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Research Article

Maps Preserving Peripheral Spectrum of Generalized Jordan Products of Self-Adjoint Operators

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Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. For $k \geq 2$, let $(i_1, \ldots, i_m)$ be a fixed sequence with $i_1, \ldots, i_m \in \{1, \ldots, k\}$ and assume that at least one of the terms in $(i_1, \ldots, i_m)$ appears exactly once. Define the generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k = T_{i_1} T_{i_2} \cdots T_{i_m} + T_{i_m} \cdots T_{i_2} T_{i_1}$ on elements in $\mathscr{A}_i$. Let $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ be a map with the range containing all rank-one projections and trace zero-rank two self-adjoint operators. We show that Φ satisfies that $\sigma_\pi(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma_\pi(A_1 \circ \cdots \circ A_k)$ for all $A_1, \ldots, A_k$, where $\sigma_\pi(A)$ stands for the peripheral spectrum of A, if and only if there exist a scalar $c \in \{-1, 1\}$ and a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for all $A \in \mathscr{A}_1$, where A^t is the transpose of A for an arbitrarily fixed orthonormal basis of H_1. Moreover, $c = 1$ whenever m is odd.

1. Introduction

Recently, the study of maps preserving spectrum of products of operators attracted attentions of researchers. In [1], Molnár characterized surjective maps Φ on bounded linear operators acting on a Hilbert space preserving the spectrum of the product of operators; that is, AB and $\Phi(A)\Phi(B)$ always have the same spectrum. This similar question was studied by Huang and Hou in [2] by replacing the spectrum by several spectral functions such as the left spectrum and spectral boundary. Hou et al. [3, 4] studied, respectively, further the maps Φ between certain operator algebras preserving the spectrum of a generalized product $T_1 * T_2 * \cdots * T_k$ and a generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k$ of low rank operators. Note that the linear maps between Banach algebras which preserve the spectrum are extensively studied in connection with a longstanding open problem due to Kaplansky on invertibility preserving linear maps ([5–10] and the references therein).

Moreover, there has been considerable interest in studying peripheral spectrum preserving maps on operator algebras. Recall that the peripheral spectrum of an element T in a complex Banach algebra $\mathscr{A}$ is defined by

$$\sigma_\pi(T) = \{z \in \sigma(T) : |z| = r(T)\}, \tag{1}$$

where $\sigma(T)$ and $r(T)$ stand for the spectrum and the spectral radius of T, respectively. Recall also that a set-valued map $\Lambda : \mathscr{A} \to 2^{\mathbb{C}}$ is said to be a spectral function if $\emptyset \neq \Lambda(T) \subseteq \sigma(T)$ for every $T \in \mathscr{A}$. Since $\sigma(T)$ is compact, $\sigma_\pi(T)$ is a well-defined nonempty set and is an important spectral function. Observe that it is always true that $\sigma_\pi(TS) = \sigma_\pi(ST)$. In [11], Tonev and Luttman studied maps preserving peripheral spectrum of the usual operator products on standard operator algebras. Recall that a standard operator algebra is a subalgebra of $\mathscr{B}(X)$ that contains the identity I and all finite rank operators, where $\mathscr{B}(X)$ stands for as usual the Banach algebra of all bounded linear operators on Banach space X. They studied also the corresponding problems in uniform algebras (see [12, 13]). Miura and Honma [14] generalized the result in [13] and characterized surjective maps ϕ and ψ satisfying $\sigma_\pi(\phi(T)\psi(S)) = \sigma_\pi(TS)$ on standard operator algebras. Cui and Li studied in [15] the maps preserving peripheral spectrum of Jordan products $AB+BA$ of operators on standard operator algebras. In [16] the maps preserving peripheral spectrum of Jordan semitriple products BAB of operators were characterized. The authors studied in [17, 18], respectively, further the maps between certain operator algebras which preserve peripheral spectrum of a generalized product $T_1 * T_2 * \cdots * T_k$ and a generalized Jordan product $T_1 \circ T_2 \circ \cdots \circ T_k$ as defined below.

Definition 1. Fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, \ldots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal to i_q for all other q; that is, i_p appears just one time in the sequence. For operators $T_1, \ldots, T_k$, the operators,

$$T_1 * T_2 * \cdots * T_k = T_{i_1} T_{i_2} \cdots T_{i_m}, \tag{2}$$

$$T_1 \circ T_2 \circ \cdots \circ T_k = T_{i_1} T_{i_2} \cdots T_{i_m} + T_{i_m} \cdots T_{i_2} T_{i_1} \tag{3}$$

are, respectively, called generalized product and generalized Jordan product of $T_1, \ldots, T_k$, while m is called the width of the products.

Evidently, the generalized Jordan product $T_1 \circ \cdots \circ T_k$ (the generalized product $T_1 * \cdots * T_k$) covers the Jordan product $T_1T_2 + T_2T_1$ and the Jordan triple product $T_1T_2T_3 + T_3T_2T_1$ (the usual product T_1T_2 and the Jordan semitriple product $T_1T_2T_1$), and so forth. We also remark that the notations $T_1 * T_2 * \cdots * T_k$ and $T_1 \circ T_2 \circ \cdots \circ T_k$ are not unique for $T_1, T_2, \ldots, T_k$ because they depend on the choice of the integers $k \geq 2$, $m \geq 2$, and the sequence $(i_1, i_2, \ldots, i_m)$. In this paper, we presume that k, m, and the sequence $(i_1, i_2, \ldots i_m)$ are arbitrary but fixed throughout the paper.

Let us consider the case of Hilbert spaces. Denote by $\mathscr{B}(H)$ the set of all bounded linear operators on a complex Hilbert space H and T^* the adjoint of $T \in \mathscr{B}(H)$. If $T = T^*$, T is self-adjoint. Denote by $\mathscr{B}_s(H)$ the real Jordan algebra of all self-adjoint operators in $\mathscr{B}(H)$. A real Jordan subalgebra of $\mathscr{B}_s(H)$ is said to be standard if it contains the identity I and all finite rank self-adjoint operators. In [14] Miura and Honma characterized the surjective maps between standard operator algebras on Hilbert spaces that preserve the peripheral spectrum of skew products T^*S of operators. Cui and Li studied in [15] the maps preserving peripheral spectrum of skew Jordan products $AB^* + B^*A$ of operators on standard operator algebras on complex Hilbert spaces. A characterization of maps preserving peripheral spectrum of Jordan products of self-adjoint operators $AB+BA$ on standard real Jordan subalgebras of $\mathscr{B}_s(H)$ was also given in [15]. In [16] the maps preserving peripheral spectrum of Jordan skew semitriple products BA^*B of operators were characterized, and then, the maps preserving peripheral spectrum of the skew generalized products of operators on Hilbert space H were characterized in [17].

Products of self-adjoint operators in Hilbert space play a role in several different areas of pure and applied mathematics. In this paper, we characterize the maps preserving the peripheral spectrum of generalized Jordan products of self-adjoint operators between the standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces. Let $\mathscr{A}_i$ be a standard real Jordan algebra in $\mathscr{B}_s(H_i)$, $i = 1, 2$, and $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ a map with range containing all rank-one projections and all rank-two self-adjoint operators with zero trace. We show that Φ satisfies that $\sigma_\pi(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma_\pi(A_1 \circ \cdots \circ A_k)$ for all $A_1, \ldots, A_k$ in $\mathscr{A}_1$ if and only if there exist a scalar $c \in \{-1, 1\}$ and a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = cUAU^*$ for all $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for all $A \in \mathscr{A}_1$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1. Moreover, $c = 1$ whenever m is odd. We also characterize the maps from $\mathscr{A}_1$ into $\mathscr{A}_2$ that preserves the peripheral spectrum of generalized product on $\mathscr{A}_i$.

2. Generalized Jordan Products of Self-Adjoint Operators

Let H_1 and H_2 be two complex Hilbert spaces and $\mathscr{B}_s(H_1)$ and $\mathscr{B}_s(H_2)$ the real linear spaces of all self-adjoint operators in $\mathscr{B}(H_1)$ and $\mathscr{B}(H_2)$, respectively. Then $\mathscr{B}_s(H_1)$ and $\mathscr{B}_s(H_2)$ are real Jordan algebras. Recall that a standard real Jordan algebra on H_i is a Jordan subalgebra of $\mathscr{B}_s(H_i)$ which contains all finite rank self-adjoint operators and the identity operator. In this section, we will characterize maps preserving peripheral spectrum of generalized Jordan products of self-adjoint operators.

Theorem 2. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Consider the product $T_1 \circ \cdots \circ T_k$ defined in* (3) *of Definition 1 with the width m. Assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map the range of which contains all rank-one projections and all rank-two self-adjoint operators with zero trace. Then Φ satisfies*

$$\sigma_\pi(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma_\pi(A_1 \circ \cdots \circ A_k) \tag{4}$$

for all $A_1, A_2, \ldots, A_k \in \mathscr{A}_1$ if and only if there exist a unitary operator $U \in \mathscr{B}(H_1, H_2)$ and a scalar $c \in \{-1, 1\}$ such that either

(1) *$\Phi(A) = cUAU^*$ for every $A \in \mathscr{A}_1$, or*

(2) *$\Phi(A) = cUA^tU^*$ for every $A \in \mathscr{A}_1$. Here A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1.*

Moreover, $c = 1$ whenever m is odd.

To prove Theorem 2, we consider the special case that $A_{i_p} = A$ and $A_{i_q} = B$ for all $q \neq p$. Thus there exist nonnegative integers r, s with $r + s = m - 1 \geq 1$ such that $A_1 \circ A_2 \circ \cdots \circ A_k = B^rAB^s + B^sAB^r$. For this special case we have.

Theorem 3. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map the range of which contains all rank-one projections and all rank-two* *self-adjoint operators with zero trace, and r, s are nonnegative integers with $r + s \geq 1$. Then Φ satisfies*

$$\sigma_\pi \left(B^r A B^s + B^s A B^r\right) = \sigma_\pi \left(\Phi(B)^r \Phi(A) \Phi(B)^s + \Phi(B)^s \Phi(A) \Phi(B)^r\right) \quad (5)$$

for all $A, B \in \mathscr{A}_1$ if and only if there exist a unitary operator $U \in \mathscr{B}(H_1, H_2)$ and a scalar $c \in \{-1, 1\}$ such that $\Phi(A) = cUAU^*$ for every $A \in \mathscr{A}_1$, or $\Phi(A) = cUA^tU^*$ for every $A \in \mathscr{A}_1$. Moreover, $c = 1$ whenever $r + s$ is even. Here A^t is the transpose of A with respect to an arbitrary but fixed orthonormal basis of H_1.

If Φ meets (4), then it also meets (5) for some r, s with $r + s = m - 1$ by taking $A_{i_p} = A$ and $A_{i_q} = B$ for $q \neq p$. Hence it is obvious that the truth of Theorem 3 will imply the truth of Theorem 2.

Thus we focus our attention to prove Theorem 3. We will do it by decomposing the proof in a number of steps and use of technical lemmas.

Note that, if $s = r > 0$, then the question is reduced to the generalized product $B^r A B^r$ of self-adjoint operators, which will be discussed in the next section. So, unless specified otherwise, we always assume in this section that $s > r \geq 0$.

Lemma 4. *For any unit vector $x \in H$ and nonzero $B \in \mathscr{B}_s(H)$, we have*

$$\sigma_\pi \left(B^r x \otimes x B^s + B^s x \otimes x B^r\right) = \begin{cases} \{\langle B^{r+s} x, x\rangle + \|B^r x\| \|B^s x\|\} & \text{if } \langle B^{r+s} x, x\rangle > 0; \\ \{\langle B^{r+s} x, x\rangle - \|B^r x\| \|B^s x\|\} & \text{if } \langle B^{r+s} x, x\rangle < 0; \\ \{\pm \|B^r x\| \|B^s x\|\} & \text{if } \langle B^{r+s} x, x\rangle = 0. \end{cases} \quad (6)$$

Proof. In fact, if there exist nonzero $\alpha, \beta \in \mathbb{R}$ such that $B^r x = \alpha x$, $B^s x = \beta x$, clearly (6) holds. Now assume that $B^r x$ and x or $B^s x$ and x are linearly independent. Then there exist nonzero $\gamma \in \mathbb{R}$ and $z \in H$ such that $(B^r x \otimes x B^s + B^s x \otimes x B^r) z = \gamma z$; that is,

$$\langle B^s z, x\rangle B^r x + \langle B^r z, x\rangle B^s x = \gamma z. \quad (7)$$

It follows that

$$\langle B^s z, x\rangle \langle B^r x, x\rangle + \langle B^r z, x\rangle \langle B^s x, x\rangle = \gamma \langle z, x\rangle, \quad (8)$$

$$\langle B^s z, x\rangle \langle B^{r+s} x, x\rangle + \langle B^r z, x\rangle \langle B^{2s} x, x\rangle = \gamma \langle B^s z, x\rangle, \quad (9)$$

$$\langle B^r z, x\rangle \langle B^{r+s} x, x\rangle + \langle B^s z, x\rangle \langle B^{2r} x, x\rangle = \gamma \langle B^r z, x\rangle. \quad (10)$$

We consider the following two cases.

Case 1 ($\langle B^{r+s} x, x\rangle = 0$). If $\langle B^r z, x\rangle \neq 0$, it follows from (10) that $\langle B^s z, x\rangle \neq 0$. Then (9) and (10) imply that $\gamma = \pm \|B^r x\| \|B^s x\|$. If $\langle B^r z, x\rangle = 0$, it follows from (9) that $\langle B^s z, x\rangle = 0$, but this contradicts (7). So $\sigma_\pi(B^r x \otimes x B^s + B^s x \otimes x B^r) = \{\pm \|B^r x\| \|B^s x\|\}$.

Case 2 ($\langle B^{r+s} x, x\rangle \neq 0$). In this case, there must be $\langle B^r z, x\rangle \neq 0$ and $\langle B^s z, x\rangle \neq 0$. Then it follows from (9) and (10) that

$$\left(\gamma - \langle B^{r+s} x, x\rangle\right)^2 = \|B^r x\|^2 \|B^s x\|^2, \quad (11)$$

which implies that $\gamma = \langle B^{r+s} x, x\rangle \pm \|B^r x\| \|B^s x\|$. So

$$\sigma \left(B^r x \otimes x B^s + B^s x \otimes x B^r\right) = \{0, \langle B^{r+s} x, x\rangle \pm \|B^r x\| \|B^s x\|\}. \quad (12)$$

Now the result follows immediately. □

In Lemmas 5 and 6, we always assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map satisfying (5) with range containing all rank-one projections and all rank-two self-adjoint operators of zero trace, and assume that r, s are nonnegative integers with $r + s \geq 1$. Recall that a self-adjoint operator A is said to be positive, denote by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$; while $A \geq B$ means that $A - B \geq 0$.

Lemma 5. *$\Phi(I) = I$ or $-I$. $\Phi(I) = -I$ may occur only if $r + s$ is odd.*

Proof. For any $A, B \in \mathscr{A}_1$, since

$$\sigma_\pi \left(B^r A B^s + B^s A B^r\right) = \sigma_\pi \left(\Phi(B)^r \Phi(A) \Phi(B)^s + \Phi(B)^s \Phi(A) \Phi(B)^r\right), \quad (13)$$

it follows that $r(A) = r(\Phi(A))$ holds for every $A \in \mathscr{A}_1$. Let $\Phi(I) = B$. By the assumption on the range of Φ, for any unit vector $y \in H_2$, there exists $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. We consider the following two cases.

Case 1 ($s > r = 0$). It follows from (5) that

$$\sigma_\pi \left(2A^s\right) = \sigma_\pi \left(\Phi(A)^s \Phi(I) + \Phi(I) \Phi(A)^s\right) = \sigma_\pi \left(y \otimes yB + By \otimes y\right), \quad (14)$$

which implies that $\|By \otimes y + y \otimes yB\| = 2$ for all unit vectors $y \in H_2$. Then by Lemma 4, we have

$$2 = \|By \otimes y + y \otimes yB\| = |\langle By, y\rangle| + \|By\| \leq 2 \|By\|, \quad (15)$$

and hence $\|By\| \geq 1$ for all unit vectors $y \in H_2$, and $\|B\| \geq 1$. On the other hand, for any unit vector $y \in H_2$, we have $2|\langle By, y\rangle| \leq |\langle By, y\rangle| + \|By\| = 2$. Hence $|\langle By, y\rangle| \leq 1$ holds for all unit vectors $y \in H_2$ and consequently, $\|B\| \leq 1$. So we must have $\|B\| = 1$ and $\|By\| = 1$ for all unit vectors $y \in H_2$. Now it follows from (15) that $B = \varepsilon I$ with $\varepsilon \in \{-1, 1\}$. In particular, if s is even, as $A^s \geq 0$, (14) and (15) imply that $\langle By, y\rangle = 1$ for all unit vectors $y \in H_2$ and hence $B = I$.

Case 2 ($s > r > 0$). By (5) we have

$$\sigma_\pi \left(2A^{r+s}\right) = \sigma_\pi \left(\Phi(A)^s \Phi(I) \Phi(A)^r + \Phi(A)^r \Phi(I) \Phi(A)^s\right) = \sigma_\pi \left(2y \otimes yBy \otimes y\right), \quad (16)$$

which implies that $\|y \otimes yBy \otimes y\| = \|A^{r+s}\| = 1$ for all unit vectors $y \in H_2$. Then $|\langle By, y\rangle| = 1$ holds for each unit vector y, and so $B = \varepsilon I$ with $\varepsilon \in \{-1, 1\}$. Particularly, if $r+s$ is even, then $A^{r+s} \geq 0$ and it follows from (16) that $\langle By, y\rangle = 1$ holds for every unit vector y. Hence $B = I$. □

If $\Phi(I) = -I$, then $-\Phi$ satisfies the conditions in Theorem 3, so we may as well assume $\Phi(I) = I$ in the sequel, and thus $\sigma_\pi(A) = \sigma_\pi(\Phi(A))$ holds for every $A \in \mathscr{A}_1$.

Lemma 6. *Φ preserves rank-one projections in both directions.*

Proof. We consider the following two cases.

Case 1 ($s > r = 0$). Consider the following.

Case 1.1 (s is even). For any unit vector $x \in H_1$, let $\Phi(x \otimes x) = B$ and $\Phi(I - x \otimes x) = T$. It follows from $\{0\} = \sigma_\pi((x \otimes x)^s(I - x\otimes x) + (I - x\otimes x)(x\otimes x)^s) = \sigma_\pi(B^sT + TB^s)$ that $B^sT + TB^s = 0$.

Note that if $A \geq 0$, and $AS + SA = 0$, then $AS = SA = 0$. If fact $AS + SA = 0$ implies that $A^2S = SA^2$. Since $A \geq 0$, we must have $AS = SA$ and $2AS = AS + SA = 0$, which forces $AS = SA = 0$.

Now, as $B^s \geq 0$ and $B^sT + TB^s = 0$, we see that $B^sT = TB^s = 0$. It follows from $\sigma_\pi(B) = \{1\}$ that $B^s \neq 0$, which implies that $\{0\} \neq \operatorname{ran} B^s \subseteq \ker T$, where $\operatorname{ran} T$ stands for the range of T. For any unit vector $y \in \ker T$, pick $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. It follows from $\sigma_\pi((I - x\otimes x)A^s + A^s(I - x\otimes x)) = \sigma_\pi(Ty\otimes y + y\otimes yT) = \{0\}$ that $(I - x\otimes x)A^s + A^s(I - x\otimes x) = 0$, which, together with $A^s \geq 0$, implies that $(I - x \otimes x)A^s = A^s(I - x \otimes x) = 0$. So we have $A = x \otimes x$ and $\Phi(x \otimes x) = y \otimes y$ is rank-one.

Case 1.2 (s is odd). For any unit vector $x \in H_1$, let $A = x \otimes x$ and $\Phi(A) = B$. We will prove that B is a rank-one projection.

Claim 1.2.1 ($\dim \ker(B - I) = 1$). Note that $\sigma_\pi(B) = \sigma_\pi(A) = \{1\}$. Then $1 \in \sigma(B) \subseteq (-1, 1]$. It follows that either (i) $\dim \ker(B - I) \geq 1$ or (ii) $B - I$ is injective but not surjective.

Assume that (ii) occurs. Since $1 \in \sigma_\pi(B)$, we have $\|B\| = 1$ and $B \leq I$. So, according to some suitable space decomposition of H_2, B has an operator matrix representation of the form

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & c & 0 \\ b & 0 & * & * & * \\ 0 & c & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \tag{17}$$

where $a > 1/2$ and $b, c \geq 0$. To see this, one can first choose three orthonormal vectors x_1, x_2, x_3 such that $1 - d < \langle Bx_j, x_j\rangle < 1$ for some sufficiently small $d \in (0, 1/4)$. Suppose the compression $\widehat{B}$ of B onto the span of $\{x_1, x_2, x_3\}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$. Then

$$\mu_2 \geq \frac{(\mu_2 + \mu_3)}{2} \geq \frac{[(3 - 3d) - 1]}{2} > \frac{1}{2}. \tag{18}$$

Let $\mu_2 = a$. Then $\widehat{B}$ is similar to

$$\begin{pmatrix} a & 0 & * \\ 0 & a & * \\ * & * & * \end{pmatrix}. \tag{19}$$

Thus, there exists a space decomposition such that B has an operator matrix of the form

$$\begin{pmatrix} aI_2 & B_{12} \\ B_{12}^* & * \end{pmatrix}. \tag{20}$$

Clearly, there are unitary U, V such that $UB_{12}V^*$ has operator matrix of the form

$$\begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \end{pmatrix}, \tag{21}$$

where $b, c \geq 0$. So B has the desired operator matrix form. Under the same decomposition, take $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$; then $\sigma_\pi(BS^s + S^sB)$ has two different points with $r(BS^s + S^sB) \geq 2a > 1$ and there exists $R \in \mathscr{A}_1$ such that $\Phi(R) = S$. It follows that $\sigma_\pi(R) = \{-1, 1\}$. So $\|R\| = 1$ and $\|R^s u\| \leq 1$ for all unit vectors $u \in H_1$. But $\sigma_\pi(AR^s + R^sA) = \sigma_\pi(x \otimes xR^s + R^sx \otimes x)$ is either a singleton or $\{\pm\|R^sx\|\}$ with $\|R^sx\| \leq 1$. This contradicts the fact $r(AR^s + R^sA) = r(BS^s + S^sB) \geq 2a > 1$.

So $\dim \ker(B - I) \geq 1$. Assume that $\dim \ker(B - I) = n \geq 2$. According to the space decomposition $H_2 = \ker(B - I) \oplus \ker(B - I)^\perp$, B has an operator matrix $I_n \oplus N$. Under the same space decomposition, take $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$. Similar to the previous discussion, one gets a contradiction again. So $\dim \ker(B - I) = 1$.

Claim 1.2.2. There exists a unit vector $y \in H_2$ such that $B = y \otimes y$.

If it is not true, then, by Claim 1.2.1, there exist a unit vector $y \in \ker(B - I)$ and a nonzero $B_2 \in \mathscr{A}_2$ with $B_2 y = 0$ such that $B = y \otimes y + B_2$. So there exists a unit vector $z \in [y]^\perp$ such that $B_2 z \neq 0$. Let $C_1 = y \otimes y$ and $C_2 = z \otimes z$. Then $\sigma_\pi(BC_1^s + C_1^sB) = \sigma_\pi(BC_1 + C_1B) = \{2\}$, $C_1C_2^s + C_2^sC_1 = 0$, and $BC_2^s + C_2^sB \neq 0$. Since the range of Φ contains all rank-one projections, there exist D_1 and D_2 in $\mathscr{A}_1$ such that $\Phi(D_1) = C_1$ and $\Phi(D_2) = C_2$. Then $\sigma_\pi(D_1) = \sigma_\pi(D_2) = \{1\}$, $\sigma_\pi(AD_1^s + D_1^sA) = \sigma_\pi(BC_1^s + C_1^sB) = \{2\}$, $D_1D_2^s + D_2^sD_1 = 0$, and $AD_2^s + D_2^sA \neq 0$.

Since $\{2\} = \sigma_\pi(AD_1^s + D_1^sA) = \sigma_\pi(x \otimes xD_1^s + D_1^sx \otimes x)$, it follows from (6) that $|\langle D_1^sx, x\rangle| + \|D_1^sx\| = 2$, which, together with $\|D_1\| = 1$, implies that $D_1^sx = x$. So, according to the space decomposition $H_1 = [x] \oplus [x]^\perp$, $D_1^s = [1] \oplus Z$ with $\sigma(Z) \subseteq (-1, 1]$. If D_2 has an operator matrix $\left(\begin{smallmatrix} v_{11} & V_{12} \\ V_{12}^* & V_{22} \end{smallmatrix}\right)$ accordingly, then

$$0 = D_1^sD_2 + D_2D_1^s = \begin{pmatrix} 2v_{11} & V_{12} + V_{12}Z \\ ZV_{12}^* + V_{12}^* & ZV_{22} + V_{22}Z \end{pmatrix}. \tag{22}$$

Since $I + Z$ is invertible, we see that $V_{12} = 0$. Clearly, $v_{11} = 0$. So, $D_2 = 0 \oplus V_{22}$. But then it contradicts the fact that $AD_2^s + D_2^sA \neq 0$. So Claim 1.2.2 holds and Φ preserves rank-one projections.

Conversely, assume that $\Phi(A)$ is a rank-one projection; then a similar discussion shows that A is a rank-one projection, too.

Case 2 ($s > r > 0$). Consider the following.

Case 2.1 ($r+s$ is even). For any unit vector $x \in H_1$, let $\Phi(x\otimes x) = B$ and $\Phi(I - x\otimes x) = T$. It follows from $\{0\} = \sigma_\pi((I - x\otimes x)^r(x\otimes x)(I - x\otimes x)^s + (I - x\otimes x)^s(x\otimes x)(I - x\otimes x)^r) = \sigma_\pi(T^rBT^s + T^sBT^r)$ that $T^rBT^s + T^sBT^r = 0$. Since $T^{s-r} \geq 0$ and $T^rBT^rT^{s-r} + T^{s-r}T^rBT^r = T^rBT^s + T^sBT^r = 0$, we see that $T^sBT^r = T^rBT^s = 0$. If $\ker T = \{0\}$, then $\ker T^r = \{0\}$ and $\ker T^s = \{0\}$, which, together with $T^rBT^s = 0$, imply that $BT^s = 0$, and thus $B = 0$, a contradiction.

So, $\ker T \neq \{0\}$. Take a unit vector $y \in \ker T$ and $A \in \mathscr{A}_1$ such that $\Phi(A) = y\otimes y$. It follows from $\sigma_\pi(A^r(I - x\otimes x)A^s + A^s(I - x\otimes x)A^r) = \sigma_\pi(2y\otimes yTy\otimes y) = \{0\}$ that $A^r(I - x\otimes x)A^s + A^s(I - x\otimes x)A^r = 0$, which, together with $A^{s-r} \geq 0$, implies that $A^r(I - x\otimes x)A^s = A^s(I - x\otimes x)A^r = 0$. Hence we have $A = x\otimes x$ and $\Phi(x\otimes x) = y\otimes y$.

Case 2.2 ($r+s$ is odd). For any unit vectors $x \in H_1$, let $A = x\otimes x$ and $\Phi(A) = B$. We will prove that B is a rank-one projection.

Claim 2.2.1 ($\dim\ker(B - I) = 1$). Note that $\sigma_\pi(B) = \sigma_\pi(A) = \{1\}$. Then $1 \in \sigma(B) \subseteq (-1, 1]$. It follows that either (i) $\dim\ker(B - I) \geq 1$ or (ii) $B - I$ is injective but not surjective.

Assume that (ii) occurs. Since $1 \in \sigma_\pi(B)$, we have $\|B\| = 1$ and $B \leq I$. So, like shown in Case 1.2.1, with respect to some suitable space decomposition of H_2, B has an operator matrix representation of the form

$$\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & c & 0 \\ b & 0 & * & * & * \\ 0 & c & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \tag{23}$$

where $a > 1/2$ and $b, c \geq 0$. Under the same decomposition, take $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$, and then $\{\pm 2a\} = \sigma_\pi(S^rBS^s + S^sBS^r)$. As S has rank-two and zero trace, there exists $R \in \mathscr{A}_1$ such that $\Phi(R) = S$. It follows that $\sigma_\pi(R) = \{-1, 1\}$. So $\|R\| = 1$ and $\|R^su\|\|R^ru\| \leq 1$ for all unit vectors $u \in H_1$. But $\sigma_\pi(R^rAR^s + R^sAR^r) = \sigma_\pi(R^rx\otimes xR^s + R^sx\otimes xR^r) = \{\pm\|R^rx\|\|R^sx\|\}$. This contradicts the fact $r(AR^s + R^sA) = r(BS^s + S^sB) \geq 2a > 1$.

So $\dim\ker(B - I) \geq 1$. Assume that $\dim\ker(B - I) = n \geq 2$. According to the space decomposition $H_2 = \ker(B - I) \oplus \ker(B - I)^\perp$, B has an operator matrix $I_n \oplus N$. Under the same space decomposition, take $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0$. Similar to the previous discussion, one gets a contradiction again. So $\dim\ker(B - I) = 1$.

Claim 2.2.2. There exists a unit vector $y \in H_2$ such that $B = y\otimes y$.

If it is not true, then, by Claim 2.2.1, there exist a unit vector $y \in \ker(B - I)$ and a nonzero $B_2 \in \mathscr{A}_2$ with $B_2y = 0$ such that $B = y\otimes y + B_2$. So there exists a unit vector $z \in [y]^\perp$ such that $B_2^rz \neq 0$ and $B_2^sz \neq 0$. Let $C_1 = y\otimes y$ and $C_2 = z\otimes z$. Since the range of Φ contains all rank-one projections, there exist D_1 and D_2 in $\mathscr{A}_1$ such that $\Phi(D_1) = C_1$ and $\Phi(D_2) = C_2$. Then $\sigma_\pi(D_1^rAD_1^s + D_1^sAD_1^r) = \sigma_\pi(C_1^rBC_1^s + C_1^sBC_1^r) = \sigma_\pi(2y\otimes yBy\otimes y) = \{2\}$, which, together with (6), implies that $\langle D_1^{r+s}x, x\rangle + \|D_1^rx\|\|D_1^sx\| = 2$. It follows from $\langle D_1^{r+s}x, x\rangle \leq \|D_1^rx\|\|D_1^sx\| \leq 1$ that $\langle D_1^{r+s}x, x\rangle = 1$. So $D_1^{r+s}x = x$, and according to the space decomposition $H_1 = [x] \oplus [x]^\perp$, $D_1^{r+s} = [1] \oplus Z$ with $\sigma(Z) \subseteq (-1, 1]$. Thus under the same space decomposition we have $D_1 = [1] \oplus Y$ with $\sigma(Y) \subseteq (-1, 1]$. Write D_2 in the operator matrix $\left(\begin{smallmatrix} v_{11} & V_{12} \\ V_{12}^* & V_{22} \end{smallmatrix}\right)$ accordingly; then

$$\begin{aligned} 0 &= D_1^sD_2D_1^r + D_1^rD_2D_1^s \\ &= \begin{pmatrix} 2v_{11} & V_{12}\left(Y^s + Y^r\right) \\ \left(Y^s + Y^r\right)V_{12}^* & Y^rV_{22}Y^s + Y^sV_{22}Y^r \end{pmatrix}. \end{aligned} \tag{24}$$

Clearly, $v_{11} = 0$. So, $A^rD_2A^s + A^sD_2A^r = 0$. But then this contradicts the fact that $\sigma_\pi(A^rD_2A^s + A^sD_2A^r) = \sigma_\pi(B^rC_2B^s + B^sC_2B^r) \neq \{0\}$. So Claim 2.2.2 holds and Φ preserves rank-one projections.

Conversely, assume that $\Phi(A)$ is a rank-one orthogonal projection; then, a similar discussion implies that A is a rank-one projection. □

The following lemma was proved in [19].

Lemma 7. *Let H be a complex Hilbert space and $A, B \in \mathscr{B}(H)$ self-adjoint operators. If $|\langle Ax, x\rangle| + \|Ax\|\|x\| = |\langle Bx, x\rangle| + \|Bx\|\|x\|$ holds for all $x \in H$, then $A = \pm B$.*

Now we are in a position to present our proof of Theorem 3, except the case $r = s$.

Proof of Theorem 3. The "if" part is obvious. Let us check the "only if" part.

By Lemma 6, Φ preserves rank-one projections in both directions. It follows that there exists a bijective map $T : H_1 \to H_2$ such that

$$\Phi(x\otimes x) = Tx\otimes Tx \tag{25}$$

for all unit vectors $x \in H_1$, where $\|Tx\| = \|x\|$ and $T(\lambda x) = \lambda Tx$ for any $x \in H_1$ and $\lambda \in \mathbb{C}$.

We consider the following two cases.

Case 1 ($s > r = 0$). For any unit vectors $x, y \in H_1$, we have $\sigma_\pi((y\otimes y)(x\otimes x)^s + (x\otimes x)^s(y\otimes y)) = \sigma_\pi((Ty\otimes Ty)(Tx\otimes Tx)^s + (Tx\otimes Tx)^s(Ty\otimes Ty))$. By (6), $\langle x, y\rangle = 0$ if and only if $\langle Tx, Ty\rangle = 0$, and when $\langle x, y\rangle \neq 0$,

$$\left|\langle Tx, Ty\rangle\right|^2 + \left|\langle Tx, Ty\rangle\right| = \left|\langle x, y\rangle\right|^2 + \left|\langle x, y\rangle\right|. \tag{26}$$

It follows that

$$\left|\langle Tx, Ty\rangle\right| = \left|\langle x, y\rangle\right| \tag{27}$$

holds for all $x, y \in H_1$.

Wigner's theorem [20] states that every bijective map T between Hilbert spaces H_1, H_2 satisfying (27) must have the form $Tx = \phi(x)Ux$ for any $x \in H_1$, where U is a unitary or a conjugate unitary (i.e., antiunitary) operator and ϕ is a so-called phase-function which means that its values are of modulus one. Thus, by Wigner's theorem, there exists a unitary or conjugate unitary operator $U : H_1 \to H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Assume first that U is unitary. Let $A \in \mathscr{A}_1$ be arbitrary. For any unit vector $x \in H_1$,

$$\begin{aligned}&\sigma_\pi\left(A(x\otimes x)^s + (x\otimes x)^s A\right)\\&\quad= \sigma_\pi\left(\Phi(A)(Ux\otimes Ux)^s + (Ux\otimes Ux)^s\Phi(A)\right).\end{aligned} \tag{28}$$

Applying (6), for any unit vector $x \in H_1$, one has

$$|\langle Ax, x\rangle| + \|Ax\| = \left|\langle U^*\Phi(A)Ux, x\rangle\right| + \left\|U^*\Phi(A)Ux\right\|, \tag{29}$$

and hence Lemma 7 implies that $U^*\Phi(A)U = \pm A$. Hence, $\Phi(A) = \pm UAU^*$ for every $A \in \mathscr{A}_1$. We claim that $\Phi(A) = UAU^*$ for every $A \in \mathscr{A}_1$. Otherwise, there exists some nonzero B_0 such that $\Phi(B_0) = -UB_0U^*$. Let $\mathscr{M}_+ = \{A \in \mathscr{A}_1 : \Phi(A) = UAU^*\}$ and $\mathscr{M}_- = \{B \in \mathscr{A}_1 : B \neq 0, \Phi(B) = -UBU^*\}$. Then $\mathscr{M}_+ \cap \mathscr{M}_- = \emptyset$, $\mathscr{M}_+ \cup \mathscr{M}_- = \mathscr{A}_1$, and $B_0 \in \mathscr{M}_-$. Note that, as $\Phi(I) = I$, $I \in \mathscr{M}_+$. It follows that, for any $B \in \mathscr{M}_-$, we have $\sigma_\pi(2B) = \sigma_\pi(IB + BI) = \sigma_\pi(\Phi(I)\Phi(B) + \Phi(B)\Phi(I)) = -\sigma_\pi(2B)$. Therefore, $\sigma_\pi(B) = \{-\|B\|, \|B\|\}$ holds for all $B \in \mathscr{M}_-$. Let $B = \int_{-\|B\|}^{\|B\|} \lambda dE_\lambda$ be the spectral resolution of B. Then the spectral project $E = \int_{(1/2)\|B\|}^{\|B\|} dE_\lambda \neq 0$. Though we do not know if $E \in \mathscr{A}_1$, we can take unit vector $x \in EH_1$ so that $\langle Bx, x\rangle \geq (1/2)\|B\| > 0$. Thus, $\langle Bx, x\rangle + \|Bx\| \geq \|B\| > 0$. By Lemma 4, we have $\sigma_\pi(x\otimes xB + Bx\otimes x) = \{\langle Bx, x\rangle + \|Bx\|\}$. Since $x \otimes x \in \mathscr{A}_1$ and $\sigma_\pi(x \otimes x) = \{1\}$, $x \otimes x \in \mathscr{M}_+$. But then,

$$\begin{aligned}\{\langle Bx, x\rangle + \|Bx\|\} &= \sigma_\pi(x\otimes xB + Bx\otimes x)\\&= \sigma_\pi(\Phi(x\otimes x)\Phi(B) + \Phi(B)\Phi(x\otimes x))\\&= -\sigma_\pi(x\otimes xB + Bx\otimes x)\\&= \{-\langle Bx, x\rangle - \|Bx\|\},\end{aligned} \tag{30}$$

a contradiction. So, $\Phi(A) = UAU^*$ holds for every $A \in \mathscr{A}_1$.

Now assume that U is conjugate unitary. Take arbitrarily an orthonormal basis $\{e_i\}_{i\in\Lambda}$ of H and define J by $J(\sum_{i\in\Lambda} \xi_i e_i) = \sum_{i\in\Lambda} \overline{\xi_i} e_i$. Then $J : H_1 \to H_1$ is conjugate unitary and $J^2 = I$. Let $V = UJ$. Then V is unitary and a similar discussion as above implies that $\Phi(A) = VA^tV^*$ for all $A \in \mathscr{A}_1$ and A^t is the transpose of A for the orthonormal basis $\{e_i\}_{i\in\Lambda}$ of H_1.

Case 2 ($s > r > 0$). For any unit vectors $x, y \in H_1$, we have

$$\begin{aligned}\left\{2\left|\langle x, y\rangle\right|^2\right\} &= \sigma_\pi\left((x\otimes x)^r(y\otimes y)(x\otimes x)^s\right.\\&\qquad\left.+ (x\otimes x)^s(y\otimes y)(x\otimes x)^r\right)\\&= \sigma_\pi\left((Tx\otimes Tx)^r(Ty\otimes Ty)(Tx\otimes Tx)^s\right.\\&\qquad\left.+ (Tx\otimes Tx)^s(Ty\otimes Ty)(Tx\otimes Tx)^r\right)\\&= \left\{2\left|\langle Tx, Ty\rangle\right|^2\right\}.\end{aligned} \tag{31}$$

Hence

$$\left|\langle Tx, Ty\rangle\right| = \left|\langle x, y\rangle\right| \tag{32}$$

holds for all $x, y \in H_1$. Thus, by Wigner's theorem, there exists a unitary or conjugate unitary operator $U : H_1 \to H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Now assume that U is unitary. Let $A \in \mathscr{A}_1$ be arbitrary. For any unit vector $x \in H_1$, since

$$\begin{aligned}\{2\langle Ax, x\rangle\} &= \sigma_\pi\left((x\otimes x)^r A(x\otimes x)^s + (x\otimes x)^s A(x\otimes x)^r\right)\\&= \sigma_\pi\left((Ux\otimes Ux)^r\Phi(A)(Ux\otimes Ux)^s\right.\\&\qquad\left.+ (Ux\otimes Ux)^s\Phi(A)(Ux\otimes Ux)^r\right)\\&= \{2\langle\Phi(A)Ux, Ux\rangle\},\end{aligned} \tag{33}$$

we have

$$\langle Ax, x\rangle = \langle\Phi(A)Ux, Ux\rangle \quad \forall\text{unit vectors } x \in H_1. \tag{34}$$

So we get $\Phi(A) = UAU^*$ for every $A \in \mathscr{A}_1$.

Similar to the case $s > r = 0$, if U is conjugate unitary, then there exists a unitary operator V such that $\Phi(A) = VA^tV^*$ for all $A \in \mathscr{A}_1$.

Hence we have shown that, in the case $\Phi(I) = I$, there exists a unitary U such that either $\Phi(A) = UAU^*$ for every $A \in \mathscr{A}_1$; or $\Phi(A) = UA^tU^*$ for every $A \in \mathscr{A}_1$, where A^t is the transpose of A with respect to an arbitrarily given orthonormal basis of H_1.

If $\Phi(I) = -I$, considering $\Psi = -\Phi$ gives $\Phi(A) = -UAU^*$ for every $A \in \mathscr{A}_1$ or $\Phi(A) = -UA^tU^*$ for every $A \in \mathscr{A}_1$. It is clear that this case does not occur if $r + s$ is even. □

3. Generalized Products of Self-Adjoint Operators on Hilbert Spaces

In this section, we will characterize maps preserving peripheral spectrum of generalized products of self-adjoint operators. Its special case, Theorem 10, makes up for the gap for the case $s = r$ in the proof of Theorem 3.

Let $\mathscr{A}$ be a real Jordan algebra in $\mathscr{B}_s(H)$. If a generalized product $T_1 * T_2 * \cdots * T_k$ defined in (2) satisfies that $T_1 * T_2 * \cdots * T_k \in \mathscr{A}$ for any $T_1, T_2, \ldots, T_k \in \mathscr{A}$, that is, the general product is closed in $\mathscr{A}$, we say that $T_1 * T_2 * \cdots * T_k$ is a generalized product on $\mathscr{A}$. The following lemma was proved in [3].

Lemma 8. *Let $T_1 * T_2 * \cdots * T_k = T_{i_1} \cdots T_{i_p} \cdots T_{i_m}$ be a generalized product on a standard real Jordan algebra $\mathscr{A} \subseteq \mathscr{B}_s(H)$ defined as in (2) of Definition 1. Then there exists a positive integer n with $m = 2n - 1$ such that $i_p = n$, and $i_j = i_{2n-j}$ for all $j = 1, \ldots, n$.*

The following is the main result in this section. Observe that we do not need the assumption that the range of the map contains all rank-two self-adjoint operators with zero trace.

Theorem 9. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of self-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Consider the generalized product $T_1 * \cdots * T_k$ on $\mathscr{B}_s(H_i)$ as in Lemma 8 with width m. Assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map the range of which contains all rank-one projections. Then Φ satisfies*

$$\sigma_\pi\left(\Phi\left(A_1\right) * \cdots * \Phi\left(A_k\right)\right) = \sigma_\pi\left(A_1 * \cdots * A_k\right) \tag{35}$$

for all $A_1, A_2, \ldots, A_k \in \mathscr{A}_1$ if and only if one of the following conditions holds.

(1) *There exists a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = UAU^*$ for all $A \in \mathscr{A}_1$.*

(2) *There exists a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = UA^tU^*$ for all $A \in \mathscr{A}_1$,*

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H_1.

To prove Theorem 9, we consider the special case by taking $A_{i_p} = A$ and $A_{i_q} = B$ if $q \neq p$. By Lemma 8, there exists positive integer $r(= n)$ with $2r = m - 1 \geq 2$ such that $A_1 * A_2 * \cdots * A_k = B^rAB^r$. It is clear that Theorem 9 is an immediate consequence of the following result.

Theorem 10. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be standard real Jordan algebras of delf-adjoint operators on complex Hilbert spaces H_1 and H_2, respectively. Assume that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map the range of which contains all rank-one projections and r is nonnegative integer with $r \geq 1$. Then Φ satisfies*

$$\sigma_\pi\left(B^rAB^r\right) = \sigma_\pi\left(\Phi(B)^r\Phi\left(A\right)\Phi(B)^r\right) \quad \forall A, B \in \mathscr{A}_1 \tag{36}$$

if and only if one of the following conditions holds.

(1) *There exists a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = UAU^*$ for all $A \in \mathscr{A}_1$.*

(2) *There exists a unitary operator $U : H_1 \to H_2$ such that $\Phi(A) = UA^tU^*$ for all $A \in \mathscr{A}_1$,*

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis of H_1.

To prove Theorem 10, it suffices to check the "only if" part. Assume in the following that $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ is a map satisfying (36) with range containing all rank-one projections.

Lemma 11. $\Phi(I) = I$.

Proof. It follows from (36) that $r(A) = r(\Phi(A))$ holds for every $A \in \mathscr{A}_1$. Let $\Phi(I) = B$. For any unit vector $y \in H_2$, there exists $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. Then

$$\sigma_\pi\left(A^{2r}\right) = \sigma_\pi\left(\Phi(A)^r\Phi\left(I\right)\Phi(A)^r\right) = \sigma_\pi\left(y \otimes yBy \otimes y\right), \tag{37}$$

which, together with $A^{2r} \geq 0$, implies that $\langle By, y\rangle = 1$ for all unit vectors $y \in H_2$. So $B = I$. □

Lemma 12. *Φ preserves rank-one projections in both directions.*

Proof. For any unit vector $x \in H_1$, let $\Phi(x \otimes x) = B$ and $\Phi(I - x \otimes x) = T$. It follows from $\{0\} = \sigma_\pi((x \otimes x)^r(I - x \otimes x)(x \otimes x)^r) = \sigma_\pi(B^rTB^r)$ that $B^rTB^r = 0$. Similarly, we have $T^rBT^r = 0$. If $\ker T = \{0\}$, then $\ker T^r = \{0\}$, which, together with $T^rBT^r = 0$, implies that $BT^r = 0$, and thus $B = 0$, a contradiction.

So, there exist a unit vector $y \in \ker T$. Take $A \in \mathscr{A}_1$ such that $\Phi(A) = y \otimes y$. It follows from $\sigma_\pi(A^r(I - x \otimes x)A^r) = \sigma_\pi(y \otimes yTy \otimes y) = \{0\}$ that $A^r(I - x \otimes x)A^r = 0$, which implies that $A = x \otimes x$. Thus $\Phi(x \otimes x) = y \otimes y$ and, therefore, Φ preserves rank-one projections.

Similarly one can show that $\Phi(A)$ is a rank-one projection will imply that A is a rank-one projection. □

Proof of Theorem 10. By Lemma 12, Φ preserves rank-one projections in both directions. It follows that there exists a bijective map $T : H_1 \to H_2$ such that

$$\Phi\left(x \otimes x\right) = Tx \otimes Tx \tag{38}$$

for all unit vectors $x \in H_1$, where $\|Tx\| = \|x\|$ and $T(\lambda x) = \lambda Tx$ for any $x \in H_1$ and $\lambda \in \mathbb{C}$.

For any unit vectors $x, y \in H_1$, we have

$$\begin{aligned}\left\{\left|\langle x, y\rangle\right|^2\right\} &= \sigma_\pi\left((x \otimes x)^r\left(y \otimes y\right)(x \otimes x)^r\right) \\ &= \sigma_\pi\left((Tx \otimes Tx)^r\left(Ty \otimes Ty\right)(Tx \otimes Tx)^r\right) \\ &= \left\{\left|\langle Tx, Ty\rangle\right|^2\right\}.\end{aligned} \tag{39}$$

Hence

$$\left|\langle Tx, Ty\rangle\right| = \left|\langle x, y\rangle\right| \tag{40}$$

holds for all $x, y \in H_1$. Thus, by Wigner's theorem again, there exists a unitary or conjugate unitary operator $U : H_1 \to H_2$ such that $\Phi(x \otimes x) = Ux \otimes Ux$ for every unit vector $x \in H_1$.

Now assume that U is unitary. Let $A \in \mathscr{A}_1$ be arbitrary. For any unit vector $x \in H_1$, since

$$\begin{aligned}\{\langle Ax, x\rangle\} &= \sigma_\pi\left((x \otimes x)^rA(x \otimes x)^r\right) \\ &= \sigma_\pi\left((Ux \otimes Ux)^r\Phi\left(A\right)(Ux \otimes Ux)^r\right) \\ &= \{\langle\Phi\left(A\right)Ux, Ux\rangle\},\end{aligned} \tag{41}$$

we have

$$\langle Ax, x\rangle = \langle\Phi\left(A\right)Ux, Ux\rangle \quad \forall \text{unit vectors } x \in H_1. \tag{42}$$

Hence we get $\Phi(A) = UAU^*$ for every $A \in \mathscr{A}_1$.

Similarly, U is conjugate unitary implies that there exists a unitary operator such that $\Phi(A) = VA^tV^*$ for all $A \in \mathscr{A}_1$. □

Remark 13. Finally, we remark that if we do not require that the generalized product is closed in the involved standard real Jordan algebras $\mathscr{A}_i$, $i = 1, 2$, we can still obtain a characterization of the maps Φ from $\mathscr{A}_1$ into $\mathscr{A}_2$ with range containing all rank-one projections which preserves the peripheral spectrum of an arbitrarily given generalized product. In fact, such maps have the same form stated in Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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