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*Research Article*

# Almost Periodic Solution of a Modified Leslie-Gower Predator-Prey Model with Beddington-DeAngelis Functional Response and Feedback Controls

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We consider a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis functional response and feedback controls as follows: $\dot{x}(t) = x(t)\left(a_1(t) - b(t)x(t) - c(t)y(t)/\left(\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)\right) - e_1(t)u(t)\right)$, $\dot{u}(t) = -d_1(t)u(t) + p_1(t)x(t-\tau)$, $\dot{y}(t) = y(t)\left(a_2(t) - r(t)y(t)/(x(t)+k(t)) - e_2(t)v(t)\right)$, and $\dot{v}(t) = -d_2(t)v(t) + p_2(t)y(t-\tau)$. Sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained.

## 1. Introduction

In recent years, the modified predator-prey systems with periodic or almost periodic coefficients have been studied extensively.

Leslie [1] proposed the famous Leslie predator-prey system as follows:

$$\begin{aligned} \dot{x}(t) &= x(a - bx) - p(x)y, \\ \dot{y} &= y\left(e - f\frac{y}{x}\right), \end{aligned} \tag{1}$$

where $x$ and $y$ stand for the population of the prey and the predator at time $t$, respectively, and $p(x)$ is the so-called predator functional response to the prey. The term $y/x$ is the Leslie-Gower term which measures the loss in the predator population due to rarity of its favorite food.

Global stability of the positive locally asymptotically stable equilibrium in a class of predator-prey systems has been introduced by Hsu and Huang [2], and the system is as follows:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k} - yp(x)\right), \\ \frac{dy}{dx} &= y\left[s\left(1 - \frac{hy}{s}\right)\right], \\ x(0) &> 0, \quad y(0) > 0, \quad r, s, k, h > 0. \end{aligned} \tag{2}$$

When the functional response $p(x)$ equals $mx$, then (2) turns into a Leslie-Gower system [3].

On the other hand, the periodic solution (almost periodic solution) and some other properties of Leslie-Gower predator-prey models were studied (see [4–9]). In particular, Zhang [10] discussed the almost periodic solution of a modified Leslie-Gower predator-prey model with the Beddington-DeAngelis function response as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r_1(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t) + \beta(t)x(t) + \gamma(t)y(t)}\right), \\ \dot{y}(t) &= y(t)\left(r_2(t) - \frac{d(t)y(t)}{x(t) + k(t)}\right), \end{aligned} \tag{3}$$

where $x(t)$ is the size of prey population and $y(t)$ is the size of predator population.

Stimulated by the above reasons, in this paper, we incorporate the feedback control into model (3) and consider the following model:

$$\begin{aligned}
\dot{x}(t) &= x(t)\Big(a_1(t) - b(t)x(t) \\
&\qquad - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)} - e_1(t)u(t)\Big), \\
\dot{u}(t) &= -d_1(t)u(t) + p_1(t)x(t-\tau), \\
\dot{y}(t) &= y(t)\left(a_2(t) - \frac{r(t)y(t)}{x(t)+k(t)} - e_2(t)v(t)\right), \\
\dot{v}(t) &= -d_2(t)v(t) + p_2(t)y(t-\tau),
\end{aligned} \tag{4}$$

where $\tau > 0$ and all the coefficients $b(t)$, $c(t)$, $r(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $a_i(t)$, $d_i(t)$, $p_i(t)$, and $e_i(t)$ $(i = 1,2)$ are all continuous, almost periodic functions on $R$.

Associated with (4), we consider a group of initial conditions with the following form (we assume, without loss of generality, that the initial time $t_0 = 0$):

$$\begin{aligned}
x(s) &= \phi(s) \geq 0, \quad s \in [-\tau, 0], \ \phi(0) > 0, \\
y(s) &= \varphi(s) \geq 0, \quad s \in [-\tau, 0], \ \varphi(0) > 0, \\
u(0) &> 0, \qquad v(0) > 0.
\end{aligned} \tag{5}$$

Let $f$ be a continuous bounded function on $R$ and we set

$$f^l = \inf_{t\in R} f(t), \qquad f^u = \sup_{t\in R} f(t). \tag{6}$$

Throughout this paper, we assume that the coefficients of the almost periodic system (4) satisfy

$$\begin{aligned}
&\min_{i=1,2}\left\{b^l, c^l, \alpha^l, \beta^l, \gamma^l, r^l, k^l, a_i^l, d_i^l, p_i^l, e_i^l\right\} > 0, \\
&\max_{i=1,2}\left\{b^u, c^u, \alpha^u, \beta^u, \gamma^u, r^u, k^u, a_i^u, d_i^u, p_i^u, e_i^u\right\} < +\infty.
\end{aligned} \tag{7}$$

By constructing a suitable Lyapunov functional, we obtain some sufficient conditions for the existence of a globally attractive positive almost periodic solution of system (4) with initial conditions (5).

## 2. Permanence

In this section, we give some definitions and results that we will use in the rest of the paper.

**Lemma 1** (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq)\, x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}\right). \tag{8}$$

**Lemma 2** (see [11]). *If $a > 0$, $b > 0$, and $\dot{x} \geq (\leq)\, b - ax$, when $t \geq 0$ and $x(0) > 0$, one has*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}, \quad \left(\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}\right). \tag{9}$$

Set the following:

$$\begin{aligned}
M_1 &= \frac{a_1^u}{b^l}, \qquad L_1 = \frac{p_1^u M_1}{d_1^l}, \\
M_2 &= \frac{a_2^u(M_1 + k^u)}{r^l}, \qquad L_2 = \frac{p_2^u M_2}{d_2^l}, \\
m_1 &= \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u}, \qquad l_1 = \frac{p_1^l m_1}{d_1^u}, \\
m_2 &= \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right), \qquad l_2 = \frac{p_2^l m_2}{d_2^u}.
\end{aligned} \tag{10}$$

**Theorem 3.** *Suppose that system* (4) *with initial condition* (5) *satisfies the following condition:*

$$a_1^l - \frac{c^u}{r^l} - e_1^u L_1 > 0, \qquad a_2^l - e_2^u L_2 > 0. \tag{11}$$

*Then system* (4) *is permanent; that is, any positive solution* $(x(t), u(t), y(t), v(t))^T$ *of the system* (4) *satisfies*

$$\begin{aligned}
0 < m_1 &\leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \\
0 < l_1 &\leq \liminf_{t\to+\infty} u(t) \leq \limsup_{t\to+\infty} u(t) \leq L_1, \\
0 < m_2 &\leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2, \\
0 < l_2 &\leq \liminf_{t\to+\infty} v(t) \leq \limsup_{t\to+\infty} v(t) \leq L_2.
\end{aligned} \tag{12}$$

*Proof.* From the first equation of (4), we have the following:

$$\dot{x}(t) \leq x(t)\left(a_1^u - b^l x(t)\right). \tag{13}$$

Applying Lemma 1 to (13) leads to

$$\limsup_{t\to+\infty} x(t) \leq \frac{a_1^u}{b^l} = M_1. \tag{14}$$

From (14), we know that there exists an enough large $T_1 > 0$ such that

$$x(t) \leq M_1, \quad t \geq T_1 > 0, \tag{15}$$

so there exists an enough large $T_2 = T_1 + \tau$ such that

$$x(t-\tau) \leq M_1, \quad t \geq T_2 > 0. \tag{16}$$

It follows from (16) and the second equation of system (4) that, for $t \geq T_2$,

$$\dot{u}(t) \leq -d^l u(t) + p_1^u M_1. \tag{17}$$

Applying Lemma 2 to (17) leads to

$$\limsup_{t\to+\infty} u(t) \leq \frac{p_1^u M_1}{d_1^l} = L_1. \tag{18}$$

By using a similar argument as that in the proof of (14) and (18), we can get the following:

$$\begin{aligned} \limsup_{t\to+\infty} y(t) &\le \frac{a_2^u(M_1+k^u)}{r^l} = M_2, \\ \limsup_{t\to+\infty} \nu(t) &\le \frac{p_2^u M_2}{d_2^l} = L_2. \end{aligned} \tag{19}$$

From (18) and the first equation of system (4) we know

$$\dot{x}(t) \ge x(t)\left(a_1^l - \frac{c^u}{\gamma^l} - e_1^u L_1 - b^u x(t)\right). \tag{20}$$

Applying Lemma 1 and (11) to the above leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{a_1^l - c^u/r^l - e_1^u L_1}{b^u} = m_1. \tag{21}$$

Therefore, we know that there exists an enough large $T_3$ such that

$$x(t) \ge m_1, \quad t \ge T_3 > 0. \tag{22}$$

From the second equation of system (4) we have the following:

$$\dot{u}(t) \ge -d_1^u u(t) + p_1^l m_1. \tag{23}$$

Applying Lemma 2 to the above, we obtain the following:

$$\liminf_{t\to+\infty} u(t) \ge \frac{p_1^l m_1}{d_1^u} = l_1. \tag{24}$$

By using a similar method as that in the proof of (21) and (24), it follows that

$$\begin{aligned} \liminf_{t\to+\infty} y(t) &\ge \frac{1}{r^u}\left(a_2^l - e_2^u L_2\right)\left(m_1 + k^l\right) = m_2 \\ \liminf_{t\to+\infty} \nu(t) &\ge \frac{p_2^l m_2}{d_2^u} = l_2. \end{aligned} \tag{25}$$

This completes the proof. □

We denote by $\Omega$ the set of all solutions $z(t) = (x(t), u(t), y(t), \nu(t))^T$ of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le \nu(t) \le L_2$ for all $t > 0$.

**Theorem 4.** *Consider the following:* $\Omega \ne \emptyset$.

*Proof.* From the properties of almost periodic function there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\begin{aligned} &a_i(t+t_n) \longrightarrow a_i(t), \qquad d_i(t+t_n) \longrightarrow d_i(t), \\ &e_i(t+t_n) \longrightarrow e_i(t), \qquad p_i(t+t_n) \longrightarrow p_i(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i=1,2), \\ &b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t), \\ &r(t+t_n) \longrightarrow r(t), \qquad k(t+t_n) \longrightarrow k(t), \\ &\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t), \\ &\qquad\qquad \gamma(t+t_n) \longrightarrow \gamma(t), \end{aligned} \tag{26}$$

as $n \to \infty$ uniformly on $R$. Let $z(t) = (x(t), u(t), y(t), \nu(t))^T$ be a solution of system (4) satisfying $m_1 \le x(t) \le M_1$, $l_1 \le u(t) \le L_1$, $m_2 \le y(t) \le M_2$, and $l_2 \le \nu(t) \le L_2$ for $t > T$. Clearly, the sequence $z(t+t_n)$ is uniformly bounded and equicontinuous on each bounded subset of $R$. Therefore, by the Arzelà-Ascoli theorem, there exists a subsequence $z(t+t_k)$ which converges to a continuous function $z^*(t) = (x^*(t), u^*(t), y^*(t), \nu^*(t))^T$ as $k \to +\infty$ uniformly on each bounded subset of $R$. Let $T_0 \in R$ be given. We may assume that $t_k + T_0 \ge T$ for all $k$. For $t \ge 0$, we have the following:

$$\begin{aligned} &x(t+t_k+T_0) \\ &= x(t_k+T_0) \\ &\quad + \int_{T_0}^{t+T_0} x(s+t_k)\big(a_1(s+t_k) - b(s+t_k)x(s+t_k) \\ &\qquad - (c(s+t_k)y(s+t_k)) \\ &\qquad \times (\alpha(s+t_k) + \beta(s+t_k)x(s+t_k) \\ &\qquad + \gamma(s+t_k)y(s+t_k))^{-1} \\ &\qquad - e_1(s+t_k)u(s+t_k)\big)\,ds, \\ &u(t+t_k+T_0) \\ &= u(t_k+T_0) \\ &\quad - \int_{T_0}^{t+T_0} d_1(s+t_k)u(s+t_k) + p_1(s+t_k)x(s+t_k-\tau)\,ds, \\ &y(t+t_k+T_0) \\ &= y(t_k+T_0) \\ &\quad + \int_{T_0}^{t+T_0} y(s+t_k)\left(a_2(s+t_k) - \frac{r(s+t_k)y(s+t_k)}{x(s+t_k)+k(s+t_k)}\right. \\ &\qquad \left. - e_2(s+t_k)\nu(s+t_k)\right)ds, \\ &\dot{\nu}(t+t_k+T_0) \\ &= \nu(t_k+T_0) \\ &\quad + \int_{T_0}^{t+T_0} -d_2(s+t_k)\nu(s+t_k) + p_2(s+t_k)y(s+t_k-\tau)\,ds. \end{aligned} \tag{27}$$

Applying Lebesgue's dominated convergence theorem and letting $k \to +\infty$ in (27), we obtain the following:

$$\begin{aligned} &x^*(t+T_0) \\ &= x^*(T_0) \\ &\quad + \int_{T_0}^{t+T_0} x^*(s)\big(a_1(s) - b(s)x^*(s) \\ &\qquad - \frac{c(s)y^*(s)}{\alpha(s)+\beta(s)x^*(s)+\gamma(s)y^*(s)} \\ &\qquad - e_1(s)u^*(s)\big)\,ds, \end{aligned}$$

$$\begin{aligned}
u^*(t+T_0) &= u^*(T_0) \\
&\quad - \int_{T_0}^{t+T_0} d_1(s)\,u^*(s) + p_1(s)\,x^*(s-\tau)\,ds, \\
y^*(t+T_0) &= y^*(T_0) \\
&\quad + \int_{T_0}^{t+T_0} y^*(s)\left(a_2(s) - \frac{r(s)\,y^*(s)}{x^*(s)+k(s)}\right. \\
&\qquad \left. - e_2(s)\,v^*(s)\right) ds, \\
v^*(t+T_0) &= v^*(T_0) \\
&\quad + \int_{T_0}^{t+T_0} -d_2(s)\,v^*(s) + p_2(s)\,y^*(s-\tau)\,ds.
\end{aligned} \tag{28}$$

Since $T_0 \in R$ is arbitrarily given, $z^*(t) = (x^*(t), u^*(t), y^*(t), v^*(t))^T$ is a solution of system (4) on $R$. It is clear that $m_1 \le x^*(t) \le M_1$, $l_1 \le u^*(t) \le L_1$, $m_2 \le y^*(t) \le M_2$, $l_2 \le v^*(t) \le L_2$ for $t \in R$. Thus $z^*(t) \in \Omega$. This completes the proof. □

## 3. Existence of a Unique Almost Periodic Solution

Now let us state several definitions and lemmas which will be useful in the proving of the main result of this section.

*Definition 5* (see [12]). A function $f(t,x)$, where $f$ is an $m$-vector, $t$ is a real scalar, and $x$ is an $n$-vector, is said to be almost periodic in $t$ uniformly with respect to $x \in S \subset R^n$, if $f(t,x)$ is continuous in $t \in R$ and $x \in S$ and if, for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$ there exists a $\varsigma$ such that the inequality

$$\left|f(t+\varsigma, x) - f(t,x)\right| < \varepsilon \tag{29}$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in S$. The number $\varsigma$ is called an $\varepsilon$-*translation* number of $f(t,x)$.

*Definition 6* (see [12]). A function $f : R \to R$ is said to be asymptotically almost periodic function, if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that $f(t) = q(t) + r(t)$, $t \in R$ and $r(t) \to 0$ as $t \to \infty$.

**Lemma 7** (see [13]). *Let $f$ be a nonnegative, integral, and uniformly continuous function defined on $[0,+\infty)$; then $\lim_{t\to+\infty} f(t) = 0$.*

**Theorem 8.** *Suppose that all conditions of Theorem 3 hold; furthermore assume that*

$$\begin{aligned}
&(H)\ \Theta > 0, \text{ where } \Theta = \min\{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}, \\
&\Theta_1 = b^l m_1 - p_1^u M_1 - \frac{c^u \beta^u M_1 M_2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} \\
&\qquad - \frac{r^u M_1 M_2}{(m_1 + k^l)^2} > 0, \\
&\Theta_2 = \frac{\gamma^l}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2} \\
&\qquad - \frac{c^u \gamma^u M_2^2}{(\alpha^l + \beta^l m_1 + \gamma^l m_2)^2} - p_2^u M_2 > 0, \\
&\Theta_3 = d_1^l - e_1^u, \qquad \Theta_4 = d_2^l - e_2^u.
\end{aligned} \tag{30}$$

*Then system* (4) *with initial conditions* (5) *is globally attractive.*

*Proof.* Let $x(t) = e^{x_1(t)}$, $y(t) = e^{y_1(t)}$, and then system (4) is transformed into

$$\begin{aligned}
\dot{x}_1(t) &= a_1(t) - b(t)\,e^{x_1(t)} \\
&\quad - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} - e_1(t)\,u(t), \\
\dot{u}(t) &= -d_1(t)\,u(t) + p_1(t)\,e^{x_1(t-\tau)}, \\
\dot{y}_1(t) &= a_2(t) - \frac{r(t)\,e^{y_1(t)}}{e^{x_1(t)} + k(t)} - e_2(t)\,v(t), \\
\dot{v}(t) &= -d_2(t)\,v(t) + p_2(t)\,e^{y_1(t-\tau)}.
\end{aligned} \tag{31}$$

Suppose that $z_1(t) = (x_1(t), u(t), y_1(t), v(t))^T$ and $z_1^*(t) = (x_1^*(t), u^*(t), y_1^*(t), v^*(t))^T$ are any two positive solutions of (31).

Let $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$, where

$$\begin{aligned}
V_1(t) &= \left|x_1(t) - x_1^*(t)\right|, \\
V_2(t) &= \left|u(t) - u^*(t)\right| + p_1^u \int_{t-\tau}^{t} \left|e^{x_1(t)} - e^{x_1^*(t)}\right| ds, \\
V_3(t) &= \left|y_1(t) - y_1^*(t)\right|, \\
V_4(t) &= \left|v(t) - v^*(t)\right| + p_2^u \int_{t-\tau}^{t} \left|e^{y_1(t)} - e^{y_1^*(t)}\right| ds.
\end{aligned} \tag{32}$$

Calculating the right derivative $D^+V_1(t)$ of $V_1(t)$ along the solution of (31), we have the following:

$$\begin{aligned}
D^+V_1(t) &= \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \left[-b(t)\left(e^{x_1(t)} - e^{x_1^*(t)}\right)\right. \\
&\qquad - \frac{c(t)\,e^{y_1(t)}}{\alpha(t) + \beta(t)\,e^{x_1(t)} + \gamma(t)\,e^{y_1(t)}} \\
&\qquad + \frac{c(t)\,e^{y_1^*(t)}}{\alpha(t) + \beta(t)\,e^{x_1^*(t)} + \gamma(t)\,e^{y_1^*(t)}} \\
&\qquad \left. - e_1(t)\left(u(t) - u^*(t)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \Big[-b(t) e^{\xi(t)} \left(x_1(t) - x_1^*(t)\right) \\
&\qquad - \frac{c(t) e^{y_1(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad + \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad - \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\qquad + \frac{c(t) e^{y_1^*(t)}}{\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right)\Big] \\
&\le \operatorname{sgn}\left(x_1(t) - x_1^*(t)\right) \\
&\quad \times \left[-b^l m_1 \left(x_1(t) - x_1^*(t)\right)\right] \\
&\quad - \frac{c(t)}{\alpha(t) + \beta(t) e^{x_1(t)} + \gamma(t) e^{y_1(t)}} \\
&\quad \cdot e^{\eta(t)} \left(y_1(t) - y_1^*(t)\right) \\
&\qquad + \Big(c(t) e^{y_1^*(t)} \Big[\beta(t) \left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad\qquad + \gamma(t) \left(e^{y_1(t)} - e^{y_1^*(t)}\right)\Big]\Big) \\
&\qquad \times \Big(\left(\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}\right) \\
&\qquad\quad \times \left(\alpha(t) + \beta(t) e^{x_1^*(t)} + \gamma(t) e^{y_1^*(t)}\right)\Big)^{-1} \\
&\qquad - e_1(t)\left(u(t) - u^*(t)\right) \\
&\le \left(\frac{c^u \beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - b^l m_1\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad + \left(\frac{c^u \gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} \right. \\
&\qquad \left. + \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2}\right) \\
&\quad \times \left|y_1(t) - y_1^*(t)\right| \\
&\quad + e_1^u \left|u(t) - u^*(t)\right|.
\end{aligned} \tag{33}
$$

Further, it follows that

$$
\begin{aligned}
D^+ V_2(t) &= \operatorname{sgn}\left(u(t) - u^*(t)\right) \\
&\quad \times \Big(-d_1(t)\left(u(t) - u^*(t)\right) \\
&\qquad + p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right) \\
&\qquad + p_1^u \left(e^{x_1(t)} - e^{x_1^*(t)}\right) \\
&\qquad - p_1(t)\left(e^{x_1(t-\tau)} - e^{x_1^*(t-\tau)}\right)\Big) \\
&\le -d_1^l \left|u(t) - u^*(t)\right| + p_1^u M_1 \left|x_1(t) - x_1^*(t)\right|, \\
D^+ V_3(t) &= \operatorname{sgn}\left(y_1(t) - y_1^*(t)\right) \\
&\quad \times \left[-\frac{r(t) e^{y_1(t)}}{e^{x_1(t)} + k(t)} + \frac{r(t) e^{y_1^*(t)}}{e^{x_1^*(t)} + k(t)}\right. \\
&\qquad \left. - e_2(t)\left(\nu(t) - \nu^*(t)\right)\right] \\
&\le -\frac{r^l m_2}{M_1 + k^u} \left|y_1(t) - y_1^*(t)\right| \\
&\quad + \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2} \left|x_1(t) - x_1^*(t)\right| \\
&\quad + e_2^u \left|\nu(t) - \nu^*(t)\right|, \\
D^+ V_4(t) &\le -d_2^l \left|\nu(t) - \nu^*(t)\right| + p_2^u M_2 \left|y_1(t) - y_1^*(t)\right|.
\end{aligned} \tag{34}
$$

Therefore, we have the following:

$$
\begin{aligned}
&D^+ V(t) \\
&= D^+ V_1(T) + D^+ V_2(T) + D^+ V_3(T) + D^+ V_4(T) \\
&\le -\left(b^l m_1 - p_1^u M_1 - \frac{c^u \beta^u M_1 M_2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - \frac{r^u M_1 M_1}{\left(m_1 + k^l\right)^2}\right) \\
&\quad \times \left|x_1(t) - x_1^*(t)\right| \\
&\quad - \left(\frac{r^l m_2}{M_1 + k^u} - \frac{c^l m_2}{\alpha^u + \beta^u M_1 + \gamma^u M_2}\right. \\
&\qquad \left. - \frac{c^u \gamma^u M_2^2}{\left(\alpha^l + \beta^l m_1 + \gamma^l m_2\right)^2} - p_2^u M_2\right) \left|y_1(t) - y_1^*(t)\right| \\
&\quad - \left(d_1^l - e_1^u\right) \left|u(t) - u^*(t)\right| - \left(d_2^l - e_2^u\right) \left|\nu(t) - \nu^*(t)\right| \\
&\le -\Theta \left(\left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| \right. \\
&\qquad \left. + \left|u(t) - u^*(t)\right| + \left|\nu(t) - \nu^*(t)\right|\right).
\end{aligned} \tag{35}
$$

Integrating the above inequality on internal $[0, t]$, it follows that, for $t \ge 0$,

$$
\begin{aligned}
&V(t) + \Theta \int_0^t \left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| \\
&\qquad + \left|u(t) - u^*(t)\right| + \left|\nu(t) - \nu^*(t)\right| ds \\
&\le V(0) < +\infty.
\end{aligned} \tag{36}
$$

Then, for $t > 0$, we obtain that

$$\int_0^t \left|x_1(t) - x_1^*(t)\right| + \left|y_1(t) - y_1^*(t)\right| + \left|u(t) - u^*(t)\right| + \left|v(t) - v^*(t)\right| ds \le \frac{V(0)}{\Theta} < +\infty. \quad (37)$$

By Lemma 7, we obtain

$$\lim_{t\to+\infty}\left|x_1(t) - x_1^*(t)\right| = 0, \qquad \lim_{t\to+\infty}\left|y_1(t) - y_1^*(t)\right| = 0,$$
$$\lim_{t\to+\infty}\left|u(t) - u^*(t)\right| = 0, \qquad \lim_{t\to+\infty}\left|v(t) - v^*(t)\right| = 0. \quad (38)$$

Then the solution of systems (4) and (5) is globally attractive. □

**Theorem 9.** *Suppose that all conditions of Theorem 8 hold; then there exists a unique almost periodic solution of systems* (4) *and* (5).

*Proof.* According to Theorem 4, there exists a bounded positive solution $W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ of (4) and (5). Then there exists a sequence $\{t_k'\}$, $t_k' \to \infty$ as $k \to \infty$, such that $(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T$ is a solution of the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\Big(a_1(t+t_k') - b(t+t_k')x(t) \\
&\quad - \frac{c(t+t_k')y(t)}{\alpha(t+t_k') + \beta(t+t_k')x(t) + \gamma(t+t_k')y(t)} \\
&\quad - e_1(t+t_k')u(t)\Big), \\
\dot{u}(t) &= -d_1(t+t_k')u(t) + p_1(t+t_k')x(t-\tau), \\
\dot{y}(t) &= y(t)\left(a_2(t+t_k') - \frac{r(t+t_k')y(t)}{x(t) + k(t+t_k')} - e_2(t+t_k')v(t)\right), \\
\dot{v}(t) &= -d_2(t+t_k')v(t) + p_2(t+t_k')y(t-\tau).
\end{aligned} \quad (39)$$

According to Theorem 3, we get that not only $\{(w_1(t+t_k'), w_2(t+t_k'), w_3(t+t_k'), w_4(t+t_k'))^T\}$ but also $\{(\dot{w}_1(t+t_k'), \dot{w}_2(t+t_k'), \dot{w}_3(t+t_k'), \dot{w}_4(t+t_k'))^T\}$ are uniformly bounded and equicontinuous. By Ascoli's theorem there exists a uniformly convergent subsequence $w_i(t+t_k) \subseteq w_i(t+t_k')(i = 1,2,3,4)$ such that, for any $\varepsilon > 0$, there exists a $K(\varepsilon) > 0$ with the property that if $m, k \ge K(\varepsilon)$, then

$$\left|w_i(t+t_m) - w_i(t+t_k)\right| < \varepsilon, \quad (i = 1,2,3,4). \quad (40)$$

This is to say, $w_i(t+t_k)$ $(i = 1,2,3,4)$ are asymptotically almost periodic functions; hence there exist four almost periodic functions $P_i(t+t_k)$ $(i = 1,2,3,4)$ and four continuous functions $F_i(t+t_k)$ $(i = 1,2,3,4)$ such that

$$w_i(t+t_k) = P_i(t+t_k) + F_i(t+t_k), \quad t \in R,\ i = 1,2,3,4, \quad (41)$$

where

$$\lim_{k\to+\infty} P_i(t+t_k) = P_i(t), \qquad \lim_{k\to+\infty} F_i(t+t_k) = 0, \quad i = 1,2,3,4, \quad (42)$$

$P_i(t)$ $(i = 1,2,3,4)$ are an almost periodic function.

Therefore,

$$\lim_{k\to+\infty} w_i(t+t_k) = P_i(t), \quad (i = 1,2,3,4). \quad (43)$$

On the other hand,

$$\begin{aligned}
\lim_{k\to+\infty} \dot{w}_i(t+t_k) &= \lim_{k\to+\infty}\lim_{h\to 0}\frac{w_i(t+t_k+h) - w_i(t+t_k)}{h} \\
&= \lim_{h\to+\infty}\lim_{k\to 0}\frac{w_i(t+t_k+h) - w_i(t+t_k)}{h} \\
&= \lim_{h\to 0}\frac{P_i(t+h) - P_i(t)}{h}, \quad (i = 1,2,3,4).
\end{aligned} \quad (44)$$

So $\dot{P}_i(t)$ $(i = 1,2,3,4)$ exist. Now we will prove that $(P_1(t), P_2(t), P_3(t), P_4(t))^T$ is an almost periodic solution of system (4).

From properties of almost periodic function, there exits a sequence $\{t_n\}$, $\{t_n\} \to \infty$ as $n \to \infty$, such that

$$\begin{aligned}
&a_i(t+t_n) \longrightarrow a_i(t), \qquad d_i(t+t_n) \longrightarrow d_i(t), \\
&e_i(t+t_n) \longrightarrow e_i(t), \qquad p_i(t+t_n) \longrightarrow p_i(t), \\
&\qquad (i = 1,2), \\
&b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t), \\
&r(t+t_n) \longrightarrow r(t), \qquad k(t+t_n) \longrightarrow k(t), \\
&\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t), \\
&\gamma(t+t_n) \longrightarrow \gamma(t),
\end{aligned} \quad (45)$$

as $n \to \infty$ uniformly on $R$.

It is easy to know that $w_i(t+t_n) \to P_i(t)$ $(i = 1,2,3,4)$ as $n \to \infty$, and then we have the following:

$$\begin{aligned}
&\dot{P}_1(t) \\
&= \lim_{n\to+\infty} \dot{w}_1(t+t_n) \\
&= \lim_{n\to+\infty} \Big[ w_1(t+t_n)\big(a_1(t+t_n) - b(t+t_n)w_1(t+t_n) \\
&\qquad - (c(t+t_n)w_3(t+t_n)) \\
&\qquad \times (\alpha(t+t_n) + \beta(t+t_n)w_1(t+t_n) \\
&\qquad + \gamma(t+t_n)w_3(t+t_n))^{-1} \\
&\qquad - e_1(t+t_n)w_2(t+t_n)\big) \Big] \\
&= P_1(t)\left(a_1(t) - b(t)P_1(t) \right. \\
&\qquad \left. - \frac{c(t)P_3(t)}{\alpha(t)+\beta(t)P_1(t)+\gamma(t)P_3(t)} - e_1(t)P_2(t)\right).
\end{aligned} \tag{46}$$

By using a similar argument as that in the above, we have the following:

$$\begin{aligned}
\dot{P}_2(t) &= -d_1(t)P_2(t) + p_1(t)P_1(t-\tau), \\
\dot{P}_3(t) &= P_3(t)\left(a_2(t) - \frac{r(t)P_3(t)}{P_1(t)+k(t)} - e_2(t)P_4(t)\right), \\
\dot{P}_4(t) &= -d_2(t)P_4(t) + p_2(t)P_3(t-\tau).
\end{aligned} \tag{47}$$

This proves that $P_i(t)$ $(i = 1,2,3,4)$ is a nonnegative almost periodic solution of systems (4) and (5); by Theorem 8, it follows that there exists a globally asymptotically stable nonnegative almost periodic solution of system (4). The proof is complete. □

## 4. An Example

Consider the following system:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left(4 - 2x(t) - \frac{10y(t)}{2+20x(t)+20y(t)} - 2u(t)\right), \\
\dot{u}(t) &= -3u(t) + \frac{1}{5}x(t-\tau), \\
\dot{y}(t) &= y(t)\left(\frac{1}{10} - \frac{20y(t)}{x(t)+23} - \frac{2}{5}v(t)\right), \\
\dot{v}(t) &= -2v(t) + 2y(t-\tau).
\end{aligned} \tag{48}$$

By a simple calculation, we check that all conditions in Theorems 8 and 9 are fulfilled. Therefore, by Theorems 8 and 9, system (48) has a unique globally asymptotically stable nonnegative almost periodic solution (see Figure 1).

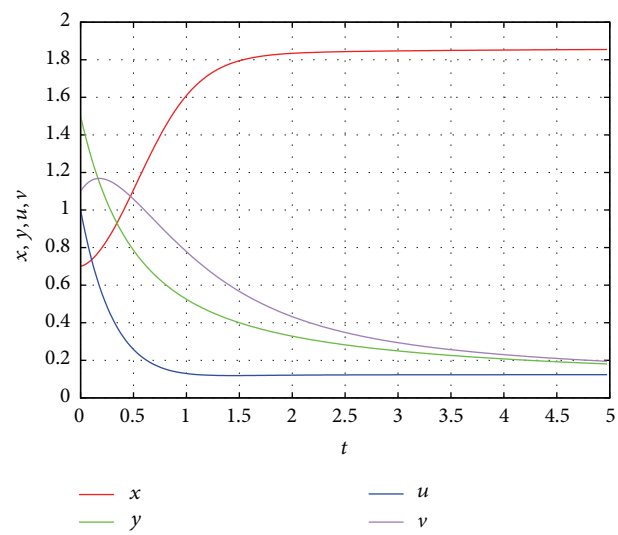


Figure 1: Dynamic behavior of system (48) with the initial $(x(0), y(0), u(0), v(0))^T = (0.7, 1.5, 1.0, 1.1)^T$, for $\tau = 0$, $t \in [0,5]$. From the figure, we could easily see that the solution $(x(t), y(t), u(t), v(t))^T$ is asymptotic to the unique almost periodic solution of the system (48).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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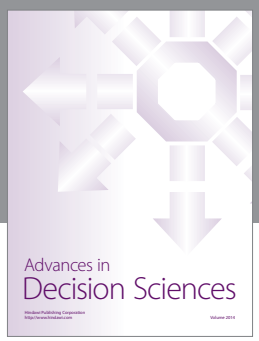

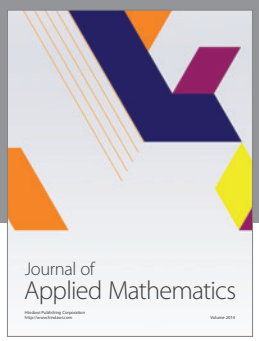

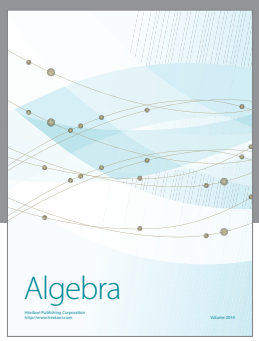

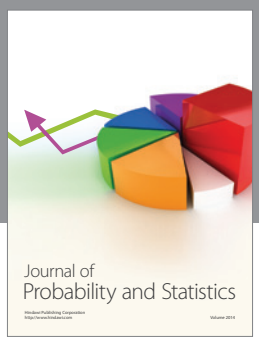

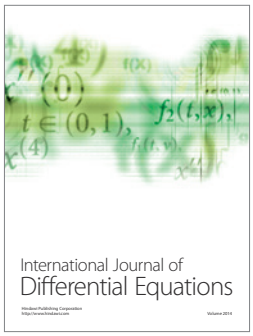

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