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Research Article

Existence of Multiple Nontrivial Solutions for a Strongly Indefinite Schrödinger-Poisson System

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We consider a Schrödinger-Poisson system in $\mathbb{R}^3$ with a strongly indefinite potential and a general nonlinearity. Its variational functional does not satisfy the global linking geometry. We obtain a nontrivial solution and, in case of odd nonlinearity, infinitely many solutions using the local linking and improved fountain theorems, respectively.

1. Introduction and Statement of Results

In this paper, we consider the Schrödinger-Poisson system:

$$\begin{gathered} -\Delta u + V(x)u + K_1(x)\phi u = Q(x) f(u), \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = K_2(x)u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1}$$

For V, K_1, K_2, Q, and f, we assume the following.

(**v**) $V \in L^q_{\mathrm{loc}}(\mathbb{R}^3)$ and $V^- = \min\{-V, 0\} \in L^\infty(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ for some $q \in [2, \infty)$. This assumption ensures that the Schrödinger operator $S = -\Delta + V$ is self-adjoint and semibounded on $L^2(\mathbb{R}^3)$ (see Theorem A.2.7 in [1]). $\sigma(S)$ denotes the spectrum of S. We assume that 0 lies in a gap of $\sigma(S)$; that is, there exist $\alpha, \beta > 0$ such that

$$\sigma(S) \cap (-\alpha, \beta) = \emptyset. \tag{2}$$

(**k**) $Q, K_i \in C(\mathbb{R}^3)$ satisfy $Q(x) > 0$, $K_i(x) > 0$ for all x,

$$\lim_{|x|\to\infty} Q(x) = 0, \quad \lim_{|x|\to\infty} K_i(x) = 0, \quad i = 1, 2, \tag{3}$$

and there exists $p \in (4, 6)$ such that

$$\int_{\mathbb{R}^3} \frac{K_i^{6p/(5p-12)}(x)}{Q^{12/(5p-12)}(x)} dx < \infty, \quad i = 1, 2. \tag{4}$$

(**f$_1$**) $f \in C(\mathbb{R})$ and there exists $C > 0$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \le C\left(|t| + |t|^{p-1}\right). \tag{5}$$

(**f$_2$**) $f(t) = o(t)$ as $t \to 0$.

(**f$_3$**) Let

$$F(t) = \int_0^t f(\tau)\, d\tau. \tag{6}$$

We assume that there exists $A > 0$ such that for all $t \in \mathbb{R}$,

$$\widetilde{F}(t) := tf(t) - 2F(t) \ge A|t|^p. \tag{7}$$

(**f$_4$**) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$.

Remark 1. Note that

$$\left(\frac{F(t)}{t^2}\right)' = \frac{tf(t) - 2F(t)}{t^3}. \tag{8}$$

This together with (7) implies that there exists $C_1 > 0$ such that for all $t \in \mathbb{R}$,

$$F(t) \ge C_1|t|^p. \tag{9}$$

Our main results are as follows.

Theorem 2. *Suppose that* (**v**), (**k**), *and* ($\mathbf{f_1}$)–($\mathbf{f_3}$) *are satisfied, then the problem* (1) *has a nontrivial solution.*

Theorem 3. *Suppose that* (**v**), (**k**), *and* ($\mathbf{f_1}$)–($\mathbf{f_4}$) *are satisfied, then the problem* (1) *has infinitely many solutions.*

Problem (1) arises in quantum mechanics and is related to the study of the nonlinear Schrödinger equation for a particle in an electromagnetic field or the Hartree-Fock equation. For a more detailed physical background of the Schrödinger-Poisson system, readers can refer to [2, 3] and the references therein.

This system has attracted considerable research attention in the recent decade, and it has been studied widely by using the modern variational method and critical point theory under various assumptions. However, many mathematical studies have been devoted to the case $\inf_{\mathbb{R}^3} V > 0$. In this case, there are many results on the existence, nonexistence, or multiplicity of solutions for (1). One can refer to [2–20].

There are very few studies devoted to (1) under the assumption that $-\Delta + V$ has a nontrivial negative eigenspace compared to the case $\inf_{\mathbb{R}^3} V > 0$. In a recent paper [21], Chen and Liu studied the problem under the assumption on V that

($\mathbf{v}'$) $V \in C(\mathbb{R}^3)$ is bounded from below and $\mu(V^{-1}(-\infty, M]) < \infty$ for every $M > 0$, where μ is the Lebesgue measure on $\mathbb{R}^3$. Moreover, the operator $-\Delta + V$ has negative eigenvalues.

They verified the existence of multiple solutions of (1) under this assumption on V and under certain 4-superlinear conditions on f. Our assumptions on V, which are different from ($\mathbf{v}'$), allow an infinitely dimensional negative eigenspace of $-\Delta + V$. This causes some difficulties. For example, it makes the verification of the compactness conditions a more delicate problem. In addition, when we search for infinitely many solutions of (1) for the case where f is odd, the classical fountain theorem of Bartsch (see [22] or [23]) cannot be applied. Fortunately, this difficulty can be overcome using a recently improved fountain theorem of Batkama and Colin [24]. To the best of our knowledge, the Schrödinger-Poisson equation with a strongly indefinite linear part has never been studied. Besides the difficulties caused by the strongly indefinite linear part, the functional related to (1) (see Section 2) involves a nonlocal term and it makes the functional not satisfy the global linking structure. To overcome this difficulty and obtain a nontrivial solution of (1), we use the local linking method (see [25]).

Throughout this paper, we denote the strong and the weak convergence by $\to$ and $\rightharpoonup$, respectively. $L^p(\mathbb{R}^3)$ denotes the standard Lebesgue space with norm $\|u\|_{L^p} = (\int_{\mathbb{R}^3} |u|^p dx)^{1/p}$. For $k \in \mathbb{N}$, $H^k(\mathbb{R}^3)$ denotes the standard Sobolev space with norm $\|u\|_{H^k} = (\int_{\mathbb{R}^3} \sum_{|\gamma| \le k} |\partial^\gamma u|^2 dx)^{1/2}$. For a Banach space E, we denote the dual space of E by E^*, and the norm of E^* is denoted by $\|\cdot\|_{E^*}$.

2. Proof of Theorem 2

Assume that (**v**) holds and let $S = -\Delta + V$ be the self-adjoint operator acting on $L^2(\mathbb{R}^3)$ with domain $D(S) = H^2(\mathbb{R}^3)$. By virtue of (**v**), we have the orthogonal decomposition

$$L^2 = L^2\left(\mathbb{R}^3\right) = L^+ + L^- \tag{10}$$

such that S is negative (resp. positive) in L^- (resp. in L^+). Let $X = D(|S|^{1/2})$ be equipped with the inner product

$$(u, v) = \left(|S|^{1/2} u, |S|^{1/2} v\right)_{L^2} \tag{11}$$

and norm $\|u\| = \||S|^{1/2} u\|_{L^2}$, where $(\cdot,\cdot)_{L^2}$ denotes the inner product of L^2. From (**v**),

$$X = H^1\left(\mathbb{R}^3\right) \tag{12}$$

with equivalent norms. Therefore, X continuously embeds in $L^q(\mathbb{R}^3)$ for all $2 \le q \le 6$. In addition, we have the decomposition

$$X = X^+ + X^-, \tag{13}$$

where $X^\pm = X \cap L^\pm$ is orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Therefore, for every $u \in X$, there is a unique decomposition

$$u = u^+ + u^-, \quad u^\pm \in X^\pm \tag{14}$$

with $(u^+, u^-) = 0$ and

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} V(x) u^2 dx = \|u^+\|^2 - \|u^-\|^2, \quad u \in X. \tag{15}$$

For $u \in X$, it is well known (see, e.g., Theorem 2.2.1 of [26]) that the Poisson equation

$$-\Delta\phi = K_2(x) u^2 \tag{16}$$

has a unique solution

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K_2(y) u^2(y)}{|x-y|} dy, \tag{17}$$

and $\phi_u \in \mathscr{D}^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 dx < \infty\}$.

Let

$$\begin{aligned}\Phi(u) = {} & \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 \\ & + \frac{1}{16\pi} \iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy \\ & - \int_{\mathbb{R}^3} Q(x) F(u)\, dx, \quad u \in X.\end{aligned} \tag{18}$$

Under the assumptions $(\mathbf{f_1})$ and $(\mathbf{k})$; Φ is a C^1 functional in X. The derivative of Φ is given by

$$\begin{aligned}\left\langle \Phi'(u), v\right\rangle &= (u^+, v) - (u^-, v) \\ &\quad + \frac{1}{8\pi}\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u(y) v(y)}{|x-y|} dx\, dy \\ &\quad + \frac{1}{8\pi}\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u(x) v(x) u^2(y)}{|x-y|} dx\, dy \\ &\quad - \int_{\mathbb{R}^3} Q(x) f(u) v dx, \quad \forall u, v \in X.\end{aligned} \tag{19}$$

It is easy to see that if u is a critical point of Φ, then (u, ϕ_u) is a solution of (1).

Our functional Φ does not satisfy the geometric assumptions of the generalized linking theorem (see, e.g., [23, Chapter 6]) because of the term

$$\frac{1}{16\pi}\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy. \tag{20}$$

To overcome this difficulty, we apply the local linking theorem to find critical points of Φ.

Recall that by definition (see [25]), a functional Ψ defined in X has a local linking at 0 with respect to the direct sum decomposition $X = X^- \oplus X^+$, if there is $\rho > 0$ such that

$$\begin{aligned}\Psi(u) &\le 0, \quad \text{for } u \in X^-,\ \|u\| \le \rho, \\ \Psi(u) &\ge 0, \quad \text{for } u \in X^+,\ \|u\| \le \rho.\end{aligned} \tag{21}$$

Let $\{e_k^\pm\}$ be the total orthonormal sequences in $X^\pm$. We consider two sequences of finite dimensional subspaces

$$X_1^\pm \subset \cdots \subset X_n^\pm \subset \cdots \subset X^\pm, \tag{22}$$

where $X_n^\pm = \operatorname{span}\{e_1^\pm, \ldots, e_n^\pm\}$. It is easy to see that

$$X^\pm = \overline{\bigcup_{n\in\mathbb{N}} X_n^\pm}. \tag{23}$$

For $n \in \mathbb{N}$, let $X_n = X_n^- \oplus X_n^+$ and Ψ_n denote the restriction of Ψ on X_n.

Definition 4. We say that $\Psi \in C^1(X)$ satisfies $(C)^*$ condition if any sequence $\{u_n\} \subset X$ such that

$$\begin{aligned}&u_n \in X_n, \\ &\sup_n \Psi(u_n) < \infty, \\ &(1 + \|u_n\|) \parallel \Psi_n'(u_n) \parallel_{X_n^*} \longrightarrow 0\end{aligned} \tag{24}$$

contains a subsequence that converges to a critical point of Ψ.

From [27, Theorem 2.2], we have the following.

Theorem 5. *Suppose that $\Psi \in C^1(X)$ has a local linking at 0, Φ satisfies $(C)^*$ condition, Ψ maps bounded sets into bounded sets, and for every $m \in \mathbb{N}$,*

$$\Psi(u) \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty,\ u \in X^- \oplus X_m^+. \tag{25}$$

Then, Ψ has a nontrivial critical point.

Lemma 6. *The functional Φ has a local linking in 0 with respect to the direct sum decomposition $X = X^+ \oplus X^-$.*

Proof. From the Hardy-littlewood-Sobolev inequality (see, e.g., [28]), we infer that there exists $C > 0$ such that for every $u \in X$,

$$\begin{aligned}&\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy \\ &\quad \le C\left\|K_1 u^2\right\|_{L^{6/5}(\mathbb{R}^3)} \left\|K_2 u^2\right\|_{L^{6/5}(\mathbb{R}^3)} \\ &\quad = C\left\|K_1^{1/2} u\right\|^2_{L^{12/5}(\mathbb{R}^3)} \left\|K_2^{1/2} u\right\|^2_{L^{12/5}(\mathbb{R}^3)}.\end{aligned} \tag{26}$$

From $(\mathbf{k})$, we deduce that K_i is bounded in $\mathbb{R}^3$, $i = 1, 2$. Therefore, by the Sobolev inequality, we have

$$\left\|K_i^{1/2} u\right\|_{L^{12/5}(\mathbb{R}^3)} \le C\|u\|_{L^{12/5}(\mathbb{R}^3)} \le C\|u\|, \quad i = 1, 2. \tag{27}$$

It follows that

$$\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy \le C\|u\|^4 \tag{28}$$

for some $C > 0$.

From $(\mathbf{f_1})$ and $(\mathbf{f_2})$, we deduce that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(t)| \le \epsilon t^2 + C_\epsilon |t|^p, \quad \forall t \in \mathbb{R}. \tag{29}$$

This together with the fact that Q is a bounded function in $\mathbb{R}^3$ (see $(\mathbf{k})$) implies that there exists $C' > 0$ such that

$$\left|\int_{\mathbb{R}^3} Q(x) F(u) dx\right| \le C'\epsilon\|u\|^2 + C' C_\epsilon \|u\|^p. \tag{30}$$

Combining (28), (30), and the definition of Φ (see (18)), we get that for any $u \in X^+$,

$$\begin{aligned}\Phi(u) &\ge \frac{1}{2}\|u\|^2 - C'\epsilon\|u\|^2 - C' C_\epsilon \|u\|^p \\ &= \left(\frac{1}{2} - C'\epsilon\right)\|u\|^2 - C' C_\epsilon \|u\|^p,\end{aligned} \tag{31}$$

and for any $u \in X^-$,

$$\begin{aligned}\Phi(u) &\le -\frac{1}{2}\|u\|^2 + \frac{C}{16\pi}\|u\|^4 - C'\epsilon\|u\|^2 - C' C_\epsilon \|u\|^p \\ &= -\left(\frac{1}{2} - C'\varepsilon\right)\|u\|^2 + \frac{C}{16\pi}\|u\|^4 + C' C_\epsilon \|u\|^p.\end{aligned} \tag{32}$$

Choose $\epsilon = 1/4C'$. Then, from the above two inequalities, we deduce that we can choose small $\rho > 0$ such that Φ satisfies (21). Therefore, Φ has a local linking in 0 with respect to the direct sum decomposition $X = X^+ \oplus X^-$. □

Lemma 7. *Under the assumptions* (**v**), (**k**), *and* ($\mathbf{f_1}$)–($\mathbf{f_3}$), *the functional* Φ *satisfies the* $(C)^*$ *condition.*

Proof. Let $\{u_n\}$ be a $(C)_c^*$ sequence; that is, $\sup_n \Phi(u_n) \le c$ and $(1+\|u_n\|)\|\Phi'(u_n)\|_{X_n^*} \to 0$.

First, we prove that $\{u_n\}$ is bounded in X. From (7), we have

$$\begin{aligned} o(1) + 2c &\ge 2\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \\ &= -\frac{1}{8\pi}\iint_{\mathbb{R}^3} \frac{K_1(x)K_2(y)u_n^2(x)u_n^2(y)}{|x-y|}dx\,dy \\ &\quad + \int_{\mathbb{R}^3} Q(x)(u_n f(u_n) - 2F(u_n))\,dx \\ &\ge -\frac{1}{8\pi}\iint_{\mathbb{R}^3} \frac{K_1(x)K_2(y)u_n^2(x)u_n^2(y)}{|x-y|}dx\,dy \\ &\quad + A\int_{\mathbb{R}^3} Q(x)|u_n|^p dx. \end{aligned} \tag{33}$$

From (26), we have

$$\begin{aligned} &\iint_{\mathbb{R}^3} \frac{K_1(x)K_2(y)u_n^2(x)u_n^2(y)}{|x-y|}dx\,dy \\ &\quad \le C\|K_1u^2\|_{L^{6/5}(\mathbb{R}^3)}\|K_2u^2\|_{L^{6/5}(\mathbb{R}^3)}. \end{aligned} \tag{34}$$

By the Hölder inequality and (4), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} K_i^{6/5}(x)|u_n(x)|^{12/5}dx \\ &\quad \le \left(\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx\right)^{12/5p} \\ &\qquad \times \left(\int_{\mathbb{R}^3} K_i^{6p/(5p-12)}(x)Q^{-12/(5p-12)}(x)\,dx\right)^{1-(12/5p)} \\ &\quad \le C\left(\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx\right)^{12/5p}, \quad i = 1,2. \end{aligned} \tag{35}$$

Combining (34) and (35), we get that

$$\begin{aligned} &\iint_{\mathbb{R}^3} \frac{K_1(x)K_2(y)u_n^2(x)u_n^2(y)}{|x-y|}dxdy \\ &\quad \le C\left(\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx\right)^{4/p}. \end{aligned} \tag{36}$$

This together with (33) yields that $\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx$ is bounded.

Second, we prove that $\|u_n\|$ is bounded. We have

$$\begin{aligned} o(1) &= \langle \Phi'(u_n), u_n^+ - u_n^- \rangle \\ &= \|u_n\|^2 - \int_{\mathbb{R}^3} Q(x) f(u_n)(u_n^+ - u_n^-)\,dx \\ &\quad + \frac{1}{8\pi}\iint_{\mathbb{R}^3} \Big(\big(K_1(x)K_2(y)u_n^2(x)u_n(y) \\ &\qquad \times (u_n^+(y) - u_n^-(y))\big) \\ &\qquad \times (|x-y|)^{-1}\Big)\,dx\,dy \\ &\quad + \frac{1}{8\pi}\iint_{\mathbb{R}^3} \Big(\big(K_1(x)K_2(y)u_n(x) \\ &\qquad \times (u_n^+(x) - u_n^-(x))u_n^2(y)\big) \\ &\qquad \times (|x-y|)^{-1}\Big)\,dx\,dy. \end{aligned} \tag{37}$$

From (5) and the Hölder inequality, we get that for any $R > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^3} Q(x) f(u_n)(u_n^+ - u_n^-)\,dx \\ &\quad \le C\int_{\mathbb{R}^3} Q(x)|u_n|\cdot|u_n^+ - u_n^-|\,dx \\ &\qquad + C\int_{\mathbb{R}^3} Q(x)|u_n|^{p-1}|u_n^+ - u_n^-|\,dx \\ &\quad \le C\int_{\{x\in\mathbb{R}^3 \mid |x|>R\}} Q(x)|u_n|\cdot|u_n^+ - u_n^-|\,dx \\ &\qquad + C\int_{\{x\in\mathbb{R}^3 \mid |x|\le R\}} Q(x)|u_n|\cdot|u_n^+ - u_n^-|\,dx \\ &\qquad + C\left(\int_{\mathbb{R}^3} Q(x)|u_n|^p dx\right)^{(p-1)/p} \\ &\qquad \times \left(\int_{\mathbb{R}^3} Q(x)|u_n^+ - u_n^-|^p dx\right)^{1/p}. \end{aligned} \tag{38}$$

Since $\int_{\mathbb{R}^3} Q(x)|u_n|^p dx$ is bounded,

$$\begin{aligned} &C\left(\int_{\mathbb{R}^3} Q(x)|u_n|^p dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^3} Q(x)|u_n^+ - u_n^-|^p dx\right)^{1/p} \\ &\quad \le C\|u_n\|. \end{aligned} \tag{39}$$

From $\lim_{|x|\to\infty} Q(x) = 0$ (see (**k**)) and the Sobolev inequality, we get that there exists $R > 0$ such that

$$\begin{aligned} &C\int_{\{x\in\mathbb{R}^3 \mid |x|>R\}} Q(x)|u_n|\cdot|u_n^+ - u_n^-|\,dx \\ &\quad \le C\sup_{|x|>R} Q\cdot\int_{\{x\in\mathbb{R}^3 \mid |x|>R\}} |u_n|\cdot|u_n^+ - u_n^-|\,dx \\ &\quad \le C\sup_{|x|>R} Q\cdot\|u_n\|^2 \le \frac{1}{2}\|u_n\|^2. \end{aligned} \tag{40}$$

From the Hölder and Sobolev inequalities and boundedness of $\int_{\mathbb{R}^3} Q(x)|u_n|^p dx$, we have

$$\begin{aligned} &C\int_{\{x\in\mathbb{R}^3|\,|x|\le R\}} Q(x)\,|u_n|\cdot|u_n^+-u_n^-|\,dx \\ &\le C\left(\int_{\{x\in\mathbb{R}^3|\,|x|\le R\}} Q(x)\,dx\right)^{p/(p-2)}\left(\int_{\mathbb{R}^3} Q(x)\ |u_n|^p\ dx\right)^{1/p} \\ &\quad\times\left(\int_{\mathbb{R}^3} Q(x)\,|u_n^+-u_n^-|^p dx\right)^{1/p} \le D_R\,\|u_n\|, \end{aligned} \tag{41}$$

where D_R is a positive constant that depends only on R. Combining (38)–(41), we infer that there exists $R>0$ such that

$$\int_{\mathbb{R}^3} Q(x)\,f(u_n)\,(u_n^+-u_n^-)\,dx \le \frac{1}{2}\|u_n\|^2+(D_R+C)\,\|u_n\|. \tag{42}$$

From the Hardy-Littlewood-Sobolev and Hölder inequalities, we get that

$$\begin{aligned} &\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n^2(x)\,u_n(y)\,(u_n^+(y)-u_n^-(y))}{|x-y|}dx\,dy \\ &\le C\left\|K_1u_n^2\right\|_{L^{6/5}(\mathbb{R}^3)}\left\|K_2u_n\,(u_n^+-u_n^-)\right\|_{L^{6/5}(\mathbb{R}^3)} \\ &\le C\left\|K_1^{1/2}u_n\right\|^2_{L^{12/5}(\mathbb{R}^3)}\left\|K_2^{1/2}u_n\right\|_{L^{12/5}(\mathbb{R}^3)} \\ &\quad\times\left\|K_2^{1/2}\,(u_n^+-u_n^-)\right\|_{L^{12/5}(\mathbb{R}^3)}. \end{aligned} \tag{43}$$

From (43), (35), and the boundedness of $\int_{\mathbb{R}^3} Q(x)|u_n|^p dx$, we get that there exists a constant $C>0$ such that

$$\begin{aligned} &\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n^2(x)\,u_n(y)\,(u_n^+(y)-u_n^-(y))}{|x-y|}dx\,dy \\ &\le C\,\left\|K_2^{1/2}\,(u_n^+-u_n^-)\right\|_{L^{12/5}(\mathbb{R}^3)} \le C\,\|u_n\|. \end{aligned} \tag{44}$$

Similarly, we have

$$\begin{aligned} &\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n(x)\,(u_n^+(x)-u_n^-(x))\,u_n^2(y)}{|x-y|}dx\,dy \\ &\le C\,\|u_n\|. \end{aligned} \tag{45}$$

Combining (37), (42), (44), and (45), we infer that $\|u_n\|$ is bounded.

Finally, we prove that $\{u_n\}$ has a convergent subsequence. Up to a subsequence, we assume that $u_n \rightharpoonup u$ in X. Then, u is a critical point of Φ. From $(\mathbf{f_1})$,

$$\int_{\mathbb{R}^3} Q(x)\,f(u_n)\,u^+dx \longrightarrow \int_{\mathbb{R}^3} Q(x)\,f(u)\,u^+dx \tag{46}$$

and from $\lim_{|x|\to\infty}Q(x)=0$ and $u_n\rightharpoonup u$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} Q(x)\,f(u_n)\,(u_n^+-u^+)\,dx\right| \\ &\quad\le C\int_{\mathbb{R}^3} Q(x)\,|u_n|^{p-1}\,|u_n^+-u^+|\,dx \longrightarrow 0. \end{aligned} \tag{47}$$

Therefore,

$$\int_{\mathbb{R}^3} Q(x)\,f(u_n)\,u_n^+dx-\int_{\mathbb{R}^3} Q(x)\,f(u)\,u^+dx \longrightarrow 0. \tag{48}$$

By $u_n\rightharpoonup u$ and $\lim_{|x|\to\infty}K_i(x)=0,\ i=1,2$, we get that

$$\begin{aligned} &\left|\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n^2(x)\,u_n(y)\,u_n^+(y)}{|x-y|}dx\,dy\right. \\ &\quad\left.-\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u^2(x)\,u(y)\,u^+(y)}{|x-y|}dx\,dy\right| \\ &\le\left\|K_1\left(u_n^2-u^2\right)\right\|_{L^{6/5}(\mathbb{R}^3)}\left\|K_2\left(u_nu_n^+-uu^+\right)\right\|_{L^{6/5}(\mathbb{R}^3)} \longrightarrow 0. \end{aligned} \tag{49}$$

Similarly,

$$\begin{aligned} &\left|\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n(x)\,u_n^+(x)\,u_n^2(y)}{|x-y|}dx\,dy\right. \\ &\quad\left.-\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u(x)\,u^+(x)\,u^2(y)}{|x-y|}dx\,dy\right| \longrightarrow 0. \end{aligned} \tag{50}$$

By (48)–(50), $\langle\Phi'(u),u^+\rangle=0$, and

$$\begin{aligned} o(1) &= \langle\Phi'(u_n),u_n^+\rangle \\ &= \|u_n^+\|^2-\int_{\mathbb{R}^3} Q(x)\,f(u_n)\,u_n^+dx \\ &\quad+\frac{1}{8\pi}\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n^2(x)\,u_n(y)\,u_n^+(y)}{|x-y|}dx\,dy \\ &\quad+\frac{1}{8\pi}\iint_{\mathbb{R}^3}\frac{K_1(x)\,K_2(y)\,u_n(x)\,u_n^+(x)\,u_n^2(y)}{|x-y|}dx\,dy, \end{aligned} \tag{51}$$

we get that $u_n^+\to u^+$ in X. The same argument implies that $u_n^-\to u^-$ in X. Therefore, $u_n\to u$ in X. □

Remark 8. From the proof of this theorem, we infer that Φ also satisfies the Cerami condition; that is, if $\{u_n\}\subset X$ satisfies

$$\sup_n\Phi(u_n)<+\infty,\qquad(1+\|u_n\|)\,\|\Phi'(u_n)\|_{X^*}\longrightarrow 0, \tag{52}$$

then $\{u_n\}$ contains a convergent subsequence.

Lemma 9. *The functional Φ satisfies* (25).

Proof. If the functional does not satisfy (25), then there exist $M > 0$, $m \in \mathbb{N}$, and $u_n \in X_m^+ \oplus X^-$ such that $\|u_n\| \to \infty$ and

$$\Phi(u_n) = \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{16\pi}\iint_{\mathbb{R}^3} \frac{K_1(x)K_2(y)u_n^2(x)u_n^2(y)}{|x-y|}dx\,dy - \int_{\mathbb{R}^3} Q(x)F(u_n)\,dx \geq -M. \quad (53)$$

This together with (36) and (9) yields

$$\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + C\left(\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx\right)^{4/p} - C_1\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx \geq -M. \quad (54)$$

Since $p > 4$, there exists $D > 0$ such that

$$D - \frac{C_1}{2}\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx \geq C\left(\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx\right)^{4/p} - C_1\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx. \quad (55)$$

Then, by (54),

$$\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \frac{C_1}{2}\int_{\mathbb{R}^3} Q(x)|u_n(x)|^p dx \geq -M - D. \quad (56)$$

We use E to denote the closure of $X_m^+ \oplus X^-$ under the norm $||u||_{Q,p} = (\int_{\mathbb{R}^3} Q(x)|u(x)|^p dx)^{1/p}$. Since there exists a continuous projection $P : E \to X_m^+$, there exists $C_3 > 0$ such that for every $u \in X_m^+ \oplus X^-$,

$$\left(\int_{\mathbb{R}^3} Q(x)|u(x)|^p dx\right)^{1/p} \geq C_3\left(\int_{\mathbb{R}^3} Q(x)|u^+(x)|^p dx\right)^{1/p}. \quad (57)$$

Since $\|u_n\| \to \infty$ and X_m^+ is finite dimensional, from (56) and (57), we get that

$$-M - D \leq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \frac{C_1C_3^p}{2}\int_{\mathbb{R}^3} Q(x)|u_n^+(x)|^p dx \longrightarrow -\infty. \quad (58)$$

This contradiction implies that Φ satisfies (25). □

Proof of Theorem 2. The desired result of Theorem 2 can be derived by combining Theorem 5 and Lemmas 6, 7, and 9. □

3. Proof of Theorem 3

Recall that $\{e_k^\pm\}$ are the total orthonormal sequences in $X^\pm$. For $k = 1, 2, \ldots$, let

$$Y_k = X^- \oplus X_k^+ = X^- \oplus \operatorname{span}\{e_1^+, \ldots, e_k^+\}, \qquad Z_k = \overline{\operatorname{span}\{e_k^+, e_{k+1}^+, \ldots\}}. \quad (59)$$

Let $P : X \to X^-$, $Q : X \to X^+$ be the orthogonal projections. As [23], on X, we define

$$|||u||| = \max\left\{\|Qu\|, \sum_{j=1}^{\infty}\frac{1}{2^{k+1}}\left|(Pu, e_k^-)\right|\right\}. \quad (60)$$

The topology generated by $|||\cdot|||$ will be denoted by τ and all topological notations related to it will include this symbol.

For the proof of Theorem 3, we use the following improved fountain theorem of Batkam and Colin [24, Theorem 12], which is a generalization of the classical fountain theorem of Bartsch [22] (see also [23]).

Theorem 10. *Assume that an even functional $\Psi \in C^1(X)$ satisfies the following:*

(**A**) *Ψ is τ-upper semicontinuous, and the gradient $\nabla\Psi$ is weakly sequentially continuous.*

If there exist $\rho_k > r_k > 0$ such that

(i) $b_k = \inf_{u\in Z_k, \|u\|=r_k}\Psi(u) \to +\infty$, *as* $k \to \infty$,

(ii) $a_k = \max_{u\in Y_k, \|u\|=\rho_k}\Psi(u) \leq 0$,

then there exist $\{c_k\} \subset \mathbb{R}$ and sequences $\{u_{k,n}\} \subset X$ such that for every $k \in \mathbb{N}$, $c_k \geq b_k$ and

$$\Psi(u_{k,n}) \longrightarrow c_k, \qquad (1 + \|u_{k,n}\|)\left\|\Psi'(u_{k,n})\right\|_{X^*} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (61)$$

Remark 11. In [24], the result of this fountain theorem is

$$\Psi(u_{k,n}) \longrightarrow c_k, \quad \left\|\Psi'(u_{k,n})\right\|_{X^*} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (62)$$

Since the deformation theorem is still valid under the Cerami condition (see, e.g., [27]), replacing the pseudogradient vector field $2\|\nabla\Psi(v)\|^{-2}\nabla\Psi(v)$ in page 442 of [24] by the Cerami-type pseudogradient vector field $(1 + \|v\|)\nabla\Psi(v)$, we see that similar to many critical point theorems, the result of the fountain theorem in [24] can be improved as (61).

Proof of Theorem 3. (1) Let us prove that Φ is τ-upper semicontinuous. Assume that $|||u_n - u||| \to 0$ and $c \leq \Phi(u_n)$. Since $u_n^+ = Qu_n \to Qu$, $u_n^- = Pu_n$ are bounded so that $Pu_n \rightharpoonup Pu$, it follows that $u_n \rightharpoonup u$ in X. Since $\lim_{|x|\to\infty}Q(x) = 0$, from ($\mathbf{f_1}$) and $u_n \rightharpoonup u$, we obtain

$$\int_{\mathbb{R}^3} Q(x)F(u_n)\,dx \longrightarrow \int_{\mathbb{R}^3} Q(x)F(u)\,dx. \quad (63)$$

In addition, from the proof of (49) we infer that

$$\left|\iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u_n^2(x) u_n^2(y)}{|x-y|} dx\, dy - \iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy\right| \longrightarrow 0. \tag{64}$$

Using the Fatou lemma, we obtain

$$\begin{aligned}
&-\Phi(u) \\
&= \frac{\|Pu\|^2}{2} - \frac{\|Qu\|^2}{2} - \frac{1}{16\pi} \\
&\quad \times \iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy \\
&\quad + \int_{\mathbb{R}^3} Q(x) F(u)\, dx \\
&\le \liminf_{n\to\infty} \left(\frac{\|Pu_n\|^2}{2} - \frac{\|Qu_n\|^2}{2} \right. \\
&\qquad - \frac{1}{16\pi} \iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u_n^2(x) u_n^2(y)}{|x-y|} dx\, dy \\
&\qquad \left. + \int_{\mathbb{R}^3} Q(x) F(u_n)\, dx \right) \\
&= \liminf_{n\to\infty} -\Phi(u_n) \le -c.
\end{aligned} \tag{65}$$

(2) The proof that $\nabla\Psi$ is weakly sequentially continuous is similar to that in the proof of Lemma 6.15 of [23].

(3) Verification of (i) for Φ: since $\lim_{|x|\to\infty} Q(x) = 0$, we infer that if $u_k \rightharpoonup 0$ in X, then

$$\int_{\mathbb{R}^3} Q(x) |u_n|^p dx \longrightarrow 0. \tag{66}$$

It follows that

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_{Q,p} \longrightarrow 0, \quad \text{as} \quad k \longrightarrow \infty, \tag{67}$$

where

$$\|u\|_{Q,q} = \left(\int_{\mathbb{R}^3} Q(x) |u|^q dx \right)^{1/q}. \tag{68}$$

From $(\mathbf{f_1})$ and $(\mathbf{f_2})$, we deduce that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(t)| \le \epsilon |t|^2 + C_\epsilon |t|^p, \quad \forall t \in \mathbb{R}. \tag{69}$$

For $u \in Z_k$, by the Sobolev inequality and (67),

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 \\
&\quad + \frac{1}{16\pi} \iint_{\mathbb{R}^3} \frac{K_1(x) K_2(y) u^2(x) u^2(y)}{|x-y|} dx\, dy \\
&\quad - \int_{\mathbb{R}^3} Q(x) F(u)\, dx \\
&\ge \frac{1}{2}\|u\|^2 - \epsilon \|u\|_{Q,2}^2 - C_\epsilon \|u\|_{Q,p}^p \\
&\ge \frac{1}{2}\|u\|^2 - C\epsilon \|u\|^2 - C_\epsilon \beta_k^p \|u\|^p.
\end{aligned} \tag{70}$$

Choosing $\epsilon = 1/4C$ and letting $r_k = (2C_\epsilon p \beta_k^p)^{1/(2-p)}$, we get from (70) that for $\|u\| = r_k$,

$$\Phi(u) \ge \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\left(2C_\epsilon p \beta_k^p\right)^{2/(2-p)}. \tag{71}$$

Since $\beta_k \to 0$ and $p > 2$, it follows that

$$b_k = \inf_{u\in Z_k, \|u\|=r_k} \Phi(u) \longrightarrow +\infty. \tag{72}$$

(4) Verification of (ii) for Φ: since $Y_k = X^- \oplus X_k^+$, (ii) is a direct consequence of Lemma 9.

Finally, from $(\mathbf{f_4})$, we obtain that Φ is an even functional. Then, from (1)–(4) and Remark 8, we deduce that for every k, there exists a critical point u_k of Φ such that $\Phi(u_k) \ge b_k$. Therefore, (1) has infinitely many solutions $\{(u_k, \phi_{u_k})\}$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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