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Research Article

Existence Theory for q-Antiperiodic Boundary Value Problems of Sequential q-Fractional Integrodifferential Equations

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We discuss the existence and uniqueness of solutions for a new class of sequential q-fractional integrodifferential equations with q-antiperiodic boundary conditions. Our results rely on the standard tools of fixed-point theory such as Krasnoselskii's fixed-point theorem, Leray-Schauder nonlinear alternative, and Banach's contraction principle. An illustrative example is also presented.

1. Introduction

We consider a q-antiperiodic boundary value problem of sequential q-fractional integrodifferential equations given by

$$\begin{gathered} {}^cD_q^{\alpha}\left({}^cD_q^{\gamma}+\lambda\right)x(t)=Af(t,x(t))+BI_q^{\rho}g(t,x(t)), \\ 0\le t\le 1,\ 0<q<1, \quad (1) \\ x(0)=-x(1), \qquad \left(t^{(1-\gamma)}D_qx(t)\right)\Big|_{t=0}=-D_qx(1), \end{gathered}$$

where ${}^cD_q^{\alpha}$ and ${}^cD_q^{\gamma}$ denote the fractional q-derivative of the Caputo type, $0<\alpha,\gamma\le 1$, $I_q^{\rho}(\cdot)$ denotes Riemann-Liouville integral with $0<\rho<1$, f,g being given continuous functions, $\lambda\in\mathbb{R}$ and A,B being real constants.

The aim of the present study is to establish some existence and uniqueness results for the problem (1) by means of Krasnoselskii's fixed-point theorem, Leray-Schauder nonlinear alternative, and Banach's contraction principle. Though the tools employed in this work are standard, yet their exposition in the framework of the given problem is new.

Fractional calculus has developed into a popular mathematical modelling tool for many real world phenomena occurring in physical and technical sciences, see, for example, [1–4]. A fractional-order differential operator distinguishes itself from an integer-order differential operator in the sense that it is nonlocal in nature and can describe the memory and hereditary properties of some important and useful materials and processes. This feature has fascinated many researchers and several results ranging from theoretical analysis to asymptotic behavior and numerical methods for fractional differential equations have been established. For some recent work on the topic, see [5–12] and references therein.

The mathematical modeling of linear control systems, concerning the controllability of systems consisting of a set of well-defined interconnected objects, is based on the linear systems of divided difference functional equations. The controllability in mathematical control theory studies the concepts such as controllability of the state, controllability of the output, controllability at the origin, and complete controllability. The q-difference equations play a key role in the control theory as these equations are always completely controllable and appear in the q-optimal control problem [13]. The variational q-calculus is known as a generalization of the continuous variational calculus due to the presence of an extra-parameter q whose nature may be physical or economical. The study of the q-uniform lattice rely on the q-Euler equations. In other words, it suffices to solve the q-Euler-Lagrange equation for finding the extremum of the functional involved instead of solving the Euler-Lagrange equation [14]. One can find more details in a series of papers [15–21].

The subject of fractional q-difference (q-fractional) equations is regarded as fractional analogue of q-difference equations and has recently gained a considerable attention. For examples and details, we refer the reader to the works [22–33] and references therein while some earlier work on the subject can be found in [34–36]. The present work is motivated by recent interest in the study of fractional-order differential equations.

2. Preliminaries on Fractional q-Calculus

Let us describe the notations and terminology for q-fractional calculus [35].

For a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2}$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \qquad (a;q)_k = \prod_{i=0}^{k-1}\left(1-aq^i\right), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{3}$$

The q-analogue of the exponent $(x-y)^k$ is

$$(x-y)^{(0)} = 1, \qquad (x-y)^{(k)} = \prod_{j=0}^{k-1}\left(x-yq^j\right), \quad k \in \mathbb{N},\ x, y \in \mathbb{R}. \tag{4}$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1-q)^{(y-1)}}{(1-q)^{y-1}}, \tag{5}$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$.

Definition 1 (see [35]). Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and

$$\begin{aligned} I_q^\beta f(t) &:= \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s)\, d_q s \\ &= t^\beta (1-q)^\beta \sum_{k=0}^{\infty} q^k \frac{\left(q^\beta;q\right)_k}{(q;q)_k} f\left(tq^k\right), \\ &\qquad \beta > 0,\ t \in [0,1]. \end{aligned} \tag{6}$$

Observe that the above q-integral reduces to the following one for $\beta = 1$.

$$I_q f(t) := \int_0^t f(s)\, d_q s = t(1-q)\sum_{k=0}^{\infty} q^k f\left(tq^k\right). \tag{7}$$

Further details of q-integrals and fractional q-integrals can be found respectively in Section 1.3 and Section 4.2 of the text [35].

Remark 2. The semigroup property holds for q-fractional integration (Proposition 4.3 [35]):

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \quad \gamma, \beta \in \mathbb{R}^+. \tag{8}$$

Further, it has been shown in Lemma 6 of [37] that

$$\begin{gathered} I_q^\beta (x)^{(\upsilon)} = \frac{\Gamma_q(\upsilon+1)}{\Gamma_q(\beta+\upsilon+1)}(x)^{(\beta+\upsilon)}, \\ 0 < x < a, \quad \beta \in \mathbb{R}^+, \quad \upsilon \in (-1, \infty). \end{gathered} \tag{9}$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative.

Let f be a real valued function defined on a q-geometric set A ($|q| \neq 1$). Then the q-derivative of a function f is defined as

$$D_q f(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \in A \setminus \{0\}. \tag{10}$$

For $0 \in A$, the q-derivative at zero is defined for $|q| < 1$ by

$$D_q f(0) = \lim_{n\to\infty} \frac{f(tq^n) - f(0)}{tq^n}, \quad t \in A \setminus \{0\}. \tag{11}$$

Provided that the limit exists and does not depend on t.

Furthermore,

$$D_q^0 f = f, \qquad D_q^n f = D_q\left(D_q^{n-1} f\right), \quad n = 1, 2, 3, \ldots. \tag{12}$$

Definition 3 (see [35]). The Caputo fractional q-derivative of order $\beta > 0$ is defined by

$${}^c D_q^\beta f(t) = I_q^{\lceil\beta\rceil-\beta} D_q^{\lceil\beta\rceil} f(t), \tag{13}$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next we enlist some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative (Theorem 5.2 [35]):

$$\begin{gathered} I_q^\beta\, {}^c D_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil-1} \frac{t^k}{\Gamma_q(k+1)}\left(D_q^k f\right)(0^+), \\ \forall t \in (0,a],\ \beta > 0; \\ {}^c D_q^\beta I_q^\beta f(t) = f(t), \quad \forall t \in (0,a],\ \beta > 0. \end{gathered} \tag{14}$$

Now we establish a lemma that plays a key role in the sequel.

Lemma 4. *For a given $h \in C([0,1],\mathbb{R})$, the boundary value problem*

$$\begin{gathered} {}^cD_q^{\alpha}\left({}^cD_q^{\gamma}+\lambda\right)x(t)=h(t), \quad 0\le t\le 1,\ 0<q<1, \\ x(0)=-x(1), \qquad \left(t^{(1-\gamma)}D_q x(t)\right)\Big|_{t=0}=-D_q x(1) \end{gathered} \tag{15}$$

is equivalent to the q-integral equation

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\quad\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu \\ &\quad+\frac{(1-2t^{\gamma})}{4[\gamma]_q} \\ &\quad\times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\ &\quad\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu \\ &\quad-\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\quad\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu. \end{aligned} \tag{16}$$

Proof. It is well known that the solution of q-fractional equation in (15) can be written as

$$\begin{aligned} x(t) = {}& \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu \\ &-\frac{t^{\gamma}}{\Gamma_q(\gamma+1)}c_0-c_1. \end{aligned} \tag{17}$$

Differentiating (17), we obtain

$$\begin{aligned} D_qx(t) = {}& \int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\ &\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu \\ &-\frac{[\gamma]_q t^{\gamma-1}}{\Gamma_q(\gamma+1)}c_0. \end{aligned} \tag{18}$$

Using the boundary conditions (15) in (17) and (18) and solving the resulting expressions for c_0 and c_1, we get

$$\begin{aligned} c_0 = {}& \frac{\Gamma_q(\gamma+1)}{2[\gamma]_q} \\ &\times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\ &\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu, \\ c_1 = {}& \frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu \\ &-\frac{1}{4[\gamma]_q} \\ &\times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\ &\times\left(\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}h(m)\,d_qm-\lambda x(u)\right)d_qu. \end{aligned} \tag{19}$$

Substituting the values of c_0 and c_1 in (17) yields the solution (16). The converse follows in a straightforward manner. This completes the proof. □

Let $\mathscr{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|x\| = \sup\{|x(t)|, t\in[0,1]\}$.

In view of Lemma 4, we define an operator $\mathscr{U}:\mathscr{C}\to\mathscr{C}$ as

$$\begin{aligned} (\mathscr{U}x)(t) = {}& \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\times\Bigg(A\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\ &\qquad\times f(m,x(m))\,d_qm \\ &\qquad+B\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\ &\qquad\times g(m,x(m))\,d_qm \\ &\qquad-\lambda x(u)\Bigg)d_qu+\frac{(1-2t^{\gamma})}{4[\gamma]_q} \end{aligned}$$

$$\times \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \times \left(A \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times f(m,x(m))\, d_q m + B \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times g(m,x(m))\, d_q m - \lambda x(u) \right) d_q u$$
$$- \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \times \left(A \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times f(m,x(m))\, d_q m + B \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times g(m,x(m))\, d_q m - \lambda x(u) \right) d_q u. \tag{20}$$

Observe that the problem (1) has solutions only if the operator equation $x = \mathscr{U}x$ has fixed points.

3. Main Results

For the forthcoming analysis, the following conditions are assumed.

(A_1) $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $|f(t,x) - f(t,y)| \le L_1|x-y|$ and $|g(t,x) - g(t,y)| \le L_2|x-y|$, for all $t \in [0,1]$, $L_1, L_2 > 0$, $x, y \in \mathbb{R}$.

(A_2) There exist $\zeta_1, \zeta_2 \in C([0,1], \mathbb{R}^+)$ with $|f(t,x)| \le \zeta_1(t)$, $|g(t,x)| \le \zeta_2(t)$, for all $(t,x) \in [0,1] \times \mathbb{R}$, where $\|\zeta_i\| = \sup_{t\in[0,1]}|\zeta_i(t)|$, $i = 1, 2$.

For computational convenience, we set

$$\delta_1 = \frac{3}{2\Gamma_q(\alpha+\gamma+1)} + \frac{1}{4[\gamma]_q \Gamma_q(\alpha+\gamma)},$$
$$\delta_2 = \frac{3}{2\Gamma_q(\alpha+\rho+\gamma+1)} + \frac{1}{4[\gamma]_q \Gamma_q(\alpha+\rho+\gamma)}, \tag{21}$$
$$\delta_3 = \frac{3}{2\Gamma_q(\gamma+1)} + \frac{1}{4\Gamma_q(\gamma+1)},$$

$$\Lambda = L\left[|A| \left(\frac{1}{4[\gamma]_q \Gamma_q(\alpha+\gamma)} + \frac{1}{2\Gamma_q(\alpha+\gamma+1)} \right) + |B| \left(\frac{1}{4[\gamma]_q \Gamma_q(\alpha+\rho+\gamma)} + \frac{1}{2\Gamma_q(\alpha+\rho+\gamma+1)} \right) \right] + |\lambda| \left[\frac{1}{4\Gamma_q(\gamma+1)} + \frac{1}{2\Gamma_q(\gamma+1)} \right]. \tag{22}$$

Our first existence result is based on Krasnoselskii's fixed point theorem.

Lemma 5 (see, Krasnoselskii [38]). *Let Y be a closed, convex, bounded, and nonempty subset of a Banach space X. Let Q_1, Q_2 be the operators such that (i) $Q_1 x + Q_2 y \in Y$ whenever $x, y \in Y$; (ii) Q_1 is compact and continuous; and (iii) Q_2 is a contraction mapping. Then there exists $z \in Y$ such that $z = Q_1 z + Q_2 z$.*

Theorem 6. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying (A_1)-(A_2). Furthermore $\Lambda < 1$, where Λ is given by (22) and $L = \max\{L_1, L_2\}$. Then the problem (1) has at least one solution on $[0,1]$.*

Proof. Consider the set $B_\sigma = \{x \in \mathscr{C} : \|x\| \le \sigma\}$, where σ is given by

$$\sigma \ge \frac{|A| \|\zeta_1\| \delta_1 + |B| \|\zeta_2\| \delta_2}{1 - |\lambda| \delta_3}, \quad 1 - |\lambda| \delta_3 > 0. \tag{23}$$

Define operators $\mathscr{U}_1$ and $\mathscr{U}_2$ on B_σ as

$$(\mathscr{U}_1 x)(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \times \left(A \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m,x(m))\, d_q m + B \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times g(m,x(m))\, d_q m - \lambda x(u) \right) d_q u, \quad t \in [0,1],$$

$$(\mathscr{U}_2 x)(t) = \frac{(1-2t^\gamma)}{4[\gamma]_q} \times \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}$$

$$\times\left(A\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times f(m,x(m))\,d_qm + B\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times g(m,x(m))\,d_qm - \lambda x(u)\right)d_qu$$
$$-\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \times\left(A\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(m,x(m))\,d_qm + B\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times g(m,x(m))\,d_qm - \lambda x(u)\right)d_qu, \quad t\in[0,1]. \tag{24}$$

For $x, y \in B_\sigma$, we find that

$$\|\mathcal{U}_1x+\mathcal{U}_2y\| \le |A|\,\|\zeta_1\|\,\delta_1 + |B|\,\|\zeta_2\|\,\delta_2 + |\lambda|\,\sigma\delta_3 \le \sigma. \tag{25}$$

Thus, $\mathcal{U}_1x + \mathcal{U}_2y \in B_\sigma$. Continuity of f and g imply that the operator $\mathcal{U}_1$ is continuous. Also, $\mathcal{U}_1$ is uniformly bounded on B_σ as

$$\|\mathcal{U}_1x\| \le \frac{|A|\,\|\zeta_1\|}{\Gamma_q(\alpha+\gamma+1)} + \frac{|B|\,\|\zeta_2\|}{\Gamma_q(\alpha+\rho+\gamma+1)} + \frac{|\lambda|\,\sigma}{\Gamma_q(\gamma+1)}. \tag{26}$$

Now, we prove the compactness of the operator $\mathcal{U}_1$. In view of (A_1), we define

$$\sup_{(t,x)\in[0,1]\times B_\sigma}|f(t,x)| = \overline{f}, \qquad \sup_{(t,x)\in[0,1]\times B_\sigma}|g(t,x)| = \overline{g}. \tag{27}$$

Consequently, for $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}&\|(\mathcal{U}_1x)(t_2)-(\mathcal{U}_1x)(t_1)\|\\ &\le \int_0^{t_1}\frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \times\left[|A|\,\overline{f}\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm + |B|\,\overline{g} \times\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}d_qm + |\lambda|\,\sigma\right]d_qu\\ &+\int_{t_1}^{t_2}\frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \times\left[|A|\,\overline{f}\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm + |B|\,\overline{g} \times\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}d_qm + |\lambda|\,\sigma\right]d_qu,\end{aligned} \tag{28}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{U}_1$ is relatively compact on B_σ. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{U}_1$ is compact on B_σ. Now, we shall show that $\mathcal{U}_2$ is a contraction.

From (A_1) and for $x, y \in B_\sigma$, we have

$$\begin{aligned}&\|\mathcal{U}_2x-\mathcal{U}_2y\|\\ &\le \sup_{t\in[0,1]}\Bigg\{\frac{1}{4[\gamma]_q} \times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \times\Bigg[|A|\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times|f(m,x(m)) - f(m,y(m))|\,d_qm\\ &\quad+|B|\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \times|g(m,x(m)) - g(m,y(m))|\,d_qm + |\lambda|\,|x(u)-y(u)|\Bigg]d_qu\end{aligned}$$

$$+\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$
$$\times\left[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right.$$
$$\times\left|f(m,x(m))\right.$$
$$\left.-f(m,y(m))\right|d_qm$$
$$+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}$$
$$\times\left|g(m,x(m))\right.$$
$$\left.-g(m,y(m))\right|d_qm$$
$$\left.\left.+|\lambda|\left|x(u)-y(u)\right|\right]d_qu\right\}$$
$$\leq \sup_{t\in[0,1]}\left\{\frac{1}{4[\gamma]_q}\right.$$
$$\times\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}$$
$$\times\left[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}L_1\right.$$
$$\times\left|x(m)-y(m)\right|d_qm$$
$$+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}$$
$$\times L_2\left|x(m)-y(m)\right|d_qm$$
$$\left.+\ |\lambda|\left|x(u)-y(u)\right|\right]d_qu$$
$$+\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$
$$\times\left[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right.$$
$$\times L_1\left|x(m)-y(m)\right|d_qm$$
$$+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}$$
$$\times L_2\left|x(m)-y(m)\right|d_qm$$
$$\left.\left.+|\lambda|\left|x(u)-y(u)\right|\right]d_qu\right\}$$
$$\leq\left[L\left[|A|\left(\frac{1}{4[\gamma]_q\Gamma_q(\alpha+\gamma)}+\frac{1}{2\Gamma_q(\alpha+\gamma+1)}\right)\right.\right.$$
$$\left.+|B|\left(\frac{1}{4[\gamma]_q\Gamma_q(\alpha+\rho+\gamma)}+\frac{1}{2\Gamma_q(\alpha+\rho+\gamma+1)}\right)\right]$$
$$\left.+|\lambda|\left[\frac{1}{4[\gamma]_q\Gamma_q(\gamma)}+\frac{1}{2\Gamma_q(\gamma+1)}\right]\right]\|x-y\|$$
$$=\Lambda\|x-y\|, \tag{29}$$

where we have used (22). In view of the assumption $\Lambda<1$, the operator $\mathcal{U}_2$ is a contraction. Thus, all the conditions of Lemma 5 are satisfied. Hence, by the conclusion of Lemma 5, the problem (1) has at least one solution on $[0,1]$. □

Our next result is based on Leray-Schauder nonlinear alternative.

Lemma 7 (nonlinear alternative for single valued maps, see [39]). *Let E be a Banach space, C a closed, convex subset of E, W an open subset of C, and $0\in W$. Suppose that $\mathcal{U}:\overline{W}\to C$ is a continuous, compact (i.e., $\mathcal{U}(\overline{W})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{W}$, or*

(ii) *there is a $x\in\partial W$ (the boundary of W in C) and $\kappa\in(0,1)$ with $x=\kappa\mathcal{U}(x)$.*

Theorem 8. *Let $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous functions and the following assumptions hold:*

(A_3) *there exist functions $\nu_1,\nu_2\in C([0,1],\mathbb{R}^+)$, and nondecreasing functions $\vartheta_1,\vartheta_2:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\leq\nu_1(t)\vartheta_1(\|x\|)$, $|g(t,x)|\leq\nu_2(t)\vartheta_2(\|x\|)$, for all $(t,x)\in[0,1]\times\mathbb{R}$;*

(A_4) *there exists a constant $\omega>0$ such that*

$$\omega>\frac{|A|\|\nu_1\|\vartheta_1(\omega)\delta_1+|B|\|\nu_2\|\vartheta_2(\omega)\delta_2}{1-|\lambda|\delta_3},\qquad 1-|\lambda|\delta_3>0. \tag{30}$$

Then the boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $\mathcal{U}:\mathcal{C}\to\mathcal{C}$ defined by (20). The proof consists of several steps.

(i) It is easy to show that $\mathcal{U}$ is continuous.

(ii) $\mathcal{U}$ maps bounded sets into bounded sets in $C([0,1]\times\mathbb{R})$.

For a positive number $\bar{r}$, let $B_{\bar{r}} = \{x \in \mathscr{C} : \|x\| \le \bar{r}\}$ be a bounded set in $C([0,1]\times\mathbb{R})$ and $x \in B_{\bar{r}}$. Then, we have

$$\begin{aligned}
&\|(\mathscr{U}x)\| \\
&\le \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m,x(m))| \, d_q m \right. \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times |g(m,x(m))| \, d_q m \\
&\quad \left. + |\lambda| \, |x(u)| \right] d_q u \\
&\quad + \frac{1}{4[\gamma]_q} \\
&\quad \times \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m,x(m))| \, d_q m \right. \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times |g(m,x(m))| \, d_q m \\
&\quad \left. + |\lambda| \, |x(u)| \right] d_q u \\
&\quad + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(m,x(m))| \, d_q m \right. \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times |g(m,x(m))| \, d_q m \\
&\quad \left. \left. + |\lambda| \, |x(u)| \right] d_q u \right\} \\
&\le \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \right. \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \nu_1(m)\, \vartheta_1(\|x\|) \, d_q m \right. \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times \nu_2(m)\, \vartheta_2(\|x\|) \, d_q m \\
&\quad \left. + |\lambda| \, |x(u)| \right] d_q u \\
&\quad + \frac{1}{4[\gamma]_q} \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right. \\
&\quad \times \nu_1(m)\, \vartheta_1(\|x\|) \, d_q m \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times \nu_2(m)\, \vartheta_2(\|x\|) \, d_q m \\
&\quad \left. + |\lambda| \, |x(u)| \right] d_q u \\
&\quad + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\quad \times \left[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right. \\
&\quad \times \nu_1(m)\, \vartheta_1(\|x\|) \, d_q m \\
&\quad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad \times \nu_2(m)\, \vartheta_2(\|x\|) \, d_q m \\
&\quad \left. \left. + |\lambda| \, |x(u)| \right] d_q u \right\} \\
&\le |A| \, \|\nu_1\| \, \vartheta_1(\|x\|) \\
&\quad \times \left\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m \right] d_q u \right. \\
&\quad + \frac{1}{4[\gamma]_q} \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\
&\quad \times \left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m \right] d_q u \\
&\quad + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}
\end{aligned}$$

$$\begin{aligned}
&\times\left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m\right] d_q u\Bigg\}\\
&+|B|\,\|\nu_2\|\,\vartheta_2(\|x\|)\\
&\times\Bigg\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m\right] d_q u\\
&+\frac{1}{4[\gamma]_q}\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\\
&\times\left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m\right] d_q u\\
&+\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\left[\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q m\right] d_q u\Bigg\}\\
&+|\lambda|\,\|x\|\Bigg\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u+\frac{1}{4[\gamma]_q}\\
&\times\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} d_q u\\
&+\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_q u\Bigg\}\\
&\le |A|\,\|\nu_1\|\,\vartheta_1(\bar{r})\,\delta_1+|B|\,\|\nu_2\|\,\vartheta_2(\bar{r})\,\delta_2+|\lambda|\,\bar{r}\delta_3\le \bar{r}.
\end{aligned}\tag{31}$$

This shows that $\mathscr{U}x \in B_{\bar{r}}$.

(iii) $\mathscr{U}$ maps bounded sets into equicontinuous sets of $C([0,1]\times\mathbb{R})$.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_{\bar{r}}$, where $B_{\bar{r}}$ is a bounded set of $C([0,1],\mathbb{R})$. Then, we obtain

$$\begin{aligned}
&\|(\mathscr{U}x)(t_2)-(\mathscr{U}x)(t_1)\|\\
&\le\Bigg|\int_0^{t_1} \frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Bigg[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu_1(m)\,\vartheta_1(\bar{r})\,d_q m\\
&+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}\\
&\times\nu_2(m)\,\vartheta_2(\bar{r})\,d_q m+|\lambda|\,\bar{r}\Bigg] d_q u\\
&+\int_{t_1}^{t_2} \frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Bigg[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu_1(m)\,\vartheta_1(\bar{r})\,d_q m\\
&+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}\\
&\times\nu_2(m)\,\vartheta_2(\bar{r})\,d_q m\\
&+|\lambda|\,\bar{r}\Bigg] d_q u\Bigg|\\
&+\frac{(t_2^{\gamma}-t_1^{\gamma})}{2[\gamma]_q}\\
&\times\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\\
&\times\Bigg[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\nu_1(m)\,\vartheta_1(\bar{r})\,d_q m\\
&+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}\\
&\times\nu_2(m)\,\vartheta_2(\bar{r})\,d_q m\\
&+|\lambda|\,\bar{r}\Bigg] d_q u.
\end{aligned}\tag{32}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_{\bar{r}}$ as $t_2 - t_1 \to 0$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathscr{U} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

(iv) Let x be a solution of the given problem such that $x = \kappa\mathscr{U}x$ for $\kappa \in (0,1)$. Then, for $t \in [0,1]$, it follows by the procedure used to establish (ii) that

$$\begin{aligned}
|x(t)| &= |\kappa(\mathscr{U}x)(t)|\\
&\le |A|\,\|\nu_1\|\,\vartheta_1(\|x\|)\,\delta_1+|B|\,\|\nu_2\|\,\vartheta_2(\|x\|)\,\delta_2\\
&\quad+|\lambda|\,\|x\|\,\delta_3.
\end{aligned}\tag{33}$$

Consequently, we have

$$\|x\| \le \frac{|A|\,\|\nu_1\|\,\vartheta_1(\|x\|)\,\delta_1+|B|\,\|\nu_2\|\,\vartheta_2(\|x\|)\,\delta_2}{1-|\lambda|\,\delta_3}.\tag{34}$$

In view of (A_4), there exists ω such that $\|x\| \neq \omega$. Let us set

$$W=\{x\in\mathscr{C} : \|x\|<\omega\}.\tag{35}$$

Note that the operator $\mathscr{U} : \overline{W} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of W, there is no $x \in \partial W$ such that $x = \kappa\mathscr{U}(x)$ for some $\kappa \in (0,1)$. In consequence, by the nonlinear alternative of Leray-Schauder type (Lemma 7), we deduce that $\mathscr{U}$has a fixed point $x \in \overline{W}$ which is a solution of the problem (1). This completes the proof.

□

Finally we show the existence of a unique solution of the given problem by applying Banach's contraction mapping principle (Banach fixed-point theorem).

Theorem 9. *Suppose that the assumption (A_1) holds and*

$$\overline{\Lambda} = (L\Lambda_1 + |\lambda|\,\delta_3) < 1, \qquad \Lambda_1 = |A|\,\delta_1 + |B|\,\delta_2, \tag{36}$$

where δ_1, δ_2, and δ_3 are given by (21) *and $L = \max\{L_1, L_2\}$. Then the boundary value problem* (1) *has a unique solution.*

Proof. Fix $M = \max\{M_1, M_2\}$, where M_1, M_2 are finite numbers given by $M_1 = \sup_{t\in[0,1]}|f(t,0)|$, $M_2 = \sup_{t\in[0,1]}|g(t,0)|$. Selecting $\sigma \geq M\Lambda_1/(1-\overline{\Lambda})$, we show that $\mathscr{U}B_\sigma \subset B_\sigma$, where $B_\sigma = \{x \in \mathscr{C} : \|x\| \leq \sigma\}$. For $x \in B_\sigma$, we have

$$\begin{aligned}
&\|(\mathscr{U}x)\| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad \times \Bigg[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f(m,x(m))\right| d_q m \\
&\qquad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\qquad\qquad \times \left|g(m,x(m))\right| d_q m \\
&\qquad + |\lambda|\,|x(u)| \Bigg] d_q u \\
&\qquad + \frac{1}{4[\gamma]_q} \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\
&\qquad\qquad \times \Bigg[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad\qquad\qquad \times \left|f(m,x(m))\right| d_q m \\
&\qquad\qquad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\qquad\qquad\qquad \times \left|g(m,x(m))\right| d_q m \\
&\qquad\qquad + |\lambda|\,|x(u)| \Bigg] d_q u \\
&\qquad + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad\qquad \times \Bigg[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad\qquad\qquad \times \left|f(m,x(m))\right| d_q m \\
&\qquad\qquad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\qquad\qquad\qquad \times \left|g(m,x(m))\right| d_q m \\
&\qquad\qquad + |\lambda|\,|x(u)| \Bigg] d_q u \Bigg\} \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad \times \Bigg[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad\qquad \times \left(\left|f(m,x(m)) - f(m,0)\right| \right. \\
&\qquad\qquad\quad \left. + \left|f(m,0)\right|\right) d_q m \\
&\qquad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\qquad\qquad \times \left(\left|g(m,x(m)) - g(m,0)\right| \right. \\
&\qquad\qquad\quad \left. + \left|g(m,0)\right|\right) d_q m \\
&\qquad + |\lambda|\,|x(u)| \Bigg] d_q u \\
&\qquad + \frac{1}{4[\gamma]_q} \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)} \\
&\qquad\qquad \times \Bigg[|A| \int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad\qquad\qquad \times \left(\left|f(m,x(m)) - f(m,0)\right| \right. \\
&\qquad\qquad\qquad\quad \left. + \left|f(m,0)\right|\right) d_q m \\
&\qquad\qquad + |B| \int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\qquad\qquad\qquad \times \left(\left|g(m,x(m)) - g(m,0)\right| \right. \\
&\qquad\qquad\qquad\quad \left. + \left|g(m,0)\right|\right) d_q m
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda||x(u)|\Bigg]d_qu\\
&+\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Bigg[|A|\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\\
&\times\left(\left|f(m,x(m))-f(m,0)\right|\right.\\
&\left.+\left|f(m,0)\right|\right)d_qm\\
&+|B|\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}\\
&\times\left(\left|g(m,x(m))-g(m,0)\right|\right.\\
&\left.+\left|g(m,0)\right|\right)d_qm\\
&+|\lambda||x(u)|\Bigg]d_qu\Bigg\}\\
&\leq|A|(L_1\sigma+M_1)\\
&\times\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm\right]d_qu\\
&+\frac{1}{4[\gamma]_q}\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm\right]d_qu\\
&+\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qm\right]d_qu\Bigg\}\\
&+|B|(L_2\sigma+M_2)\\
&\times\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}d_qm\right]d_qu\\
&+\frac{1}{4[\gamma]_q}\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}d_qm\right]d_qu\\
&+\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\left[\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}d_qm\right]d_qu\Bigg\}\\
&+|\lambda|\sigma\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu+\frac{1}{4[\gamma]_q}\\
&\times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}d_qu\\
&+\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\Bigg\}\\
&\leq M\Lambda_1+\overline{\Lambda}\sigma\leq\sigma
\end{aligned}\tag{37}
$$

This shows that $\mathcal{U}B_\sigma\subset B_\sigma$. For $x,y\in\mathbb{R}$, we obtain

$$
\begin{aligned}
&\|\mathcal{U}x-\mathcal{U}y\|\\
&\leq\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Bigg[|A|\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\\
&\times\left|f(m,x(m))-f(m,y(m))\right|d_qm\\
&+|B|\int_0^u\frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)}\\
&\times\left|g(m,x(m))-g(m,y(m))\right|d_qm\\
&+|\lambda|\left|x(u)-y(u)\right|\Bigg]d_qu+\frac{1}{4[\gamma]_q}\\
&\times\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\\
&\times\Bigg[|A|\int_0^u\frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\\
&\times\left|f(m,x(m))\right.\\
&\left.-f(m,y(m))\right|d_qm
\end{aligned}
$$

$$
\begin{aligned}
&+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad\times\left|g(m,x(m))\right. \\
&\quad\left.-g(m,y(m))\right|d_qm \\
&\left.+|\lambda|\left|x(u)-y(u)\right|\right]d_qu \\
&+\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times\left[|A|\int_0^u \frac{(u-qm)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right. \\
&\quad\times\left|f(m,x(m))\right. \\
&\quad\left.-f(m,y(m))\right|d_qm \\
&+|B|\int_0^u \frac{(u-qm)^{(\alpha+\rho-1)}}{\Gamma_q(\alpha+\rho)} \\
&\quad\times\left|g(m,x(m))\right. \\
&\quad\left.-g(m,y(m))\right|d_qm \\
&\left.\left.+|\lambda|\left|x(u)-y(u)\right|\right]d_qu\right\} \\
&\leq\overline{\Lambda}\left\|x-y\right\|.
\end{aligned}
\tag{38}
$$

Since $\overline{\Lambda} \in (0,1)$ by the given assumption, therefore $\mathcal{U}$ is a contraction. Hence, it follows by Banach's contraction principle that the problem (1) has a unique solution. □

Example 10. Consider a q-fractional integrodifferential equation with q-antiperiodic boundary conditions given by

$$
\begin{aligned}
&{}^cD_q^{\alpha}\left({}^cD_q^{\gamma}+\frac{1}{15}\right)x(t) \\
&\quad=\frac{1}{5}f(t,x(t))+\frac{1}{8}I_q^{\rho}g(t,x(t)), \\
&\qquad 0<t<1,\quad 0<q<1, \\
&x(0)=-x(1),\qquad \left(t^{(1-\gamma)}D_qx(t)\right)\Big|_{t=0}=-D_qx(1),
\end{aligned}
\tag{39}
$$

where $\alpha = \gamma = \rho = q = 1/2, \lambda = 1/15, A = 1/5, B = 1/8, f(t,x) = (t/20\pi)\sin 2\pi x + (t+1)(1+x)^4/(1+(1+x)^4), g(t,x) = (1/8\pi)\tan^{-1}x + 1/16$. With the given data, $\delta_1 \simeq 1.92678, \delta_2 \simeq 1.72332, \delta_3 \simeq 1.90037$, and

$$
\left|f(t,x)\right|\leq(t+1)\left(\frac{\|x\|}{10}+1\right),\qquad \left|g(t,x)\right|\leq\frac{1}{8}. \tag{40}
$$

Clearly, $\nu_1(t) = t+1, \vartheta_1(\|x\|) = \|x\|/10+1, \nu_2(t) = 1/8, \vartheta_1(\|x\|) = 1$, and the condition ($A_4$) implies that $\omega > 1.00176$. Thus all the assumptions of Theorem 8 are satisfied. Hence, the conclusion of Theorem 8 applies to the problem (39).

Example 11. Consider the following q-fractional q-antiperiodic boundary value problem:

$$
\begin{aligned}
&{}^cD_q^{\alpha}\left({}^cD_q^{\gamma}+\frac{1}{8}\right)x(t) \\
&\quad=\frac{1}{2}f(t,x(t))+\frac{1}{4}I_q^{\rho}g(t,x(t)), \\
&\qquad 0<t<1,\ 0<q<1, \\
&x(0)=-x(1),\qquad \left(t^{(1-\gamma)}D_qx(t)\right)\Big|_{t=0}=-D_qx(1),
\end{aligned}
\tag{41}
$$

where $\alpha = \gamma = \rho = q = A = 1/2, \lambda = 1/8, B = 1/4, f(t,x) = (1/(t^2+2))(|x|/(1+|x|))+\sin^2 t, g(t,x) = (1/4)\tan^{-1}x+\cos^2 t+t^3+5$. With the given data, it is found that $L_1 = 1/2, L_2 = 1/4$ as $|f(t,x)-f(t,y)| \leq (1/2)|x-y|, |g(t,x)-g(t,y)| \leq (1/4)|x-y|$. Clearly $L = \max\{L_1, L_2\} = 1/2$. Moreover, $\delta_1 \simeq 1.92678, \delta_2 \simeq 1.72332$, and $\delta_3 \simeq 1.90037$. Using the given values, it is found that $\overline{\Lambda} \simeq 0.934655 < 1$. Thus all the assumptions of Theorem 9 are satisfied. Hence, by the conclusion of Theorem 9, there exists a unique solution for the problem (41).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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