x_n,x_{n-1},x_{n-2})\right) \le \cdots \\ &\le \psi^n\left(G_{p_b}(x_2,x_1,x_0)\right). \end{aligned} \tag{83}$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} G_{p_b}(x_n,x_{n+1},x_{n+2}) = 0. \tag{84}$$

Step 2. We will prove that $\{x_n\}$ is a G_{p_b}-Cauchy sequence.

Suppose to the contrary that $\{x_n\}$ is not a G_{p_b}-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad G_{p_b}\left(x_{m_i},x_{n_i},x_{n_i}\right) \ge \varepsilon. \tag{85}$$

This means that

$$G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) < \varepsilon. \tag{86}$$

From (85) and using the rectangle inequality, we get

$$\begin{aligned} \varepsilon \le G_{p_b}\left(x_{m_i},x_{n_i},x_{n_i}\right) &\le sG_{p_b}\left(x_{m_i},x_{m_i+1},x_{m_i+1}\right) \\ &\quad + sG_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right). \end{aligned} \tag{87}$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} G_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right). \tag{88}$$

From the definition of $M(x,y)$ we have

$$\begin{aligned} &M\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) \\ &\quad= \max\Big\{G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right), \\ &\qquad \left(G_{p_b}\left(x_{m_i},x_{m_i},fx_{m_i}\right)\right. \\ &\qquad \left.\times\left[G_{p_b}\left(x_{n_i-1},x_{n_i-1},fx_{n_i-1}\right)\right]^2\right) \\ &\qquad \times\left(1+\left[sG_{p_b}\left(fx_{m_i},fx_{n_i-1},fx_{n_i-1}\right)\right]^2\right)^{-1}\Big\} \\ &\quad= \max\Big\{G_{p_b}\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right), \\ &\qquad \left(G_{p_b}\left(x_{m_i},x_{m_i},x_{m_i+1}\right)\right. \\ &\qquad \left.\times\left[G_{p_b}\left(x_{n_i-1},x_{n_i-1},x_{n_i}\right)\right]^2\right) \\ &\qquad \times\left(1+\left[sG_{p_b}\left(x_{m_i+1},x_{n_i},x_{n_i}\right)\right]^2\right)^{-1}\Big\} \end{aligned} \tag{89}$$

and if $i \to \infty$, by (84) and (86), we have

$$\limsup_{i\to\infty} M\left(x_{m_i},x_{n_i-1},x_{n_i-1}\right) \le \varepsilon. \tag{90}$$

Now, from (75) we have

$$sG_{p_b}\left(x_{m_i+1}, x_{n_i}, x_{n_i}\right) = sG_{p_b}\left(fx_{m_i}, fx_{n_i-1}, fx_{n_i-1}\right) \le \psi\left(M\left(x_{m_i}, x_{n_i-1}, x_{n_{i-1}}\right)\right). \tag{91}$$

Again, if $i \to \infty$ by (88) we obtain

$$\varepsilon = s\left(\frac{\varepsilon}{s}\right) \le \left(s \limsup_{i\to\infty} G_{p_b}\left(x_{m_i+1}, x_{n_i}, x_{n_i}\right)\right) \le \psi(\varepsilon) < \varepsilon, \tag{92}$$

which is a contradiction. Consequently, $\{x_n\}$ is a G_{p_b}-Cauchy sequence in X. Thus, from Lemma 25 we have proved that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space $(X, d_{G_{p_b}})$. Since (X, G_{p_b}) is G_{p_b}-complete, then, from Lemma 25, $(X, d_{G_{p_b}})$ is a b-complete b-metric space. Therefore, the sequence $\{x_n\}$ b-converges to some $u \in X$; that is, $\lim_{n\to\infty} d_{G_{p_b}}(x_n, u) = 0$. Again, from Lemma 25 and (62),

$$\lim_{n\to\infty} G_{p_b}(u, x_n, x_n) = \lim_{m,n\to\infty} G_{p_b}(x_n, x_m, x_m) = G_{p_b}(u, u, u) = 0. \tag{93}$$

Step 3. Now we show that u is a fixed point of f. Suppose to the contrary, that $fu \ne u$; then, from Lemma 20, we have $G_{p_b}(u, u, fu) > 0$.

Using the rectangle inequality, we get

$$G_{p_b}(u, u, fu) \le sG_{p_b}(fu, fx_n, fx_n) + sG_{p_b}(fx_n, u, u). \tag{94}$$

Letting $n \to \infty$ and using the continuity of f, we get

$$G_{p_b}(u, u, fu) \le sG_{p_b}(fu, fu, fu). \tag{95}$$

Note that, from (75), we have

$$sG_{p_b}(fu, fu, fu) \le \psi(M(u, u, u)), \tag{96}$$

where

$$M(u,u,u) = \max\left\{G_{p_b}(u,u,u), \frac{G_{p_b}(u,u,fu)\,G_{p_b}(u,u,fu)\,G_{p_b}(u,u,fu)}{1+\left[sG_{p_b}(fu,fu,fu)\right]^2}\right\} \le G_{p_b}(u,u,fu). \tag{97}$$

Hence, as ψ is nondecreasing, we have $sG_{p_b}(fu, fu, fu) \le G_{p_b}(u, u, fu)$. Thus, by (95) we obtain that

$$G_{p_b}(u, u, fu) = sG_{p_b}(fu, fu, fu). \tag{98}$$

Equation (96) yields that $G_{p_b}(u, u, fu) \le \psi(M(u, u, u)) \le \psi(G_{p_b}(u, u, fu))$. This is impossible, according to our assumptions on ψ. Hence, we have $fu = u$. Thus, u is a fixed point of f. □

Theorem 37. *Under the hypotheses of Theorem 36, instead of the continuity assumption of f, assume that, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Following the proof of Theorem 36, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$. Using the given assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By (75) we have

$$sG_{p_b}(fu, fu, x_n) = sG_{p_b}(fu, fu, fx_{n-1}) \le \psi(M(u, u, x_{n-1})), \tag{99}$$

where

$$M(u,u,x_{n-1}) = \max\left\{G_{p_b}(u,u,x_{n-1}), \frac{\left[G_{p_b}(u,u,fu)\right]^2 G_{p_b}(x_{n-1},x_{n-1},fx_{n-1})}{1+\left[sG_{p_b}(fu,fu,fx_{n-1})\right]^2}\right\}. \tag{100}$$

Letting $n \to \infty$ in the above, from (93), we get

$$\lim_{n\to\infty} M(u, u, x_{n-1}) = 0. \tag{101}$$

Again, taking the upper limit as $n \to \infty$ in (99) and using Lemma 29 and (101) we get

$$s\left[\frac{1}{s}G_{p_b}(u, fu, fu) - G_{p_b}(u,u,u)\right] \le s\limsup_{n\to\infty} G_{p_b}(x_n, fu, fu) \le \limsup_{n\to\infty} \psi(M(u,u,x_{n-1})) = 0. \tag{102}$$

So we get $G_{p_b}(u, fu, fu) = 0$. That is, $fu = u$. □

Corollary 38. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a generalized partial b-metric G_{p_b} on X such that (X, G_{p_b}) is a G_{p_b}-complete G_{p_b}-metric space, and let $f : X \to X$ be an increasing mapping with $x_0 \preceq f(x_0)$ for some $x_0 \in X$. Suppose that*

$$sG_{p_b}(fx, fy, fz) \le kM(x, y, z), \tag{103}$$

where $0 \le k < 1/s$ and

$$M(x,y,z) = \max\left\{G_{p_b}(x,y,z), \frac{G_{p_b}(x,x,fx)\,G_{p_b}(y,y,fy)\,G_{p_b}(z,z,fz)}{1+\left[sG_{p_b}(fx,fy,fz)\right]^2}\right\} \tag{104}$$

for all comparable elements $x, y, z \in X$. If f is continuous or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$, we have $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

We conclude this section by presenting some examples that illustrate our results.

Example 39. Let $X = [0,1]$ be equipped with the usual order and G_{p_b}-metric function G_{p_b} given by $G_{p_b}(x,y,z) = [\max\{x,y,z\}]^2 = \max\{x^2,y^2,z^2\}$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = (1/4)x(e^{-x^2})^{1/2}$ and the function $\beta \in \mathcal{F}$ given by $\beta(t) = (1/2)e^{-t}$, $t > 0$, and $\beta(0) \in [0, 1/2)$. It is easy to see that f is an increasing function on X and $0 \le f(0) = 0$. We show that f is G_{p_b}-continuous on X. By Proposition 27 it is sufficient to show that f is G_{p_b} sequentially continuous on X. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G_{p_b}(x_n,x,x) = G_{p_b}(x,x,x)$, so we have $\lim_{n\to\infty}\max\{x_n^2,x^2\} = x^2$, equally $\max\{\lim_{n\to\infty}x_n^2, x^2\} = x^2$, and hence $\lim_{n\to\infty}x_n^2 = \alpha \le x^2$. On the other hand we have

$$
\begin{aligned}
&\lim_{n\to\infty} G_{p_b}(fx_n, fx, fx)\\
&\quad = \lim_{n\to\infty}\max\left\{(fx_n)^2,(fx)^2\right\}\\
&\quad = \lim_{n\to\infty}\max\left\{\frac{1}{16}x_n^2e^{-x_n^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = \max\left\{\frac{1}{16}\lim_{n\to\infty}x_n^2e^{-x_n^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = \max\left\{\frac{1}{16}\alpha e^{-\alpha}, \frac{1}{16}x^2e^{-x^2}\right\} = \frac{1}{16}x^2e^{-x^2}\\
&\quad = \max\left\{\frac{1}{16}x^2e^{-x^2}, \frac{1}{16}x^2e^{-x^2}, \frac{1}{16}x^2e^{-x^2}\right\}\\
&\quad = G_{p_b}(fx, fx, fx).
\end{aligned}
\tag{105}
$$

So f is G_{p_b} sequentially continuous on X.

For all comparable elements $x, y, z \in X$ and the fact that $g(x) = x^2e^{-x^2}$ is an increasing function on X we have

$$
\begin{aligned}
sG_{p_b}(fx, fy, fz) &= 2\max\left\{\frac{1}{16}x^2e^{-x^2}, \frac{1}{16}y^2e^{-y^2}, \frac{1}{16}z^2e^{-z^2}\right\}\\
&= \frac{1}{8}\max\left\{x^2e^{-x^2}, y^2e^{-y^2}, z^2e^{-z^2}\right\}\\
&= \frac{1}{8}e^{-\max\{x^2,y^2,z^2\}}\max\left\{x^2,y^2,z^2\right\}\\
&\le \frac{1}{2}e^{-\max\{x^2,y^2,z^2\}}\max\left\{x^2,y^2,z^2\right\}\\
&= \beta\left(G_{p_b}(x,y,z)\right)G_{p_b}(x,y,z)\\
&\le \beta\left(G_{p_b}(x,y,z)\right)M(x,y,z).
\end{aligned}
\tag{106}
$$

Hence, f satisfies all the assumptions of Theorem 32 and thus it has a fixed point (which is $u = 0$).

Example 40. Let $X = [0,1]$ be equipped with the usual order and G_{p_b}-metric function G_{p_b} given by $G_{p_b}(x,y,z) = [\max\{x,y,z\}]^2 = \max\{x^2,y^2,z^2\}$ with $s = 2$. Consider the mapping $f : X \to X$ defined by $f(x) = (1/4)\sqrt{\ln(x^2+1)}$ and the function $\psi \in \Psi$ given by $\psi(t) = (1/8)t$, $t \ge 0$. It is easy to see that f is increasing function and $0 \le f(0) = 0$. Now we show that f is G_{p_b}-continuous function on X.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}G_{p_b}(x_n,x,x) = G_{p_b}(x,x,x)$, so we have $\lim_{n\to\infty}\max\{x_n^2,x^2\} = x^2$, equally $\max\{\lim_{n\to\infty}x_n^2,x^2\} = x^2$, and hence $\lim_{n\to\infty}x_n^2 = \alpha \le x^2$. On the other hand we have

$$
\begin{aligned}
&\lim_{n\to\infty} G_{p_b}(fx_n, fx, fx)\\
&\quad = \lim_{n\to\infty}\max\left\{(fx_n)^2,(fx)^2\right\}\\
&\quad = \lim_{n\to\infty}\max\left\{\frac{1}{16}\ln\left(x_n^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = \max\left\{\frac{1}{16}\ln\left(\lim_{n\to\infty}x_n^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = \max\left\{\frac{1}{16}\ln(\alpha+1), \frac{1}{16}\ln\left(x^2+1\right)\right\} = \frac{1}{16}\ln\left(x^2+1\right)\\
&\quad = \max\left\{\frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(x^2+1\right)\right\}\\
&\quad = G_{p_b}(fx, fx, fx).
\end{aligned}
\tag{107}
$$

So f is G_{p_b}-sequentially continuous on X.

For all comparable elements $x, y, z \in X$, we have

$$
\begin{aligned}
&sG_{p_b}(fx, fy, fz)\\
&\quad = 2\max\left\{\left(\frac{1}{4}\sqrt{\ln(x^2+1)}\right)^2, \left(\frac{1}{4}\sqrt{\ln(y^2+1)}\right)^2,\right.\\
&\qquad\qquad \left.\left(\frac{1}{4}\sqrt{\ln(z^2+1)}\right)^2\right\}\\
&\quad = 2\max\left\{\frac{1}{16}\ln\left(x^2+1\right), \frac{1}{16}\ln\left(y^2+1\right), \frac{1}{16}\ln\left(z^2+1\right)\right\}\\
&\quad = \frac{1}{8}\max\left\{\ln\left(x^2+1\right), \ln\left(y^2+1\right), \ln\left(z^2+1\right)\right\}\\
&\quad \le \frac{1}{8}\max\left\{x^2,y^2,z^2\right\} = \psi\left(G_{p_b}(x,y,z)\right)\\
&\quad \le \psi(M(x,y,z)).
\end{aligned}
\tag{108}
$$

Hence, f satisfies all the assumptions of Theorem 36 and thus it has a fixed point (which is $u = 0$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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