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Research Article

Integral Transform Methods for Solving Fractional Dynamic Equations on Time Scales

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We introduce nabla type Laplace transform and Sumudu transform on general time scale. We investigate the properties and the applicability of these integral transforms and their efficiency in solving fractional dynamic equations on time scales.

1. Introduction

It is known that the methods connected to the employment of integral transforms are very useful in mathematical analysis. Those methods are successfully applied to solve differential and integral equations, to study special functions, and to compute integrals. One of the more widely used integral transforms is the Laplace transform defined by the following formula:

$$\mathscr{L}\{f\}(z) = F(z) = \int_0^\infty f(t)\, e^{-zt} dt, \quad z \in \mathbb{C}. \tag{1}$$

The function F of a complex variable is called the Laplace transform of the function f. Watugala [1] introduced a new integral transform called Sumudu transform defined by the following formula:

$$\mathcal{S}\{f\}(u) = \frac{1}{u}\int_0^\infty f(u)\, e^{-t/u} dt, \quad u \in (-\tau_1, \tau_2), \tag{2}$$

and applied to the solution of ordinary differential equations in the control engineering problems (see also [2]). It appeared like the modification of the Laplace transform. The Sumudu transform rivals the Laplace transform in problem solving. Its main advantage is the fact that it may be used to solve problems without resorting to a new frequency domain, because it preserves scale and unit properties.

The theory of time scale calculus was initiated by Hilger [3] (see also [4]). This theory is a tool that unifies the theories of continuous and discrete time system. It is a subject of recent studies in many different fields in which a dynamic process can be described with continuous and discrete models. For the detailed information on theory of time scale calculus, we refer to [5, 6]. The delta Laplace transform on arbitrary time scale ($\mathbb{T}$) is introduced by Bohner and Peterson in [7] (see also [8]) by the following formula:

$$\mathscr{L}\{x\}(z) := \int_{t_0}^\infty x(t)\, e_{\ominus z}(\sigma(t), t_0)\, \Delta t, \quad z \in \mathscr{D}\{x\}, \tag{3}$$

where $\mathscr{D}\{x\}$ consists of all complex numbers $z \in \mathbb{C}$ for which the improper integral exists and for which $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$. In a similar fashion, Agwa et al. in [9] introduce the Sumudu transform on arbitrary time scale $\mathbb{T}$, by the following formula:

$$\mathbb{S}\{x\}(z) := \frac{1}{z}\int_{t_0}^\infty x(t)\, e^{\sigma}_{\ominus(1/z)}(t, t_0)\, \Delta t, \tag{4}$$

for $z \in \mathscr{D}\{x\}$, where $\mathscr{D}\{x\}$ consists of all complex numbers $z \in \mathbb{C}$ for which the improper integral exists and for which $1 + \mu(t)/z \neq 0$ for all $t \in \mathbb{T}$. Note that if $\mathbb{T} = \mathbb{R}$ (for real analysis), (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) at $t_0 = 0$. In the case of $\mathbb{T} = \mathbb{Z}$ (for discrete analysis), we have

$$\mathscr{L}\{x\}(z) = \frac{Z\{x\}(z+1)}{z+1}, \tag{5}$$

where $Z\{x\}(z) = \sum_{t=0}^\infty x(t) z^{-t}$ is the classical Z-transform, which will be used to solve higher order linear forward

difference equations (see [7]). Similarly, formula (3) can also be extended to other particular discrete settings such as $\mathbb{T} = q^{\mathbb{Z}}$, $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{Z}$ (in h-calculus) (see [10]), and also $\mathbb{T} = \mathbb{T}_{(q,h)}^{t_0}$ (in (q,h)-calculus) (see [11]). Likewise, the delta Sumudu transform on time scales not only can be applied on ordinary differential equations when $\mathbb{T} = \mathbb{R}$ and on forward difference equations when $\mathbb{T} = hZ$ $(h > 0)$ but also can be applied for q-difference equations when $\mathbb{T} = q^{\mathbb{Z}}$ and on different types of time scales like $\mathbb{T} = h\mathbb{Z}$ and $\mathbb{T} = \mathbb{T}_n$; for the space of the harmonic numbers, see [9].

Continuous fractional calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators. Fractional differentiation has played an important role in various areas ranging from mechanics to image processing. Their fundamental results have been surveyed, for example, in the monographs [12, 13]. On the other hand, discrete fractional calculus is a very new area for scientists. Foundation of this theory were formulated in pioneering works by Agarwal [14] and Díaz and Osler [15, 16], where basic approach, definitions, and properties of the theory of fractional sums and differences were reported (see also [17, 18]). Recently, a series of papers continuing this research has appeared (see e.g., [19–26] and the references cited therein).

The extension of basic notions of fractional calculus to other discrete settings was performed in [27, 28]. In these papers, the authors often preferred the power function notation based on the time scales theory, which easily exposes similarities among the results in q-calculus, h-calculus, (q,h)-calculus, and the continuous case. However, this notation was employed only formally, since there was no general time scale definition of the power function and therefore the achieved results could not be generalized to other time scales. On this account, some ideas regarding fundamental properties which should be met by power functions on time scales were outlined in [29]. In [30], the authors introduced fractional derivatives and integrals on time scales via the generalized Laplace transform. However, this approach suffers by some technical difficulties, connected to the inverse Laplace transform (see [8]). Recently, in [31, 32] (see also [33]), the authors independently suggested an axiomatic definition of power functions on arbitrary time scale.

The aim of this paper is to introduce the nabla type Laplace transform and Sumudu transform, their properties, and applicability and its efficiency in solving fractional dynamic equations on arbitrary time scale. Of course, it is possible to consider also the delta type Laplace and Sumudu transforms (3) and (4), respectively; however, the nabla version seems be more suitable for fractional calculus as outlined, for example, in [27, 28, 34].

This paper is organized as follows. In Section 2, we recall basics of the time scale theory and the foundation of fractional calculus on time scales. Section 3 is devoted to nabla Laplace transform, its properties, convolution theorem, and examples of solution of fractional dynamic equations on time scales in terms of Mittag-Leffler function. Finally, in Section 4, we introduce nabla Sumudu transform and its properties on arbitrary time scales. A close relationship between nabla Sumudu transform and nabla Laplace transform and several important results were obtained. This section ended up with solving some fractional dynamic equation with nabla Sumudu transform method.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T} = \mathbb{R}$, $\mathbb{Z}$, and $\overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$. Let $\mathbb{T}$ have a right-scattered minimum m and define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $a, b \in \mathbb{T}$ with $a < b$, we denote by $[a,b]_{\mathbb{T}}$ the closed interval $[a,b] \cap \mathbb{T}$.

The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{6}$$

and the backward graininess function $\nu : \mathbb{T}_\kappa \to [0,\infty)$ is defined by $\nu(t) := t - \rho(t)$. For details and advancement on time scales, see the monographs [3, 5, 7, 35–37].

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla-derivative (briefly, the ∇-derivative) [21] of f at t, denoted by $f^\nabla(t)$, is the number (provided it exists) with the property that, given any $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\right| \le \epsilon\left|\rho(t) - s\right| \quad \forall s \in U. \tag{7}$$

For $\mathbb{T} = \mathbb{R}$, $f^\nabla(t) = f'(t))$ is the usual derivative; for $\mathbb{T} = \mathbb{Z}$, the ∇-derivative is the backward difference operator, $f^\nabla(t) = f(t) - f(t-1) = \nabla f(t)$.

A function $f : \mathbb{T} \to \mathbb{C}$ is left-dense continuous or ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous.

The set of ld-continuous functions $f : \mathbb{T} \to \mathbb{C}$ will be denoted by $C_{\mathrm{ld}}(\mathbb{T},\mathbb{C})$ and the set of functions $f : \mathbb{T} \to \mathbb{C}$ that are ∇-differentiable and whose derivatives are ld-continuous is denoted by $C_{\mathrm{ld}}^1(\mathbb{T},\mathbb{C})$.

It is known from [5] that if $f \in C_{\mathrm{ld}}(\mathbb{T},\mathbb{C})$, then there exists a function F such that $F^\nabla(t) = f(t)$. In this case, we define the *Cauchy integral* by

$$\int_a^b f(t)\,\nabla t = F(b) - F(a), \quad \text{for } a, b \in \mathbb{T}. \tag{8}$$

Let $[a,b] \subseteq \mathbb{T}$ and $f \in C_{\mathrm{ld}}(\mathbb{T},\mathbb{C})$. If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\nabla t = \int_a^b f(t)\,dt, \tag{9}$$

where the right-hand side integral is the Riemann integral from calculus and if $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\nabla t = \sum_{t=a+1}^{b} f(t). \tag{10}$$

For $f, g \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$ and $a, b \in \mathbb{T}$, the integration by parts formula is given by

$$\int_a^b f(\rho(t)) g^{\nabla}(t) \nabla t = [f(t)(t)]_a^b - \int_a^b f^{\nabla}(t) g(t) \nabla t. \quad (11)$$

A function $f \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$ is called ν-regressive if $1 - \nu f \neq 0$ on $\mathbb{T}_\kappa$ and positively ν-regressive if it is real valued and $1 - \nu f > 0$ on $\mathbb{T}_\kappa$. The set of ν-regressive functions and the set of positively ν-regressive functions are denoted by $\mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ and $\mathscr{R}_\nu^+(\mathbb{T}, \mathbb{C})$, respectively, and $\mathscr{R}_\nu^-(\mathbb{T}, \mathbb{C})$ is defined similarly. For simplicity, we denote by ${}_c\mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ the set of complex ν-regressive constants and, similarly, we define the sets ${}_cR_\nu^+(\mathbb{T}, \mathbb{C})$ and ${}_cR_\nu^-(\mathbb{T}, \mathbb{C})$.

Let $f \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$. The the nabla exponential function $\widehat{e}_f(\cdot, s)$ is defined to be the unique solution of the following initial value problem:

$$\begin{gathered} x^{\nabla} = fx \quad \text{on } \mathbb{T}_\kappa, \\ x(s) = 1 \end{gathered} \quad (12)$$

for some fixed $s \in \mathbb{T}$. Let $h > 0$; set

$$\begin{gathered} \mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq \frac{1}{h} \right\}, \\ \mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h} \right\}, \end{gathered} \quad (13)$$

and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h > 0$, the Hilger real part and imaginary part of a complex number are given by

$$\mathfrak{R}_h(z) := \frac{1}{h}(1 - |1 - hz|), \quad (14)$$

$$\mathfrak{I}_h(z) := \frac{1}{h}\operatorname{Arg}(1 - hz), \quad (15)$$

respectively, where Arg denotes the principle argument function; that is, $\operatorname{Arg} : \mathbb{C} \to (-\pi, \pi]_{\mathbb{R}}$, and let $\mathfrak{R}_0(z) : \mathfrak{R}(z)$ and $\mathfrak{I}_0(z) := \mathfrak{I}(z)$. For any fixed complex number z, the Hilger real part $\mathfrak{R}_h(z)$ is a nondecreasing function of $h \in [0, \infty)$ (see [38]).

For $h \geq 0$, we define the ν-cylinder transformation $\widehat{\xi}_h : \mathbb{C}_h \to \mathbb{Z}_b$ by

$$\widehat{\xi}_h(z) := \begin{cases} z, & h = 0, \\ -\dfrac{1}{h}\operatorname{Log}(1 - hz), & h > 0 \end{cases} \quad (16)$$

for $z \in \mathbb{C}_h$. Then, the nabla exponential function can also be written in the following form:

$$\widehat{e}_f(t, s) := \exp\left\{ \int_s^t \widehat{\xi}_{\nu(\tau)}(f(\tau)) \nabla \tau \right\} \quad \text{for } s, t \in \mathbb{T}. \quad (17)$$

It is known that the nabla exponential function $\widehat{e}_f(\cdot, s)$ is strictly positive on $\mathbb{T}$, provided $f \in \mathscr{R}_\nu^+(\mathbb{T}, \mathbb{C})$ (see Theorem 3.18 [6]). For $f, g \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$, the ν-circul plus and the ν-circle minus are defined by

$$f \oplus_\nu g := f + g - \nu f g, \qquad f \ominus_\nu g := \frac{f - g}{1 - \nu g}, \quad (18)$$

respectively. For further details on nabla exponential function, we refer to [5].

We recall the notion of Taylor monomials introduced in [39] (see also [7]). These monomials $\widehat{h}_n : \mathbb{T} \times \mathbb{T} \to \mathbb{C}, n \in \mathbb{N}_0$, are defined recursively as follows:

$$\widehat{h}_0(t, s) = 1 \quad \forall s, t \in \mathbb{T} \quad (19)$$

and, given $\widehat{h}_n$ for $n \in \mathbb{N}_0$, we have

$$\widehat{h}_{n+1}(t, s) = \int_s^t \widehat{h}_n(\tau, s) \nabla \tau \quad \forall s, t \in \mathbb{T}. \quad (20)$$

Example 1. For the case $\mathbb{T} = \mathbb{R}$, we have

$$\widehat{h}_n(t, s) = \frac{(t - s)^k}{k!}, \quad \forall s, t \in \mathbb{R}. \quad (21)$$

For the case $\mathbb{T} = \mathbb{Z}$, we have

$$\widehat{h}_n(t, s) = \frac{(t - s)^{(n)}}{n!} = \frac{\prod_{j=0}^{n-1}(t - s - j)}{n!}, \quad \forall s, t \in \mathbb{Z}, \quad (22)$$

where $t^{(n)} = t(t + 1)(t + 2) \cdots (t + n - 1)$.

For the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for some $q > 1$, we have

$$\widehat{h}_n(t, s) = \prod_{k=0}^{n-1} \frac{q^k t - s}{\sum_{j=0}^k q^j}, \quad \forall s, t \in \overline{q^{\mathbb{Z}}}. \quad (23)$$

Lemma 2 (nabla Cauchy formula [37]). *Let $n \in \mathbb{Z}^+$, $a, b \in \mathbb{T}$, and let $f : \mathbb{T} \to \mathbb{C}$ be ∇-integrable on $\widetilde{\mathbb{T}} := [a, b] \cap \mathbb{T}$. If $t \in \widetilde{\mathbb{T}}$, then*

$${}_a\nabla^{-n} f(t) = \int_a^t \widehat{h}_{n-1}(t, \rho(\tau)) f(\tau) \nabla \tau. \quad (24)$$

The formula (24) is a corner stone in the introduction of the nabla fractional integral ${}_a\nabla^{-\alpha} f(t)$ for $\alpha > 0$. However, it requires a reasonable and natural extension of a discrete system of monomials $(\widehat{h}_n, n \in \mathbb{N}_0)$ to a continuous system $(\widehat{h}_\alpha, \alpha \in \mathbb{R}^+)$. However, the calculation of $\widehat{h}_n$ for $n > 1$ is a difficult task which seems to be answerable only in some particular cases (see Example 1).

Recently, [31, 32] independently suggested quite similar axiomatic definitions of time scales power functions. In [40], the author have considered the power functions and essentials of fractional calculus on isolated time scales. The definitions below follow from [31].

Definition 3. Let $s, t \in \mathbb{T}$ and $\alpha, \beta > -1$. The time scales power functions $\widehat{h}_\alpha(t, s)$ are defined as a family of nonnegative functions satisfying

(i) $\int_s^t \widehat{h}_\alpha(t, \rho(\tau)) \widehat{h}_\beta(\tau, s) \nabla \tau = \widehat{h}_{\alpha+\beta+1}(t, s)$ for $t \geq s$;

(ii) $\widehat{h}_0(t, s) = 1$ for $t \geq s$;

(iii) $\widehat{h}_\alpha(t, t) = 0$ for $\alpha \in (0, 1)$.

Further, we have the following.

Definition 4. Let $\alpha \geq 0$, $\beta > 0$, and $a, b \in \mathbb{T}$. Then for $f \in C_{\mathrm{ld}}([a,b]_{\mathbb{T}}, \mathbb{C})$ one defines the following.

(i) The fractional integral of order $\alpha > 0$ with the lower limit a as

$$\left({}_a\nabla^{-\alpha} f\right)(t) := \int_a^t \widehat{h}_{\alpha-1}\left(t, \rho(\tau)\right) f(\tau)\, \nabla\tau \tag{25}$$

and for $\alpha = 0$ one puts $({}_a\nabla^0 f)(t) = f(t)$.

(ii) The Riemann-Liouville fractional derivative of order $\beta > 0$ with lower limit a as

$$\left({}_a\nabla^{\beta} f\right)(t) := \left[{}_a\nabla^{-(m-\beta)} f\right]^{\nabla^m}(t), \quad t \in [\sigma(a), b]_{\mathbb{T}}, \tag{26}$$

where $m = [\beta] + 1$.

(iii) The Caputo fractional derivative ${}_a^C\nabla^{\gamma} f(t)$ $(\gamma > 0)$ on $[\sigma(a), b]_{\mathbb{T}}$ is defined via the Riemann-Liouville fractional derivative by

$${}_a^C\nabla^{\gamma} f(t) := \left({}_a\nabla^{-(m-\gamma)} f^{\nabla^m}\right)(t), \tag{27}$$

where $m = [\gamma] + 1$.

3. Nabla Laplace Transform

Note that below we assume that $z \in_c \mathscr{R}_\nu$; then $(\ominus_\nu z) \in_c \mathscr{R}_\nu$ and therefore $\widehat{e}_{\ominus_\nu z}(\cdot, t_0)$ is well defined on $\mathbb{T}$. From now on we assume that $\mathbb{T}$ is unbounded above.

The following theorem is concerning the asymptotic nature of the nabla exponential function. To this end, we define the minimal graininess function $\nu_* : \mathbb{T} \to [0, \infty)$ by

$$\nu_*(s) := \inf_{\tau \in [s,\infty)_{\mathbb{T}}} \nu(\tau) \quad \text{for } s \in \mathbb{T}, \tag{28}$$

and for $h \geq 0$ and $\lambda \in \mathbb{R}$, we define

$$\mathbb{C}_h(\lambda) := \left\{z \in \mathbb{C}_h : \mathfrak{R}_h(z) > \lambda\right\}. \tag{29}$$

Theorem 5 (decay of the nabla exponential function). *Let*

$$s \in \mathbb{T}, \quad \lambda \in_c R_\nu^+\left([s,\infty)_{\mathbb{T}}, \mathbb{C}\right). \tag{30}$$

Then, for any $z \in \mathbb{C}_{\nu_*(s)}(\lambda)$, we have the following properties:

(i) $|\widehat{e}_{\lambda\ominus_\nu z}(t,s)| \leq \widehat{e}_{\lambda\ominus_\nu \mathfrak{R}_{\nu_*(s)}(z)}(t,s)$ *for all* $t \in [s,\infty)_{\mathbb{T}}$,

(ii) $\lim_{t\to\infty} \widehat{e}_{\lambda\ominus_\nu \mathfrak{R}_{\nu_*(s)(z)}}(t,s) = 0$,

(iii) $\lim_{t\to\infty} \widehat{e}_{\lambda\ominus_\nu z}(t,s) = 0$.

Proof. The proof is similar to Theorem 3.4 of [38]. □

Definition 6 (exponential order). Let $s \in \mathbb{T}$. A function $f \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$ has exponential order α on $[s,\infty)_{\mathbb{T}}$, if

(i) $\alpha \in {}_cR_\nu^+([s,\infty)_{\mathbb{T}}, \mathbb{C})$,

(ii) there exists $K > 0$, such that $|f(t)| \leq K\widehat{e}_\alpha(t,s)$ for all $t \in [s,\infty)_{\mathbb{T}}$.

Lemma 7. *Let $s \in \mathbb{T}$ and $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be a function of exponential order α. Then,*

$$\lim_{t\to\infty} f(t)\, \widehat{e}_{\ominus_\nu z}(t,s) = 0, \tag{31}$$

where $z \in \mathbb{C}_{\nu_*(s)}(\alpha)$.

Proof. It follows that

$$\left|f(t)\, \widehat{e}_{\ominus_\nu z}(t,s)\right| \leq K\widehat{e}_\alpha(t,s)\left|\widehat{e}_{\ominus z}(t,s)\right| = K\left|\widehat{e}_{\alpha\ominus z}(t,s)\right| \tag{32}$$

for all $t \in [s,\infty)_{\mathbb{T}}$ and some $K > 0$. By Theorem 5(iii) and letting $t \to \infty$ in (32), we get (31). This completes the proof. □

Definition 8. Let $f \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$ be a function. Then, the ∇-Laplace transform ${}_s\mathscr{L}_\nabla\{f\}(\cdot)$ about the point $s \in \mathbb{T}$ of the function f is defined by

$${}_s\mathscr{L}_\nabla\{f\}(z) := \int_s^\infty \widehat{e}_{\ominus_\nu z}\left(\rho(t), s\right) f(t)\, \nabla t \quad \text{for } z \in \mathscr{D}_\nu\{f\}, \tag{33}$$

where $\mathscr{D}_\nu\{f\}$ consists of all complex numbers $z \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ for which the improper integral exists.

Theorem 9. *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be of exponential order α. Then, the ∇-Laplace transform ${}_s\mathscr{L}_\nabla\{f\}(\cdot)$ exists on $\mathbb{C}_{\nu^*(s)}(\alpha)$ and converges absolutely.*

Proof. The proof is similar to Theorem 5.1 in [38]. □

Theorem 10 (linearity of the transform). *Let $f_1, f_2 \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be of exponential order α_1, α_2, respectively. Then, for any $c_1, c_2 \in \mathbb{R}$, we have*

$${}_s\mathscr{L}_\nabla\{c_1 f_1 + c_2 f_2\}(z) = c_1\, {}_s\mathscr{L}_\nabla\{f_1\}(z) + c_2\, {}_s\mathscr{L}_\nabla\{f_2\}(z) \tag{34}$$

for all $z \in \mathbb{C}_{\nu_*(s)}(\max\{\alpha_1, \alpha_2\})$.

Proof. The proof follows from the linearity property of the ∇-integral (see Theorem 8.47(i) in [5]). □

Theorem 11 (transform of derivative). *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be a function of exponential order α. Then, one has*

$${}_s\mathscr{L}_\nabla\left\{f^\nabla\right\}(z) = z\widehat{F}(z) - f(s), \tag{35}$$

for all $z \in \mathbb{C}_{\nu_*(s)}(\alpha)$, where $\widehat{F}$ denotes ${}_s\mathscr{L}_\nabla\{f\}$.

Proof. By using integration by parts formula (11), we get

$$\begin{aligned} &{}_s\mathscr{L}_\nabla\left\{f^\nabla\right\}(z) \\ &\quad = \int_s^\infty \widehat{e}_{\ominus_\nu z}\left(\rho(t), s\right) f^\nabla(t)\, \nabla t \\ &\quad = \left[\widehat{e}_{\ominus_\nu z}(t,s) f(t)\right]_s^\infty + z\int_s^\infty \widehat{e}_{\ominus_\nu z}\left(\rho(t), s\right) f(t)\, \nabla t \\ &\quad = -f(s) + z\widehat{F}(z), \end{aligned} \tag{36}$$

for all $z \in \mathbb{C}_{\nu_*(s)}(\alpha)$. This completes the proof. □

By induction, we have the following result.

Corollary 12. *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}},\mathbb{C})$ be a function of exponential order α. Then for any $n \in \mathbb{N}$, one has*

$$ {}_s\mathscr{L}_\nabla\left\{f^{\nabla^n}\right\}(z) = z^n\widehat{F}(z) - \sum_{k=0}^{n-1} z^{n-k-1} f^{\nabla^k}(s) \tag{37}$$

for all $z \in \mathbb{C}_{\nu_*(s)}(\alpha)$.

Definition 13 (see [41]). For a given $f : [t_0,\infty)_{\mathbb{T}} \to \mathbb{C}$, the solution of the shifting problem

$$\begin{gathered} u^{\nabla_t}(t,\rho(s)) = -u^{\nabla_s}(t,s), \quad t,s \in \mathbb{T},\ t \geq s \geq t_0, \\ u(t,t_0) = f(t), \quad t \in \mathbb{T},\ t \geq t_0 \end{gathered} \tag{38}$$

is denoted by $\widehat{f}$ and is called the shift (or delay) of f.

In this section, we will assume that the problem (38) has a unique solution $\widehat{f}$ for a given initial function f and that the functions f, g, and the complex number z are such that the operations fulfilled are valid.

Definition 14 (see [41]). For given functions $f,g : \mathbb{T} \to \mathbb{C}$, their convolution $f * g$ is defined by

$$f(t) * g(t) = \int_s^t \widehat{f}(t,\rho(\tau))\, g(\tau)\,\nabla\tau, \quad s,t \in \mathbb{T}, \tag{39}$$

where $\widehat{f}$ is the shift of f introduced in Definition 8.

We state the following results without proof, since the proofs of them are similar to those in [6].

Theorem 15. *The convolution is associative; that is,*

$$(f * g) * h = f * (g * h). \tag{40}$$

Theorem 16. *If f is ∇-differentiable, then*

$$(f * g)^{\nabla} = f^{\nabla} * g + f(s)\, g \tag{41}$$

and if g is nabla-differentiable, then

$$(f * g)^{\nabla} = f * g^{\nabla} + fg(s). \tag{42}$$

Corollary 17. *The following formula holds:*

$$\int_s^t \widehat{f}(t,\rho(\tau))\,\nabla\tau = \int_s^t f(\tau)\,\nabla\tau. \tag{43}$$

Theorem 18 (convolution theorem). *Suppose $f,g : \mathbb{T} \to \mathbb{R}$ are locally ∇-integrable functions on $\mathbb{T}$ and their convolution $f * g$ is defined by* (39). *Then,*

$$\begin{gathered} {}_s\mathscr{L}_\nabla\{f * g\}(z) = {}_s\mathscr{L}_\nabla\{f\}(z) \cdot {}_s\mathscr{L}_\nabla\{g\}(z), \\ z \in \mathscr{D}_\nu\{f\} \cap \mathscr{D}_\nu\{g\}. \end{gathered} \tag{44}$$

Let $\Phi_\alpha(t) = \widehat{h}_{\alpha-1}(t,s)$. Then, by means of convolution, the nabla operators in Definition 4 can be restated as

$$\begin{gathered} \left({}_a\nabla^{-\alpha} f\right)(t) = \Phi_\alpha(t) * f(t), \\ \left({}_a\nabla^{\beta} f\right)(t) = \left(\Phi_{m-\beta}(t) * f(t)\right)^{\nabla^n}, \quad m = [\beta] + 1, \\ \left({}_a^C\nabla^{\gamma} f\right)(t) = \Phi_{m-\gamma}(t) * f^{\nabla^m}, \quad m = [\gamma] + 1. \end{gathered} \tag{45}$$

It is known [5, 7] that, for all $k \in \mathbb{N}_0$ and $z \in \mathscr{D}_\nu$,

$$ {}_{t_0}\mathscr{L}_\nabla\left\{\widehat{h}_k(\cdot,t_0)\right\}(z) = \frac{1}{z^{k+1}}. \tag{46}$$

In general, we have

Theorem 19. *For $\alpha > 0$ and $z \in \mathscr{D}_\nu$,*

$$ {}_s\mathscr{L}_\nabla\left\{\widehat{h}_\alpha(t,s)\right\}(z) = \frac{1}{z^{\alpha+1}} \tag{47}$$

holds.

Proof. First, we write Definition 3(i) in convolution form; that is,

$$\int_s^t \widehat{h}_\alpha(t,\rho(\tau))\,\widehat{h}_\beta(\tau,s)\,\nabla\tau = \widehat{h}_\alpha(t,s) * \widehat{h}_\beta(t,s). \tag{48}$$

Then, obviously

$$\widehat{h}_\alpha(t,s) * \widehat{h}_\beta(t,s) = \widehat{h}_{\alpha+\beta+1}(t,s). \tag{49}$$

We show that (47) satisfies the Laplace transform of (49). Let $\beta = 0$. Taking Laplace transform to the left-hand side followed by applying convolution theorem (39) yields

$$ {}_s\mathscr{L}_\nabla\left\{\widehat{h}_\alpha(t,s) * \widehat{h}_0(t,s)\right\} = \left(\frac{1}{z^{\alpha+1}}\right)\left(\frac{1}{z}\right) = \frac{1}{z^{\alpha+2}}. \tag{50}$$

But, from the right side of (50), we have

$$\frac{1}{z^{\alpha+2}} = \frac{1}{z^{(\alpha+1)+1}} = {}_s\mathscr{L}_\nabla\left\{\widehat{h}_{\alpha+1}(t,s)\right\}(z). \tag{51}$$

Hence the result follow from (50) and (51). This completes the proof. □

From (45), knowing

$$ {}_s\mathscr{L}_\nabla\{\Phi_\alpha\}(z) = {}_s\mathscr{L}_\nabla\left\{\widehat{h}_{\alpha-1}(\cdot,s)\right\}(z) = \frac{1}{z^{\alpha}}, \tag{52}$$

we have (by taking $s = a$) the following result.

Theorem 20. *For $\alpha > 0$,*

$$ {}_a\mathscr{L}_\nabla\left\{{}_a\nabla^{-\alpha} f\right\}(z) = z^{-\alpha}\widehat{F}(z). \tag{53}$$

For the Riemann-Liouville fractional derivative derivative (26), we have the following result.

Theorem 21. *For $\beta > 0$ and $m = [\beta] + 1$,*

$$\mathscr{L}_\nabla\left\{{}_a\nabla^{\beta} f\right\}(z) = z^{\beta}\widehat{F}(z) - \sum_{k=0}^{m-1} z^{m-k-1}\left[{}_a\nabla^{-(m-\beta)} f\right]^{\nabla^k}(a). \tag{54}$$

Proof. Write (26) as

$$\left({}_a\nabla^{\beta} f\right)(t) = g^{\nabla^m}(t), \quad \text{where } g(t) = \left({}_a\nabla^{-(m-\beta)} f\right)(t). \tag{55}$$

From (53), we have

$$\widehat{G}(z) = z^{-(m-\beta)}\widehat{F}(z). \tag{56}$$

Thus, by (37) and (56), we have

$$\begin{aligned}\mathscr{L}_\nabla\left\{{}_a\nabla^\beta f\right\}(z) &= \mathscr{L}_\nabla\left\{g^{\nabla^m}\right\}(z)\\ &= z^m\widehat{G}(z) - \sum_{k=0}^{m-1} z^{m-k-1} g^{\nabla^k}(a)\\ &= z^\beta\widehat{F}(z) - \sum_{k=0}^{m-1} z^{m-k-1}\left[{}_a\nabla^{-(m-\beta)} f\right]^{\nabla^k}(a).\end{aligned} \tag{57}$$

□

The Laplace transform (54) is equivalent to the following one:

$$\mathscr{L}_\nabla\left\{{}_a\nabla^\beta f\right\}(z) = z^\beta\widehat{F}(z) - \sum_{j=1}^{\ell} z^{j-1}\left({}_a\nabla^{\beta-j} f\right)(a), \quad \ell-1<\beta\le\ell. \tag{58}$$

The nabla Laplace transform of Caputo fractional derivative of order α is given as follows.

Theorem 22. *For $\alpha > 0$ and $m = [\alpha] + 1$,*

$$\mathscr{L}_\nabla\left\{{}_a^C\nabla^\alpha f\right\}(z) = z^\alpha F(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(a). \tag{59}$$

Proof. Write (27) as

$$\left({}_a^C\nabla^\alpha f\right)(t) = \left({}_a\nabla^{-(m-\alpha)} g\right)(t), \quad \text{where } g(t) = f^{\nabla^m}(t). \tag{60}$$

By following (53) and (37), we get

$$\begin{aligned}\mathscr{L}_\nabla\left\{{}_a^C\nabla^\alpha f\right\}(z) &= \mathscr{L}_\nabla\left\{{}_a\nabla^{-(m-\alpha)} g\right\}(z)\\ &= z^{-(m-\alpha)}\widehat{G}(z)\\ &= z^{-(m-\alpha)}\left[z^m\widehat{F}(z) - \sum_{k=0}^{m-1} z^{m-k-1} f^{\nabla^k}(a)\right]\\ &= z^\alpha F(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(a).\end{aligned} \tag{61}$$

□

Now, let us consider the generalized Mittag-Leffler function on time scales (see [28, 42]).

Definition 23. Let $\alpha, \beta, \lambda \in \mathbb{R}$ and $s, t \in \mathbb{T}$. The time scales Mittag-Leffler function, $E_{\alpha,\beta}^{s,\lambda}(t)$, is defined by the following series expansion:

$$E_{\alpha,\beta}^{s,\lambda}(t) = \sum_{k=0}^{\infty}\lambda^k\widehat{h}_{\alpha k+\beta-1}(t,s). \tag{62}$$

In the following theorem, we give the Laplace transform of generalized Mittag-Leffler function on time scales.

Theorem 24. *For $\alpha, \beta, \lambda \in \mathbb{R}$ and $a, t \in \mathbb{T}$, it holds that*

$${}_a\mathscr{L}_\nabla\left\{E_{\alpha,\beta}^{a,\lambda}(t)\right\}(z) = \frac{z^{-\beta}}{1-\lambda z^{-\alpha}}, \tag{63}$$

provided $|\lambda/z^\alpha| < 1$.

Proof. By using Theorem 10 and the relation (47), we obtain

$$\begin{aligned}{}_a\mathscr{L}_\nabla\left\{E_{\alpha,\beta}^{a,\lambda}(t)\right\}(z) &= {}_a\mathscr{L}_\nabla\left\{\sum_{k=0}^{\infty}\lambda^k\widehat{h}_{\alpha k+\beta-1}(t,a)\right\}\\ &= \sum_{k=0}^{\infty}\lambda^k\,{}_a\mathscr{L}_\nabla\left\{\widehat{h}_{\alpha k+\beta-1}(t,a)\right\}\\ &= \sum_{k=0}^{\infty}\frac{\lambda^k}{z^{k\alpha+\beta}} = \frac{1}{z^\beta}\left\{1+\frac{\lambda}{z^\alpha}+\frac{\lambda^2}{z^{2\alpha}}+\cdots\right\}\\ &= \frac{z^{-\beta}}{1-\lambda z^{-\alpha}}, \quad \left|\frac{\lambda}{z^\alpha}\right| < 1.\end{aligned} \tag{64}$$

□

Example 25. Consider the following initial value problem:

$$\left({}_a\nabla^\alpha y\right)(t) - \lambda y(t) = f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}}, \tag{65}$$

$$\left({}_a\nabla^{\alpha-j} y\right)(a) = b_j, \quad \left(b_j \in \mathbb{R}; j = 1, 2, \ldots, n = [\alpha]\right). \tag{66}$$

By taking Laplace transform of both sides of (65) and using (58), we get

$$\begin{aligned}&z^\alpha\,{}_a\mathscr{L}_\nabla\{y\}(z) - \sum_{j=1}^{n} z^{j-1}\left({}_a\nabla^{\alpha-j} y\right)(a) - \lambda\,{}_a\mathscr{L}_\nabla\{y\}(z)\\ &\quad = {}_a\mathscr{L}_\nabla\{f\}(z),\\ &{}_a\mathscr{L}_\nabla\{y\}(z)\\ &\quad = \frac{z^{-\alpha}}{1-\lambda z^{-\alpha}}\sum_{j=1}^{n} z^{j-1}\left({}_a\nabla^{\alpha-j} y\right)(a) + \frac{z^{-\alpha}}{1-\lambda z^{-\alpha}}\,{}_a\mathscr{L}_\nabla\{f\}(z)\\ &\quad = \sum_{j=1}^{n} b_j\,{}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha-j+1}^{a,\lambda}(t)\right\}(z)\\ &\qquad + {}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t)\right\}(z)\,{}_a\mathscr{L}_\nabla\{f\}(z)\\ &\quad = \sum_{j=1}^{n} b_j\,{}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha-j+1}^{a,\lambda}(t)\right\}(z) + {}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t) * f(t)\right\}.\end{aligned} \tag{67}$$

Thus, we have

$$\begin{aligned} y(t) &= \sum_{j=1}^{n} b_j E_{\alpha,\alpha-j+1}^{a,\lambda}(t) + E_{\alpha,\alpha}^{a,\lambda}(t) * f(t) \\ &= \sum_{j=1}^{n} b_j E_{\alpha,\alpha-j+1}^{a,\lambda}(t) + \int_a^t E_{\alpha,\alpha}^{\rho(\tau),\lambda}(t) f(\tau) \nabla\tau. \end{aligned} \tag{68}$$

The above example coincides with the case $\mathbb{T} = \mathbb{R}$ (see [43]).

Now, we consider the Cauchy problem for dynamic equations with the nabla type Caputo fractional derivatives.

Example 26. Consider the following initial value problem:

$$\left({}_a^C\nabla^\alpha y\right)(t) - \lambda y(t) = f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}}, \tag{69}$$

$$y^{\nabla^j}(a) = c_j, \quad \left(c_j \in \mathbb{R}; j = 0, 1, 2, \ldots, n-1\right). \tag{70}$$

By taking Laplace transform of both sides of (69) and using (59), we get

$$\begin{aligned} & z^\alpha {}_a\mathscr{L}_\nabla\{y\}(z) - \sum_{j=0}^{n-1} z^{\alpha-j-1} y^{\nabla^j}(a) - \lambda {}_a\mathscr{L}_\nabla\{y\}(z) \\ &\quad = {}_a\mathscr{L}_\nabla\{f\}(z), \\ & {}_a\mathscr{L}_\nabla\{y\}(z) \\ &\quad = \frac{z^{-\alpha}}{1-\lambda z^{-\alpha}} \sum_{j=0}^{n-1} c_j z^{\alpha-j-1} + \frac{z^{-\alpha}}{1-\lambda z^{-\alpha}} {}_a\mathscr{L}_\nabla\{f\}(z) \\ &\quad = \sum_{j=0}^{n-1} c_j {}_a\mathscr{L}_\nabla\left\{E_{\alpha,j+1}^{a,\lambda}(t)\right\}(z) \\ &\qquad + {}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t)\right\}(z)\ {}_a\mathscr{L}_\nabla\{f\}(z) \\ &\quad = \sum_{j=0}^{n} c_j {}_a\mathscr{L}_\nabla\left\{E_{\alpha,j+1}^{a,\lambda}(t)\right\}(z) + {}_a\mathscr{L}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t) * f(t)\right\}. \end{aligned} \tag{71}$$

Thus, we have

$$\begin{aligned} y(t) &= \sum_{j=0}^{n-1} c_j E_{\alpha,j+1}^{a,\lambda}(t) + E_{\alpha,\alpha}^{a,\lambda}(t) * f(t) \\ &= \sum_{j=0}^{n-1} c_j E_{\alpha,j+1}^{a,\lambda}(t) + \int_a^t E_{\alpha,\alpha}^{\rho(\tau),\lambda}(t) f(\tau) \nabla\tau. \end{aligned} \tag{72}$$

The last example clearly coincides with the real counter part; see [43].

4. Nabla Sumudu Transform

In [9], the authors introduced and studied the (delta) Sumudu transform on time scales. Many important results were produced and applied on dynamic equation on time scales. In this section, we will consider the nabla Sumudu transform. Most of the results were coated from [9, 44, 45] without proof since their proofs are similar.

Definition 27. Let $f \in C_{\mathrm{ld}}(\mathbb{T}, \mathbb{C})$ be a function. Then, the ∇-Sumudu transform ${}_s\mathcal{S}_\nabla\{f\}(\cdot)$ about the the point $s \in \mathbb{T}$ of the function f is defined by

$$\begin{aligned} & {}_s\mathbb{S}_\nabla\{f\}(z) \\ &\quad := \frac{1}{z}\int_s^\infty \hat{e}_{\ominus_\nu(1/z)}(\rho(t), s) f(t) \nabla t \quad \text{for } z \in \mathscr{D}\{f\}, \end{aligned} \tag{73}$$

where $\mathscr{D}\{f\}$ consists of all complex numbers $z \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ for which the improper integral exists.

Let us define the set

$$\overline{\mathbb{C}}_h(\lambda) := \left\{z \in \mathbb{C}_h : \mathfrak{R}_h\left(\frac{1}{z}\right) > \lambda\right\}. \tag{74}$$

We notice that, following Lemma 7, if $f \in C_{\mathrm{ld}}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ is a function of exponential order α, then

$$\lim_{t\to\infty} f(t)\, \hat{e}_{\ominus_\nu(1/z)}(t, s) = 0, \tag{75}$$

where $z \in \overline{\mathbb{C}}_h(\alpha)$. Hence, we have the following.

Theorem 28. *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be of exponential order α. Then, the ∇-Sumudu transform ${}_s\mathcal{S}_\nabla\{f\}(\cdot)$ exists on $\overline{\mathbb{C}}_{\nu^*(s)}(\alpha)$ and converges absolutely.*

In the special case $\mathbb{T} = \mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $a \in \mathbb{R}$ fixed (see [44]), we have

$${}_a\mathbb{S}_\nabla\{f\}(z) = \frac{1}{z}\sum_{k=1}^{\infty}\left(\frac{z-1}{z}\right)^{k-1} f(a+k) \tag{76}$$

for each $z \in \mathbb{C} \setminus \{0, 1\}$ for which the series converges.

The following theorem states the close relationship between nabla Sumudu transform and nabla Laplace transform.

Theorem 29. *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be a function. Then*

$${}_s\mathbb{S}_\nabla\{f\}(z) = \frac{1}{z}\, {}_s\mathscr{L}_\nabla\{f\}\left(\frac{1}{z}\right) = \frac{1}{z}\hat{F}\left(\frac{1}{z}\right). \tag{77}$$

The following theorem can be easily verified using induction.

Theorem 30. *Let $f \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$ be of exponential order α. Then,*

$${}_s\mathbb{S}_\nabla\left\{f^{\nabla^n}\right\}(z) = \frac{1}{z^n}\, {}_sS_\nabla\{f\}(z) - \sum_{k=0}^{n-1}\frac{1}{z^{n-k}} f^{\nabla^k}(s), \tag{78}$$

where $z \in \overline{\mathbb{C}}_{\nu^*(s)}(\alpha)$.

The following theorem presents the *nabla*-Sumudu transformation of convolution of two functions on time scales.

Theorem 31. *Let $f, g \in C_{ld}([s,\infty)_{\mathbb{T}}, \mathbb{C})$. Then*

$$ {}_s\mathbb{S}_\nabla \{f * g\}(z) = z\left[{}_{t_0}\mathbb{S}_\nabla \{f\}(z) \cdot {}_{t_0}\mathbb{S}_\nabla \{g\}(z)\right]. \quad (79)$$

Proof. The proof is a direct consequence of relation (77) and Theorem 18. □

Now, we consider the ∇-Sumudu transform on time scale fractional calculus. We begin with ∇-Sumudu transform of power function on $\mathbb{T}$.

Theorem 32. *Let $s \in \mathbb{T}$. For $\alpha > -1$, one has*

$$ {}_s\mathbb{S}_\nabla \left\{\widehat{h}_\alpha(\cdot, s)\right\}(z) = z^\alpha. \quad (80)$$

Proof. Using Theorem 28 and the result (41), we have

$$\begin{aligned} {}_s\mathbb{S}_\nabla \left\{\widehat{h}_\alpha(\cdot, s)\right\}(z) &= \frac{1}{z}\, {}_s\mathscr{L}_\nabla \left\{\widehat{h}_\alpha(\cdot, s) f\right\}\left(\frac{1}{z}\right) \\ &= \frac{1}{z}\left(\frac{1}{(1/z)^{\alpha+1}}\right) \\ &= z^\alpha. \end{aligned} \quad (81)$$

This completes the proof. □

In particular, $\widehat{h}_0(t,s) = 1$ and hence the ∇-Sumudu transform of $f(t) = 1$ is given as follows.

Corollary 33. *The ∇-Sumudu transform of $f(x) = 1$ is given by*

$$ {}_s\mathbb{S}_\nabla \{1\}(z) = {}_s\mathbb{S}_\nabla \left\{\widehat{h}_0(\cdot, s)\right\}(z) = z^0 = 1. \quad (82)$$

In the following theorem, we give the Sumudu transform of generalized Mittag-Leffler function on time scales.

Theorem 34. *For $\alpha, \beta, \lambda \in \mathbb{R}$ and $a, t \in \mathbb{T}$, it holds that*

$$ {}_a\mathbb{S}_\nabla \left\{E^{a,\lambda}_{\alpha,\beta}(t)\right\}(z) = \frac{z^{\beta-1}}{1-\lambda z^\alpha}, \quad (83)$$

provided $|\lambda z^\alpha| < 1$.

Proof. Using the relation (77) and the result (63), we get

$$\begin{aligned} {}_a\mathbb{S}_\nabla \left\{E^{a,\lambda}_{\alpha,\beta}(t)\right\}(z) &= \frac{1}{z}\, {}_a\mathscr{L}_\nabla \left\{E^{a,\lambda}_{\alpha,\beta}(t)\right\}\left(\frac{1}{z}\right) \\ &= \frac{1}{z}\left(\frac{(1/z)^{-\beta}}{1-\lambda(1/z)^{-\alpha}}\right) \\ &= \frac{z^{\beta-1}}{1-\lambda z^\alpha}, \quad \left|\lambda z^\alpha\right| < 1. \end{aligned} \quad (84)$$

□

The nabla Sumudu transform of fractional integral and fractional derivatives are as follows.

Theorem 35. *(i) For $\alpha > 0$,*

$$ {}_a\mathbb{S}_\nabla \left\{{}_a\nabla^{-\alpha} f\right\}(z) = z^\alpha\, {}_a\mathbb{S}_\nabla \{f\}(z). \quad (85)$$

(ii) For $\beta > 0$ and $m = [\beta] + 1$,

$$\begin{aligned} &{}_a\mathbb{S}_\nabla \left\{{}_a\nabla^\beta f\right\}(z) \\ &\quad = \frac{1}{z^\beta}\, {}_a\mathbb{S}_\nabla \{f\}(z) - \sum_{k=0}^{m-1} \frac{1}{z^{m-k}}\left[{}_a\nabla^{-(m-\beta)} f\right]^{\nabla^k}(a). \end{aligned} \quad (86)$$

(iii) For $\gamma > 0$ and $m = [\gamma] + 1$,

$$ {}_a\mathbb{S}_\nabla \left\{{}^C_a\nabla^\gamma f\right\}(z) = \frac{1}{z^\gamma}\, {}_a\mathbb{S}_\nabla \{f\}(z) - \sum_{k=0}^{m-1} \frac{1}{z^{\gamma-k}} f^{\nabla^k}(a). \quad (87)$$

Proof. The proof to each part follows immediately after applying (77) and the respective Laplace transforms (53), (54), and (59). □

As in the case of Laplace transform (see relation (58)), the ∇-Sumudu transform in Theorem 35(ii) is equivalent to

$$\begin{aligned} \mathbb{S}_\nabla \left\{{}_a\nabla^\beta f\right\}(z) &= \frac{1}{z^\beta}\, {}_a\mathbb{S}_\nabla \{f\}(z) - \sum_{j=1}^{\ell} \frac{1}{z^j}\left({}_a\nabla^{\beta-j} f\right)(a), \\ &\qquad \ell - 1 < \beta \le \ell. \end{aligned} \quad (88)$$

In the following example we will illustrate the use of the ∇-Sumudu transform by applying it to solve initial value problems.

Example 36. Consider the following initial value problem:

$$\left({}_a\nabla^\alpha y\right)(t) - \lambda y(t) = f(t), \quad t \in \mathbb{T},\ 0 < \alpha \le 1, \quad (89)$$

$$\left({}_a\nabla^{\alpha-1} y\right)(a) = y_0, \quad y_0 \in \mathbb{R}. \quad (90)$$

We begin by taking the ∇-Sumudu transform of both sides of (89). By using Theorem 35(ii) for $0 < \alpha \le 1$, we get

$$\begin{aligned} &z^{-\alpha}\, {}_a\mathbb{S}_\nabla \{y\}(z) - z_a^{-1}\nabla^{-(1-\alpha)} y(a) - \lambda \mathbb{S}_\nabla \{y\}(z) \\ &\quad = {}_a\mathbb{S}_\nabla \{f\}(z). \end{aligned} \quad (91)$$

Hence,

$$\begin{aligned} &{}_a\mathbb{S}_\nabla \{y\}(z) \\ &\quad = \frac{z^{-1}}{z^{-\alpha}-\lambda}\, {}_a\nabla^{\alpha-1} y(a) + \frac{1}{z^{-\alpha}-\lambda}\, {}_a\mathbb{S}_\nabla \{f\}(z) \\ &\quad = \left(\frac{z^{\alpha-1}}{1-\lambda z^\alpha}\right) y_0 + z\left(\frac{z^{\alpha-1}}{1-\lambda z^\alpha}\right) {}_a\mathbb{S}_\nabla \{f\}(z) \\ &\quad = y_0 \cdot {}_a\mathbb{S}_\nabla \left\{E^{a,\lambda}_{\alpha,\alpha-1}(t)\right\}(z) \\ &\qquad + z\left\{{}_a\mathbb{S}_\nabla \left\{E^{a,\lambda}_{\alpha,\alpha-1}(t)\right\}(z) \cdot {}_a\mathbb{S}_\nabla \{f\}(z)\right\} \\ &\quad = y_0 \cdot {}_a\mathbb{S}_\nabla \left\{E^{a,\lambda}_{\alpha,\alpha-1}(t)\right\}(z) \\ &\qquad + {}_a\mathbb{S}_\nabla \left\{\left\{E^{a,\lambda}_{\alpha,\alpha-1}(t)\right\}(z) * \{f\}(z)\right\}. \end{aligned} \quad (92)$$

Thus, we have

$$ y(t) = y_0 E^{a,\lambda}_{\alpha,\alpha-1}(t) + E^{a,\lambda}_{\alpha,\alpha-1}(t) * f(t). \quad (93)$$

Example 37. Consider the following Caputo type initial value problem:

$$\left({}_a^C\nabla^\alpha y\right)(t) - \lambda y(t) = f(t), \quad (t \in \mathbb{T}, n-1 < \alpha < n, n \in \mathbb{N}, \lambda \in \mathbb{R}), \tag{94}$$

$$y^{\nabla^k}(a) = b_k, \quad k = 0, 1, 2, \ldots, n-1. \tag{95}$$

By taking the ∇-Sumudu transform of both sides of (94) and using Theorem 35(iii), we get

$$\begin{aligned}
&\frac{1}{z^\alpha}\,{}_a\mathbb{S}_\nabla\{y\}(z) - \sum_{k=0}^{n-1}\frac{1}{z^{\alpha-k}}y^{\nabla^k}(a) - \lambda\mathbb{S}_\nabla\{y\}(z) \\
&\quad = {}_a\mathbb{S}_\nabla\{f\}(z), \\
&{}_a\mathbb{S}_\nabla\{y\}(z) \\
&\quad = \frac{z^\alpha}{1-\lambda z^\alpha}\sum_{k=0}^{n-1}\frac{b_k}{z^{\alpha-k}} + \frac{z^\alpha}{1-\lambda z^\alpha}\,{}_a\mathbb{S}_\nabla\{f\}(z) \\
&\quad = \sum_{k=0}^{n-1} b_k\frac{z^k}{1-\lambda z^\alpha} + \frac{z^\alpha}{1-\lambda z^\alpha}\,{}_a\mathbb{S}_\nabla\{f\}(z) \\
&\quad = \sum_{k=0}^{n-1} b_k\,{}_a\mathbb{S}_\nabla\left\{E_{\alpha,k+1}^{a,\lambda}(t)\right\}(z) \\
&\qquad + z\left({}_a\mathbb{S}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t)\right\}(z)\cdot{}_a\mathbb{S}_\nabla\{f\}(z)\right) \\
&\quad = \sum_{k=0}^{n-1} b_k\,{}_a\mathbb{S}_\nabla\left\{E_{\alpha,k+1}^{a,\lambda}(t)\right\}(z) + {}_a\mathbb{S}_\nabla\left\{E_{\alpha,\alpha}^{a,\lambda}(t) * f(t)\right\}.
\end{aligned} \tag{96}$$

Thus, we have

$$y(t) = \sum_{k=0}^{n-1} b_k E_{\alpha,k+1}^{a,\lambda}(t) + E_{\alpha,\alpha}^{a,\lambda}(t) * f(t). \tag{97}$$

In particular, when $0 < \alpha < 1$, the initial value problem

$$\begin{aligned}
&{}_a^C\nabla^\alpha y(t) - \lambda y(t) = f(t), \quad t \in \mathbb{T}, \\
&y(a) = b_0, \quad y_0 \in \mathbb{R}
\end{aligned} \tag{98}$$

has a solution of the following form:

$$y(t) = b_0 E_{\alpha,1}^{a,\lambda}(t) + E_{\alpha,\alpha}^{a,\lambda}(t) * f(t). \tag{99}$$

Example 38. Consider the following Caputo type initial value problem:

$$\left({}_a^C\nabla^\alpha y\right)(t) = f(t), \quad t \in \mathbb{T},\ 0 < \alpha \le 1, \tag{100}$$

$$y(a) = y_0, \quad y_0 \in \mathbb{R}. \tag{101}$$

By taking the ∇-Sumudu transform of both sides of (100) and using Theorem 35(iii) for $0 < \alpha \le 1$, we get

$$z^{-\alpha}\,{}_a\mathbb{S}_\nabla\{y\}(z) - z^{-\alpha}y(a) = {}_a\mathbb{S}_\nabla\{f\}(z). \tag{102}$$

Hence,

$$\begin{aligned}
{}_a\mathbb{S}_\nabla\{y\}(z) &= y_0 + z^\alpha\,{}_a\mathbb{S}_\nabla\{f\}(z) \\
&= y_0\,{}_a\mathbb{S}_\nabla\left\{\widehat{h}_0(t,a)\right\}(z) \\
&\quad + z\,{}_a\mathbb{S}_\nabla\left\{\widehat{h}_{\alpha-1}(t,a)\right\}(z)\,{}_a\mathbb{S}_\nabla\{f\}(z).
\end{aligned} \tag{103}$$

Thus, we have

$$\begin{aligned}
y(t) &= y_0 + \widehat{h}_{\alpha-1}(t,a) * f(t) \\
&= y_0 + {}_a\nabla^{-\alpha}f(t).
\end{aligned} \tag{104}$$

Remark 39. Following Theorem 29 and the examples on solving fractional dynamic equations, one can conclude that

(a) if the solution of fractional dynamic equation exists by ∇-Sumudud transform, then the solution exists by ∇-Laplace transform, and vise versa;

(b) if the solution of fractional dynamic equation exists by ∇-Sumudud transform, then the solution exists by Sumudu and Laplace transform (here $\mathbb{T} = \mathbb{R}$).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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