Research Article

Global Hopf Bifurcation Analysis for an Avian Influenza Virus Propagation Model with Nonlinear Incidence Rate and Delay

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The paper investigated an avian influenza virus propagation model with nonlinear incidence rate and delay based on SIR epidemic model. We regard delay as bifurcating parameter to study the dynamical behaviors. At first, local asymptotical stability and existence of Hopf bifurcation are studied; Hopf bifurcation occurs when time delay passes through a sequence of critical values. An explicit algorithm for determining the direction of the Hopf bifurcations and stability of the bifurcation periodic solutions is derived by applying the normal form theory and center manifold theorem. What is more, the global existence of periodic solutions is established by using a global Hopf bifurcation result.

1. Introduction

In March 2013, new avian-origin influenza $A(H7N9)$ virus $(A - OIV)$ broke out in Shanghai and the surrounding provinces of China [1]. During the first week of April, this virus had been detected in six provinces and municipal cities; this virus has caused global concern as a potential pandemic threat [2]. The virus fast took people's life without timely treatment. Therefore, strong measures should be taken to control the spread of H7N9 viruses.

$H7N9$ is an infectious disease caused by influenza A virus. Moreover, it is essential to study and to dominate the spread of $H7N9$. Mathematical models become important instruments in the analysis and control of infectious diseases. The present study evaluates the possible application of SIR model for $H7N9$ spreading.

Let $S(t)$, $I(t)$, and $R(t)$ be the population densities of susceptible, infective, and recovered, respectively. Recruitment of new individuals is into the susceptible class at a constant rate B [3]. Parameters μ_1, μ_2, and μ_3 are positive constants which represent the death rate of the classes, respectively. τ is the length of the infectious period; $1/\gamma$ is the average time spent in class I before recovery [3].

In 1979, Cooke [4] used mass action incidence $\beta S(t)I(t - \tau)$. In 2009, Xu and Ma [5] developed the model with the force

of infection given by $\beta S(t)(I(t - \tau)/(1 + \alpha I(t - \tau)))$, where α determines the level at which the force of infection saturates and β is a contract [5]. Then, the avian influenza virus propagation model based on SIR model has the following form:

$$
\begin{aligned}
\dot{S}(t) &= B - \mu_1 S(t) - \frac{\beta S(t) I(t - \tau)}{1 + \alpha I(t - \tau)}, \\
\dot{I}(t) &= \frac{\beta S(t) I(t - \tau)}{1 + \alpha I(t - \tau)} - (\mu_2 + \gamma) I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu_3 R(t).
\end{aligned} \tag{1}
$$

Since R does not appear in the first two equations, and avoid excessive use of parentheses in some of the latter calculations, the avian influenza virus propagation model is transformed into the following form

$$
\begin{aligned}
\dot{S}(t) &= B - \mu_1 S(t) - \frac{\beta S(t) I(t - \tau)}{1 + \alpha I(t - \tau)}, \\
\dot{I}(t) &= \frac{\beta S(t) I(t - \tau)}{1 + \alpha I(t - \tau)} - (\mu_2 + \gamma) I(t),
\end{aligned} \tag{2}
$$

$$
\dot{R}(t) = \gamma I(t) - \mu_3 R(t), \tag{3}
$$

with the following initial condition:

$$S(0) \in R^+, \; I(\theta) = \phi(\theta) \quad \text{for } \theta \in [-\tau, 0],$$

$$\text{where } \phi \in C\left([-\tau, 0], R^+\right), \tag{4}$$

which was presented and studied in [3].

The steady state of the model and the stability of epidemic models have been studied in many papers. Zhang and Li [6] studied the global stability of an SIR epidemic model with constant infectious periods. Xu and Ma [5] showed the global stability of the endemic equilibrium for the case of the reproduction number $R_0 > 1$. McCluskey [3] shown that the endemic equilibrium is globally asymptotically stable whenever it exists. In this paper, we investigated the Hopf bifurcation and the global existence of periodic solutions of model (2), which have not been reported yet.

The organization of this paper is as follows. In Section 2, we will investigate the local asymptotical stability and existence of Hopf bifurcation by analyzing the associated characteristic equation. In Section 3, an explicit algorithm for determining the direction of the Hopf bifurcations and stability of the bifurcation periodic solutions will be derived by applying the normal form theory and center manifold theorem. In Section 4, existence of global periodic solutions will be established by using a global Hopf bifurcation result. In Section 5, a brief discussion is offered to conclude this work.

2. Local Stability and Hopf Bifurcation

Some results can be directly obtained from [3, 5]. The basic reproduction number for the model is $R_0 = B\beta/\mu_1(\mu_2 + \gamma)$. System (2) always has a disease-free equilibrium $E_1 = (B/\mu_1, 0)$. If $B\beta > \mu_1(\mu_2 + \gamma)$, system (2) has a unique endemic equilibrium $E^* = (S^*, I^*) = ((B\alpha + \mu_2 + \gamma)/(\beta + \alpha\mu_1), (B\beta - \mu_1(\mu_2 + \gamma))/(\mu_2 + \gamma)(\beta + \alpha\mu_1))$ [3]. The characteristic equation of system (2) at the endemic equilibrium E^* is

$$\lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0, \tag{5}$$

where $p_0 = (\mu_2 + \gamma)(\mu_1 + \beta I^*/(1 + \alpha I^*))$, $p_1 = \mu_1 + \mu_2 + \gamma + \beta I^*/(1 + \alpha I^*)$, $q_0 = -\beta\mu_1 S^*/(1 + \alpha I^*)^2$, and $q_1 = -\beta S^*/(1 + \alpha I^*)^2$. If

$$R_0 > 1 \tag{P_1}$$

hold, when $\tau = 0$, the endemic equilibrium E^* of system (2) is locally stable [5].

If $i\omega \; (\omega > 0)$ is a solution of system (2), separating real and imaginary parts, we obtain the following:

$$p_1\omega = q_0 \sin\omega\tau - q_1\omega\cos\omega\tau,$$

$$\omega^2 - p_0 = q_0\cos\omega\tau + q_1\omega\sin\omega\tau. \tag{6}$$

Then, we get

$$\cos\omega\tau = \frac{(q_0 - p_1 q_1)\omega^2 - p_0 q_0}{q_0^2 + q_1^2\omega^2},$$

$$\sin\omega\tau = \frac{p_1 q_0\omega + (\omega^2 - p_0)q_1\omega}{q_0^2 + q_1^2\omega^2}. \tag{7}$$

It follows that

$$\omega^4 + \left(p_1^2 - 2p_0 - q_1^2\right)\omega^2 + p_0^2 - q_0^2 = 0. \tag{8}$$

Letting $z = \omega^2$, we get

$$z^2 + \left(p_1^2 - 2p_0 - q_1^2\right)z + p_0^2 - q_0^2 = 0. \tag{9}$$

It is easy to show that

$$p_1^2 - 2p_0 - q_1^2 = \left(\mu_1 + \frac{\beta I^*}{1 + \alpha I^*}\right)^2$$

$$+ (\mu_2 + \gamma)^2 - \frac{(\mu_2 + \gamma)^2}{(1 + \alpha I^*)^2} > 0,$$

$$p_0^2 - q_0^2 \tag{10}$$

$$= (\mu_2 + \gamma)\left[(\mu_2 + \gamma)\left(\mu_1 + \frac{\beta I^*}{1 + \alpha I^*}\right) + \frac{\beta\mu_1 S^*}{(1 + \alpha I^*)^2}\right]$$

$$\times \left(\mu_1 - \frac{\mu_1}{1 + \alpha I^*} + \frac{\beta I^*}{1 + \alpha I^*}\right).$$

The case of

$$\beta \geq \mu_1\alpha \tag{H_1}$$

has been discussed in [5]. We obtain global asymptotic stability of the endemic equilibrium when $R_0 > 1$. If

$$\beta < \mu_1\alpha \tag{H_2}$$

hold, that is, $(\beta - \mu_1\alpha)I^*/(1 + \alpha I^*) < 0$, we have $p_0^2 - q_0^2 < 0$. Following the theorem given by Ruan [7], there exists critical value

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\frac{(\omega_k^2 - p_0)q_0 - p_1 q_1\omega_k^2}{q_0^2 + q_1^2\omega_k^2} + \frac{2j\pi}{\omega_k}, \tag{11}$$

with

$$\omega_j$$

$$= \left[\frac{2p_0 + p_1^2 - q_1^2 + \sqrt{\left(2p_0 + p_1^2 - q_1^2\right)^2 - 4\left(p_0^2 - q_0^2\right)}}{2}\right]^{1/2}, \tag{12}$$

where $k = 1, 2, \ldots,$ $j = 0, 1, 2, \ldots$. If (P_1) and (H_2) are satisfied, (6) has a pair of purely imaginary roots $\pm\omega_0 i$ when

$\tau = \tau_0$. Additionally, all roots of (6) have negative real parts when $\tau \in [0, \tau_0]$ and when $\tau > \tau_0$ (5) has at least a pair of roots with positive real part. In order to give the main results, it is necessary to prove the transversality condition $\mathrm{Re}(d\lambda/d\tau)^{-1} > 0$ holds. Denote $\lambda = \alpha(\tau) + i\omega(\tau)$ as the root of (5) with $\alpha(\tau) = 0$, $\omega(\tau) = \omega_0$. Differentiating (5) with respect to τ yields

$$\left[2\lambda + p_1 + \left(q_1 - \tau\left(q_1\lambda + q_0\right)e^{-\lambda\tau}\right)\right]\frac{d\lambda}{d\tau} = \lambda\left(q_1\lambda + q_0\right)e^{-\lambda\tau}. \tag{13}$$

For the sake of simplicity denoting ω_0 and τ_0 by ω, τ, respectively,

$$\frac{d\lambda}{d\tau} = \frac{\left(q_1\lambda + q_0\right)\lambda e^{-\lambda\tau}}{2\lambda + p_1 + q_1 e^{-\lambda\tau} - \left(\left(q_1\lambda + q_0\right)\tau e^{-\lambda\tau}\right)} \tag{14}$$

in the following:

$$\begin{aligned}
\mathrm{Re}&\left(\frac{d\lambda}{d\tau}\right)^{-1} \\
&= \frac{2\lambda + p_1 + q_1 e^{-\lambda\tau}}{\left(q_1\lambda + q_0\right)\lambda e^{-\lambda\tau}} \\
&= \left(\left(p_1\cos\omega\tau - 2\omega\sin\omega\tau + q_1\right)\right. \\
&\qquad + i\left(2\omega\cos\omega\tau + p_1\sin\omega\tau\right)\right) \\
&\qquad \times \left(-q_1\omega^2 + iq_0\omega\right)^{-1} \\
&= \left(-p_1 q_1\omega^2\cos\omega\tau + 2q_1\omega^3\sin\omega\tau\right. \\
&\qquad - q_1^2\omega^2 + 2q_0\omega^2\cos\omega\tau + p_1 q_0\omega\sin\omega\tau\right) \\
&\qquad \times \left(q_0\omega^2 + \omega^4\right)^{-1} \\
&= \left(\left(-p_1 q_1\omega^2 + 2q_0\omega^2\right)\left(q_0 - p_1 q_1\right)\omega^2 - p_0 q_0\right. \\
&\qquad + \left(2q_1\omega^3 + p_1 q_0\omega\right) \\
&\qquad \times \left[p_1 q_0\omega + \left(\omega^2 - p_0\right)q_1\omega\right] - q_0^2 q_1^2\omega^2 - q_1^4\omega^2\right) \\
&\qquad \times \left(\left(q_0^2 + q_1^2\omega^2\right)\left(q_0\omega^2 + \omega^4\right)\right)^{-1} \\
&= \left(2q_1^2\omega^6 + \left[2q_0^2 + \left(p_1^2 - 2p_0 - q_1^2\right)q_1^2\right]\omega^4\right. \\
&\qquad + \left(p_1^2 + 2p_0 - q_1^2\right)q_0^2\omega^2\right) \\
&\qquad \times \left(\left(q_0^2 + q_1^2\omega^2\right)\left(q_0\omega^2 + \omega^4\right)\right)^{-1}.
\end{aligned} \tag{15}$$

From (10), we know $p_1^2 - 2p_0 - q_1^2 > 0$; then, $\mathrm{Re}\,(d\lambda/d\tau)^{-1} > 0$ hold. Under this condition, we have the following theorem.

Theorem 1. *(i) If (P_1) and (H_1) holds, the equilibrium (S^*, I^*) of system (2) is asymptotically stable for any $\tau > 0$.*

(ii) If (P_1) and (H_2) holds, (S^*, I^*) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau \in (\tau_0, +\infty)$. System (2)

exhibits the Hopf bifurcation at the equilibrium (S^*, I^*) for $\tau = \tau_j$, $j = 0, 1, 2, \ldots.$

3. Direction and Stability of the Bifurcating Periodic Solutions

In Section 2, we obtain the conditions under which a family of periodic solutions bifurcate from the steady state at the critical value of τ. In this section, we investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solution at critical values τ_0, using techniques of the normal form theory and center manifold theorem.

Let $u_1 = S(t) - S^*$ and let $u_2 = I(t) - I^*$. The Taylor expansion of system (2) at E^* is

$$\begin{aligned}
\dot{u}_1(t) &= a_1 u_1(t) - a_2 u_2(t - \tau) \\
&\quad + a_6 u_2(t)^2(t - \tau) + a_7 u_1(t)u_2(t - \tau), \\
\dot{u}_2(t) &= a_3 u_1(t) + a_4 u_2(t) + a_5 u_2(t - \tau) \\
&\quad - a_7 u_1(t)u_2(t - \tau) - a_6 u_2^2(t - \tau),
\end{aligned} \tag{16}$$

where $a_1 = -\mu_1 + \beta I^*/(1 + \alpha I^*)$, $a_2 = -\beta S^2/(1 + \alpha I^*)$, $a_3 = \beta I^*/(1 + \alpha I^*)$, $a_4 = -\mu_2 - \gamma$, $a_5 = \beta S^2/(1 + \alpha I^*)$, $a_6 = -\alpha\beta S^*/(1 + \alpha I^*)^3$, $a_7 = \beta/(1 + \alpha I^*)^2$, $\tau = \tau_0 + \mu$, and $u_t = u(t + \theta) \in C_1$ for $\theta \in [-1, 0]$. System (2) is transformed into FDE as

$$\dot{u}(t) = L_\mu + F(u_t, \mu), \tag{17}$$

with

$$L_\mu(\phi) = (\tau_0 + \mu)\left[B_1\phi(0) + B_2\phi(-1)\right],$$

$$F(\phi, \mu) = (\tau_0 + \mu)\begin{pmatrix} a_6\phi_2^2(-1) + a_7\phi_1(0)\phi_2(-1) \\ -a_6\phi_1(0)\phi_2(-1) - a_7\phi_2^2(-1) \end{pmatrix}, \tag{18}$$

where

$$B_1 = \begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & a_2 \\ 0 & a_5 \end{pmatrix}. \tag{19}$$

By Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation, for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{20}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\left[B_1\delta(\theta) + B_2\delta(\theta + 1)\right], \tag{21}$$

where $\delta(\theta)$ is a delta function.

For $\phi \in C'[-1, 0]$, the operators A and R are defined as follows:

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^0 d\left(\eta(t, \mu)\phi(t)\right), & \theta = 0, \end{cases} \tag{22}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \theta), & \theta = 0. \end{cases} \tag{23}$$

The adjoint operator $A^*(0)$ corresponding to $A(0)$ is defined as follows:

$$A^* \psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\[2mm] \displaystyle\int_{-1}^{0} d\left(\eta^T(t,0)\,\psi(-t)\right), & s = 0 \end{cases} \tag{24}$$

and an adjoint bilinear is as follows:

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\,\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \overline{\psi}(\xi - \theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi, \tag{25}$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the preceding discussion, we know that $q(\theta)$ and $q^*(\theta)$ be the eigenvectors of A and A^* corresponding to $i\tau_0\omega_0$ and $-i\tau_0\omega_0$, respectively. Next, we calculate $q(\theta)$ and $q^*(s)$ to determine the normal form of operator A.

Proposition 2. *Let $q(\theta)$ and $q^*(s)$ be eigenvectors of A and A^* corresponding to $i\tau_0\omega_0$ and $-i\tau_0\omega_0$, respectively, satisfying $\langle q^*, q \rangle = 1$ and $\langle q^*, \overline{q} \rangle = 0$.*

Then,

$$q(\theta) = (1, \alpha)^T e^{i\omega_0\tau_0\theta} = \left(1, \frac{\omega_0 i - a_1}{a_2 e^{-i\omega_0\tau_0}}\right)^T e^{i\omega_0\tau_0\theta},$$

$$q^*(s) = D(1, \beta^*)\, e^{-i\omega_0\tau_0 s} = \left(1, \frac{-\omega_0 i - a_1}{a_3}\right)^T e^{-i\omega_0\tau_0 s}, \tag{26}$$

where

$$D = \frac{1}{1 + \alpha\overline{\beta}^* - \tau_0\alpha\left(a_2 + \overline{\beta}^* a_5\right)e^{-i\omega_0\tau_0}}. \tag{27}$$

Proof. Without loss of generality, we just consider the eigenvector $q(\theta)$. By the definition of A and $q(\theta)$ with $\theta \in [-1, 0)$, we get $q(\theta) = (1, \alpha)^T e^{i\omega_0\tau_0}$ (here, α is a parameter). In what follows, notice that $q(0) = (1, \alpha)^T$ and $Aq(0) = \int_{-1}^{0} d(\eta(t, \mu)\phi(t)) = i\omega_0\tau_0 q(0)$; we have $\alpha = (\omega_0 i - a_1)/a_2 e^{-i\omega_0\tau_0}$. Using a proof procedure similar to that in [8], by direct computation, we get $q(\theta)$ and $q^*(s)$. Bring $q(\theta)$ and $q^*(s)$ into $\langle q^*, q \rangle = 1$; it is not hard to obtain the parameter $\overline{D}$. The detailed procedure of proof refers to [9]. The proof is completed. $\square$

Then, we construct the coordinates of the center manifold C_0 at $\mu = 0$. Let

$$z(t) = \langle q^*, u_t \rangle, \qquad W(t, \theta) = u_t(\theta) - 2\,\mathrm{Re}\,\{z(t)\, q(\theta)\}. \tag{28}$$

On the center manifold C_0, we have

$$W(t, \theta) = W\left(z(t), \overline{z(t)}, \theta\right), \tag{29}$$

where

$$W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}\frac{z^3}{6}, \ldots; \tag{30}$$

and z and $\overline{z}$ are local coordinates for the center manifold C_0 in the direction of q and q^*, respectively. Since $\mu = 0$, we have

$$
\begin{aligned}
z'(t) &= i\tau_0\omega_0 z(t) + \langle q^*(\theta), f(W + 2\,\mathrm{Re}\,\{z(t)\,q(\theta)\})\rangle \\
&= i\tau_0\omega_0 z(t) + \overline{q^*(0)} f\left(W(z, \overline{z}, 0) + 2\,\mathrm{Re}\,\{z(t)\,q(0)\}\right) \\
&\triangleq i\tau_0\omega_0 z(t) + \overline{q^*(0)} f_0(z, \overline{z}),
\end{aligned}
\tag{31}
$$

where

$$f_0(z, \overline{z}) = f_{z^2}\frac{z^2}{2} + f_{\overline{z}^2}\frac{\overline{z}^2}{2} + f_{z\overline{z}}z\overline{z} + \cdots. \tag{32}$$

We rewrite this as

$$z'(t) = i\tau_0\omega_0 z + g(z, \overline{z}), \tag{33}$$

with

$$
\begin{aligned}
g(z, \overline{z}) &= \overline{q^*}(0)\, f_0(z, \overline{z}) \\
&= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots, \\
g(z, \overline{z}) &= D\tau_0\left(1, \overline{\beta}^*\right)\begin{pmatrix} a_6\phi_2^2(-1) + a_7\phi_1(0)\phi_2(-1) \\ -a_6\phi_1(0)\phi_2(-1) - a_7\phi_2^2(-1) \end{pmatrix},
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
\phi_1(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} \\
&\quad + W_{02}^{(1)}(0)\frac{\overline{z}}{2}, \\
\phi_2(-1) &= z\alpha e^{-i\omega_0\tau_0} + \overline{z}\,\overline{\alpha}e^{i\omega_0\tau_0} + W_{20}^{(2)}(-1)\frac{z^2}{2} \\
&\quad + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}}{2}.
\end{aligned}
\tag{35}
$$

Comparing the coefficients of the above equation with (22), we obtain

$$
g_{20} = 2D\tau_0\left[\left(a_6 - \overline{\beta}^* a_7\right)\alpha^2 e^{-2i\omega_0\tau_0} + \left(a_7 - \overline{\beta}^* a_6\right)\alpha e^{-i\omega_0\tau_0}\right],
$$

$$
\begin{aligned}
g_{11} &= D\tau_0\left[2\left(a_6 - \overline{\beta}^* a_7\right)\alpha\overline{\alpha} + \left(a_7 - \overline{\beta}^* a_6\right) \right. \\
&\quad \left. \times \left(\overline{\alpha}e^{i\omega_0\tau_0} + \alpha e^{-i\omega_0\tau_0}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
g_{02} &= 2D\tau_0\left[\left(a_6 - \overline{\beta}^* a_7\right)\overline{\alpha}^2 e^{2i\omega_0\tau_0} \right. \\
&\quad \left. + \left(a_7 - \overline{\beta}^* a_6\right)\overline{\alpha}e^{-i\omega_0\tau_0}\right],
\end{aligned}
$$

$$g_{21} = 2D\tau_0 \left[\left(a_6 - \overline{\beta^*} a_7 \right) \right.$$

$$\times \left[\overline{\alpha} e^{i\omega_0 \tau_0} W_{20}^{(2)}(-1) + 2\alpha e^{-i\omega_0 \tau_0} W_{11}^{(2)}(-1) \right]$$

$$+ \left(a_7 - \overline{\beta^*} a_6 \right)$$

$$\times \left[\frac{1}{2} \overline{\alpha} e^{i\omega_0 \tau_0} W_{20}^{(1)}(0) + \alpha e^{-i\omega_0 \tau_0} W_{11}^{(1)}(0) \right.$$

$$\left. + \frac{1}{2} W_{20}^{(2)}(-1) + W_{11}^{(2)}(-1) \right],$$

$$(36)$$

$$\dot{W} = \dot{u}_t - \dot{z} q - \dot{\overline{z}} \overline{q}$$

$$= \begin{cases} AW - 2 \operatorname{Re} \overline{q^*}(0) f_0 q(\theta), & \theta \in [-1, 0], \\ AW - 2 \operatorname{Re} \overline{q^*}(0) f_0 q(\theta) + f_0, & \theta = 0. \end{cases}$$

$$(37)$$

$$\triangleq AW + H(z, \overline{z}, \theta),$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}(\theta) \frac{\overline{z}^2}{2} + \cdots .$$

$$(38)$$

Expanding the above series and comparing the coefficients, we get

$$(A - 2i\omega_0 \tau_0 I) W_{20}(\theta) = -H_{20}(\theta),$$

$$AW_{11}(\theta) = -H_{11}(\theta),$$

$$(A + 2i\omega_0 \tau_0 I) W_{02}(\theta) = -H_{02}(\theta).$$

$$(39)$$

Comparing the coefficients with (38), we obtain

$$H_{20}(\theta) = -g_{20} q(\theta) - \overline{g}_{02} \overline{q}(\theta),$$

$$H_{11}(\theta) = -g_{11} q(\theta) - \overline{g}_{11} \overline{q}(\theta).$$

$$(40)$$

It follows from (39), (40), and the definition of A that we have

$$\dot{W}_{20}(\theta) = 2i\tau_0 \omega_0 W_{20}(\theta) + g_{20} q(\theta) + \overline{g}_{20} \overline{q}(\theta),$$

$$\dot{W}_{11}(\theta) = g_{11} q(\theta) + \overline{g}_{11} \overline{q}(\theta).$$

$$(41)$$

So,

$$W_{20}(\theta) = \frac{ig_{20}}{\tau_0 \omega_0} q(0) e^{i\tau_0 \omega_0 \theta}$$

$$- \frac{\overline{g}_{02}}{3i\tau_0 \omega_0} \overline{q}(0) e^{-i\tau_0 \omega_0 \theta} + E_1 e^{2i\tau_0 \omega_0 \theta}, \quad (42)$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau_0 \omega_0} q(0) e^{i\tau_0 \omega_0 \theta} + \frac{i\overline{g}_{11}}{\tau_0 \omega_0} \overline{q}(0) e^{-i\tau_0 \omega_0 \theta} + E_2,$$

where

$$E_1 = 2 \begin{pmatrix} 2i\omega_0 - a_1 & -a_2 e^{-2i\omega_0 \tau_0} \\ -a_3 & 2i\omega_0 - a_4 - a_5 e^{-2i\omega_0 \tau_0} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} a_6 \alpha^2 e^{-2i\omega_0 \tau_0} + a_7 \alpha e^{-i\omega_0 \tau_0} \\ -a_7 \alpha^2 e^{-2i\omega_0 \tau_0} - a_6 \alpha e^{-i\omega_0 \tau_0} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 - a_5 \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} 2a_6 \alpha\overline{\alpha} + a_7 \left(\overline{\alpha} e^{i\omega_0 \tau_0} + \alpha e^{-i\omega_0 \tau_0} \right) \\ -2a_7 \alpha\overline{\alpha} - a_6 \left(\overline{\alpha} e^{i\omega_0 \tau_0} + \alpha e^{-i\omega_0 \tau_0} \right) \end{pmatrix}.$$

$$(43)$$

According to the discussion above, we can compute the following parameters:

$$C_1(0) = \frac{i}{2\tau_0 \omega_0} \left(g_{20} g_{11} - 2 |g_{11}| - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2 \operatorname{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \left(\operatorname{Im}\{\lambda'(\tau_0)\} \right)}{\omega_0},$$

$$(44)$$

where μ_2 determines the directions of the Hopf bifurcations, β_2 determines the stability of the bifurcation periodic solutions, and T_2 determines the period of the bifurcating periodic solutions [9]. By lemma (5), we know that $\operatorname{Re}\{\lambda'(\tau_0)\} > 0$; we have the following theorem.

Theorem 3. *If* $\operatorname{Re}\{C_1(0)\} < 0(> 0)$*, the direction of the Hopf bifurcation of the system* (1) *at the equilibrium* $(0, 0)$ *when* $\tau = \tau_0$ *is supercritical (subcritical) and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).*

4. Global Existence of Periodic Solution

From the above discussion, we know that system (2) undergoes a local Hopf bifurcation at $E^* = (S^*, I^*)$ when $\tau = \tau_j$ $(j = 0, 1, 2, \ldots)$. A natural question is that if the bifurcating periodic solutions of system (2) exist when is τ far away from critical values? In this section, we will study the global existence of periodic solutions of system (2). Through use of a global Hopf bifurcation theorem given by Wu [10], we obtain the global continuation of periodic solutions bifurcating from the points (E^*, τ_j) $(j = 0, 1, 2, \ldots)$. First of all, we define

$$X = C([-\tau, 0], R),$$

$$\Sigma = Cl(x, \tau, l) : (x, \tau, l) \in X \times R_+ \times R_+,$$

$$x \text{ is a } l\text{-periodic solution of system},$$

$$N = (\widehat{x}, \tau, l) : \widehat{x} = 0 \quad \text{or} \quad \nu.$$

$$(45)$$

Let $C(E^*, \tau_j, 2\pi/\omega_0)$ denote the connected component of $C(E^*, \tau_j, 2\pi/\omega_0)$ in Σ and $\operatorname{Proj}_\tau(E^*, \tau_j, 2\pi/\omega_0)$ its projection on τ component. From theorem (5), we know that

$C(E^*, \tau_j, 2\pi/\omega_0)$ is nonempty. ω_0 and τ_j are defined in (10) and (11).

Lemma 4. *All periodic solutions of system* (2) *are uniformly bounded.*

Proof. Let $(S(t), I(t))$ be a nonconstant periodic solution of system (2), and let $S(t_1), S(t_2)$ and $I(t_3), I(t_4)$ be the maximum and minimum of $S(t)$ and $I(t)$, respectively. Using a proof procedure similar to that in [8], we can obtain

$$0 < S(t) < \frac{B}{\mu_1}, \qquad 0 < I(t) < \frac{B}{\mu_2 + \gamma}. \tag{46}$$

It is shown that all periodic solutions of system (2) are uniformly bounded. This completes the proof. □

Lemma 5. *System* (2) *has no nonconstant periodic solution of period* τ.

Proof. For a contradiction, if system (5) has a τ-periodic solution, say $(S(t), I(t))$, then it satisfies the ODES as follows:

$$\dot{S}(t) = B - \mu_1 S(t) - \frac{\beta S(t) I(t)}{1 + \alpha I(t)} = P(S, I),$$

$$\dot{I}(t) = \frac{\beta S(t) I(t)}{1 + \alpha I(t)} - (\mu_2 + \gamma) I(t) = Q(S, I). \tag{47}$$

We can get

$$\frac{\partial P}{\partial S} + \frac{\partial Q}{\partial I} = -\mu_1 - \frac{\beta I(t)}{1 + \alpha I(t)} - \frac{1}{(1 + \alpha I(t))^2} - (\mu_2 + \gamma) < 0. \tag{48}$$

By Bendixson's criterion, we know that system (2) has no nonconstant periodic solutions, which prove the lemma. □

Theorem 6. *Suppose that the condition* (H_1) *and* (P_1) *is satisfied. Then, for each* $\tau > \tau_j$, $j = 0, 1, 2, \ldots$, *system* (2) *has at least* $j - 1$ *periodic solutions.*

Proof. The characteristic matrix of system (2) at the equilibrium $\bar{z} = [\bar{z}^{(1)}, \bar{z}^{(2)}] \in R^2$ is in the following form:

$$\Delta(\bar{z}, \tau, l)(\lambda) = \lambda I - D_\phi F(\bar{z}, \bar{\tau}, \bar{l})(e^\lambda Id); \tag{49}$$

that is,

$$\Delta(\bar{z}, \tau, l)$$

$$= \begin{pmatrix} \lambda + \mu_1 + \dfrac{\beta \bar{z}^{(2)} e^{-\lambda\tau}}{1 + \alpha \bar{z}^{(2)} e^{-\lambda\tau}} & \dfrac{\beta \bar{z}^{(1)}}{\left(1 + \alpha \bar{z}^{(2)} e^{-\lambda\tau}\right)^2} \\[3mm] -\dfrac{\beta \bar{z}^{(2)} e^{-\lambda\tau}}{1 + \alpha \bar{z}^{(2)} e^{-\lambda\tau}} & \lambda + \dfrac{\beta \bar{z}^{(1)}}{\left(1 + \alpha \bar{z}^{(2)} e^{-\lambda\tau}\right)^2} \end{pmatrix}. \tag{50}$$

Using a proof procedure similar to that in [9], it is easy to obtain that $(E^*, \tau_j, 2\pi/\omega_0)$, $j = 0, 1, 2, \ldots$, is an isolated center.

From the proof procedure of Lemmas 4 and 5, it is easy to know that there exist $\varepsilon > 0$, $\delta > 0$, smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to C$ such that

$$\Delta(\lambda(\tau)) = \Delta_{(v,\tau,T)}(\lambda(\tau)) = 0, \qquad |\lambda(\tau) - i\omega_0| < \varepsilon, \tag{51}$$

for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$, and

$$\lambda(\tau_j) = i\omega_0, \qquad \frac{d\operatorname{Re}(\lambda(\tau))}{d\tau}\bigg|_{\tau=\tau_j} > 0. \tag{52}$$

Define $l_j = 2\pi/\omega_0$, and let $\Omega_\varepsilon = \{(0, l) : 0 < u < \varepsilon, |l - l_j| < \varepsilon\}$. Obviously, if $|\tau - \tau_j| \leq \delta$ and $(u, l) \in \partial\Omega_\varepsilon$ such that $\Delta_{(E^*,\tau,l)}(u + 2\pi i/l) = 0$, if and only if $\tau = \tau_j, u = 0, l = l_j$, set

$$H^\pm\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right)(u, l) = \Delta_{(E^*,\tau_j\pm\delta,l)}\left(u + \frac{2\pi i}{l}\right). \tag{53}$$

We obtain the crossing number as follows:

$$\gamma_1\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right) = \deg_B\left(H^-\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_\varepsilon\right)$$

$$- \deg_B\left(H^+\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_\varepsilon\right) = -1. \tag{54}$$

We conclude that

$$\sum_{(E^*,\tau,l)\in C\left(E^*,\tau_j,2\pi/\omega_0\right)} \gamma_1\left(E^*, \tau, l\right) < 0. \tag{55}$$

Since the first crossing number of each center is always -1, by [10, Theorem 3.3], we conclude that $C(E^*, \tau_j, 2\pi/\omega_0)$ is unbounded. By the definition of τ_j given in (10), we know that, for $j \geq 1$, $(\tau_j/(j+1)) < 2\pi/\omega_0 < \tau_j$ automatically hold.

Again, the population densities of susceptible and infective are ultimately uniformly bounded, implying that the projection of $C(E^*, \tau_j, 2\pi/\omega_0)$ onto the τ-space is bounded. Meanwhile, system (2) with $\tau = 0$ has no nonconstant periodic solutions; if there exits $\tau^* > 0$ such that the projection of $C(E^*, \tau_j, 2\pi/\omega_0)$ onto the τ-space ins $(0, \tau_0)$ with $\tau^* > \tau_j$, then, the projection of $C(E^*, \tau_j, 2\pi/\omega_0)$ onto the τ-space is bounded. Since $(2\pi/\omega_0) < \tau_j$ and from Lemma 5, we can obtain $0 < l < \tau^*$ for $(E, \tau, T) \in C(E^*, \tau_j, 2\pi/\omega_0)$ with $l < \tau^*$; that is to say, $C(E^*, \tau_j, 2\pi/\omega_0)$ onto l-space is also bounded. Because $C(E^*, \tau_j, 2\pi/\omega_0)$ is unbounded, $\operatorname{Proj}_\tau(E, \tau_j, 2\pi/\omega_0)$ must be unbounded. Consequently, $\operatorname{Proj}_\tau(E, \tau_j, 2\pi/\omega_0)$ include $[\tau_j, \infty)$ for $j \geq 1$. That is to say, for each $\tau > \tau_j$ $(j \geq 1)$, system (2) at least has $j - 1$ nonconstant period solutions. The proof is complete. □

5. Conclusion

In this paper we have analytically studied an avian influenza virus propagation model with nonlinear incidence rate and time delay depending on SIR epidemic model. Some previous efforts in epidemic models have been mainly concerned with the global stability and asymptotical stability. However, it

is a new idea to study the bifurcation periodic solutions and global existence of periodic solutions. The theoretical analysis for the avian influenza virus propagation models is given. Then, Hopf bifurcation occurs when time delay passes through a sequence of critical values. Furthermore, bifurcations and stability of the bifurcation periodic solutions are derived. Finally, global existence of periodic solutions is established.

Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work; there is no professional or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, in this paper.

References

[1] J. Li, X. Yu, X. Pu et al., "Environmental connections of novel avian-origin H7N9 influenza virus infection and virus adaptation to the human," *Science China Life Sciences*, vol. 56, no. 6, pp. 485–492, 2013.

[2] J. He, L. Ning, and Y. Tong, "Origins and evolutionary genomics of the novel 2013 avian-origin H7N9 influenza A virus in China: early findings," http://arxiv.org/abs/1304.1985.

[3] C. C. McCluskey, "Global stability for an SIR epidemic model with delay and nonlinear incidence," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 3106–3109, 2010.

[4] K. L. Cooke, "Stability analysis for a vector disease model," *The Rocky Mountain Journal of Mathematics*, vol. 9, no. 1, pp. 31–42, 1979.

[5] R. Xu and Z. Ma, "Global stability of a SIR epidemic model with nonlinear incidence rate and time delay," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 5, pp. 3175–3189, 2009.

[6] F. Zhang and Z. Li, "Global stability of an SIR epidemic model with constant infectious period," *Applied Mathematics and Computation*, vol. 199, no. 1, pp. 285–291, 2008.

[7] S. Ruan, "Absolute stability, conditional stability and bifurcation in Kolmogorov-type predator-prey systems with discrete delays," *Quarterly of Applied Mathematics*, vol. 59, no. 1, pp. 159–173, 2001.

[8] J. Wei and C. Yu, "Hopf bifurcation analysis in a model of oscillatory gene expression with delay," *Proceedings of the Royal Society of Edinburgh A. Mathematics*, vol. 139, no. 4, pp. 879–895, 2009.

[9] X. Meng, H. Huo, and H. Xiang, "Hopf bifurcation in a three-species system with delays," *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 635–661, 2011.

[10] J. Wu, "Symmetric functional-differential equations and neural networks with memory," *Transactions of the American Mathematical Society*, vol. 350, no. 12, pp. 4799–4838, 1998.

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