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Research Article

Value Distribution of Certain Type of Difference Polynomials

Nan Li and Lianzhong Yang

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Correspondence should be addressed to Lianzhong Yang; lzyang@sdu.edu.cn

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We investigate the value distribution of difference product $f(z)^n \sum_{i=1}^{k} a_i f(z + c_i)$, for $n \geq 2$ and $n = 1$, respectively, where $f(z)$ is a transcendental entire function of finite order and a_i, c_i are constants satisfying $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$.

1. Introduction

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see [1–3]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite linear measures. In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Hayman proved the following theorem in [4].

Theorem 1. *Let $f(z)$ be a transcendental integral function and let $n \geq 2$ be an integer; then $f^n f'(z)$ assumes all values except possibly zero infinitely often.*

Clunie proved that if $n = 1$, then Theorem 1 remains valid.

Recently, many papers (see [5–17]) focus on complex difference. They obtain many new results on difference using the value distribution theory of meromorphic functions.

In [12], Laine and Yang found a difference analogue of Hayman's result as follows.

Theorem 2. *Let $f(z)$ be a transcendental entire function of finite order and c a nonzero complex constant. Then for $n \geq 2$, $f(z)^n f(z + c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Liu and Yang [14] proved the following theorem.

Theorem 3. *Let $f(z)$ be a transcendental entire function of finite order and let c be a nonzero complex constant, $\Delta f(z) = f(z + c) - f(z) \not\equiv 0$. Then for $n \geq 2$, $f(z)^n \Delta f(z) - p(z)$ has infinitely many zeros, where $p(z) \not\equiv 0$ is a polynomial in z.*

Chen [6] proved the following theorem.

Theorem 4. *Let $f(z)$ be a transcendental entire function of finite order and let $c \in \mathbb{C} \backslash \{0\}$ be a constant satisfying $f(z+c) \not\equiv f(z)$. Set $H_n(z) = f(z)^n \Delta f(z)$ where $\Delta f(z) = f(z+c) - f(z)$, and $n \geq 2$ is an integer. Then the following statements hold.*

(i) *If $f(z)$ satisfies $\sigma(f) \neq 1$ or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H_n(z)$ has only finitely many zeros.*

It is natural to ask what condition will guarantee that

$$f(z)^n L(f) \tag{1}$$

assumes every nonzero and zero value infinitely often, where $L(f)$ is a linear kth order difference operator with varying shifts, operating on a transcendental entire function of finite order.

In this paper, we consider the above question for $n \geq 2$ and $n = 1$, respectively, and obtain the following results.

Theorem 5. *Let f be a transcendental entire function of finite order and let a_i, c_i $(i = 1, \ldots, k)$ be constant satisfying $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$ and $c_i \neq c_j$ when $i \neq j$. Set $H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f(z + c_i)$, where $n, k \geq 2$ are integers. Then the following statements hold.*

(i) *If $f(z)$ satisfies $\sigma(f)\neq 1$ or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f)=1$, then $H_n(z)$ has only finitely many zeros.*

(iii) *$H_n(z)-\alpha(z)$ has infinitely many zeros, and $\lambda(H_n(z)-\alpha(z))=\sigma(f)$, where $\alpha(z)\not\equiv 0$ is a small function of f.*

Remark 6. The result of Theorem 5 may be false if $k=1$. For example, if $f(z)=e^{z^2}$, we have that $f(z)^2f(z+c)=e^{3z^2+2cz+c^2}$ (where $c\in\mathbf{C}\setminus\{0\}$ is a constant satisfying $f(z+c)\not\equiv f(z)$) has no zero, but $f(z)^2(f(z+c)-f(z))=e^{3z^2}(e^{2cz+c^2}-1)$ has infinitely many zeros. This also shows that the restriction $\sigma(f)=1$ in Theorem 5(ii) is sharp. The following example shows that the assumption $\sigma(f)\neq 1$ in Theorem 5(i) cannot be deleted. In fact, let $f(z)=e^z$; we have $H_2=f^2(f(z+c)-f(z))=e^{2z}(e^{z+1}-e^z)=e^{3z}(e-1)\neq 0$.

By (i) and (iii) of Theorem 5, we can easily obtain the following corollary.

Corollary 7. *Let f be a transcendental entire function of finite order and let a_i, c_i $(i=1,\dots,k)$ be constants satisfying $\sum_{i=1}^k a_if(z+c_i)\not\equiv 0$ and $c_i\neq c_j$ when $i\neq j$. Set $H_n(z)=f(z)^n\sum_{i=1}^k a_if(z+c_i)$, where $n,k\geq 2$ are integers. If $\sigma(f)\neq 1$ or has infinitely many zeros, then $H_n(z)$ takes every value $a\in\mathbf{C}$ infinitely often.*

Theorem 8. *Let f be a finite-order transcendental entire function with a finite Borel exceptional value d, and let a_i, c_i be constants satisfying $\sum_{i=1}^k a_if(z+c_i)\not\equiv 0$ where $\sum_{i=1}^k a_i=0$. Set $H(z)=f(z)\sum_{i=1}^k a_if(z+c_i)$. Then the following statements hold.*

(i) *$H(z)$ takes every nonzero value $a\in\mathbb{C}$ infinitely often and satisfies $\lambda(H-a)=\sigma(f)$.*

(ii) *If $d\neq 0$, then $H(z)$ has no finite Borel exceptional value.*

(iii) *If $d=0$, then 0 is also the Borel exceptional value of $H(z)$. So that $H(z)$ has no nonzero finite Borel exceptional value.*

Theorem 9. *Let f be a transcendental entire function of finite order and let a_i, c_i be constants satisfying $\sum_{i=1}^k a_if(z+c_i)\not\equiv 0$. Set $H(z)=f(z)\sum_{i=1}^k a_if(z+c_i)$.*

If there exists an infinite sequence $\{z_n\}$ satisfying $f(z_n)=\sum_{i=1}^k a_if(z_n+c_i)=0$, then $H(z)$ takes every value $a\in\mathbb{C}$ (including $a=0$) infinitely often.

Theorem 10. *Let f be a transcendental entire function of finite order and let c_i be distinct constants satisfying $\sum_{i=1}^k a_if(z+c_i)\not\equiv 0$. Set $H(z)=f(z)\sum_{i=1}^k a_if(z+c_i)$, where $k\geq 2$ is an integer.*

(i) *If $f(z)$ has only finitely many zeros and $\sigma(f)\neq 1$ or has infinitely many zeros, then $H(z)$ has infinitely many zeros.*

(ii) *If $f(z)$ has only finitely many zeros and $\sigma(f)=1$, then $H(z)$ has only finitely many zeros.*

Example 11. An entire function $f(z)=e^{z^2}$ satisfies Theorem 8 (iii), it has Borel exceptional value 0, and let $a_1=a_2=1$, $a_3=-2$, $a_4=\cdots=a_k=0$, $c_1=1$, $c_2=-1$, and $c_3=0$. Then

$$\begin{aligned}H(z)&=f(z)\left(f(z+1)+f(z-1)-2f(z)\right)\\&=e^{2z^2}\left(\left(e+\frac{1}{e}\right)e^{2z}-2\right)\end{aligned}\tag{2}$$

has also the Borel exceptional value 0 since $\lambda(H)=1<\sigma(H)=2$.

Simultaneously, $f(z)=e^{z^2}$ also satisfies Theorem 10(i), although $f(z)$ has no zero, we can also get $H(z)$ has infinitely many zeros since $\sigma(f)\neq 1$.

Example 12. An entire function $f(z)=e^z+1$ satisfies Theorem 8(ii), it has Borel exceptional value 1, and let $a_1=a_2=1$, $a_3=-2$, $a_4=\cdots=a_k=0$, $c_1=1$, $c_2=-1$, and $c_3=0$. Then

$$\begin{aligned}H(z)&=f(z)\left(f(z+1)+f(z-1)-2f(z)\right)\\&=e^z\left(e^z+1\right)\left(e+\frac{1}{e}-2\right)\end{aligned}\tag{3}$$

has no finite Borel exceptional value.

2. Some Lemmas

Lemma 13 (see [9]). *Let $f(z)$ be a meromorphic function of finite order, $c\in\mathbb{C}\setminus\{0\}$, $\delta<1$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=o\left(\frac{T(r+|c|,f)}{r^\delta}\right)=S(r,f),\tag{4}$$

for all r outside an exceptional set of finite logarithmic measures.

Lemma 14 (see [7]). *Let $f(z)$ be a nonconstant, finite-order meromorphic solution of*

$$f^nP_1(z,f)=Q_1(z,f),\tag{5}$$

where $P_1(z,f)$, $Q_1(z,f)$ are difference polynomials in $f(z)$ with meromorphic coefficients $a_j(z)$ $(j=1,\dots,s)$, and let $\delta<1$. If the degree of $Q_1(r,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$\begin{aligned}m\left(r,P_1(z,f)\right)&=o\left(\frac{T(r+|c|,f)}{r^\delta}\right)+o\left(T(r,f)\right)\\&\quad+O\left(\sum_{j=1}^s m\left(r,a_j\right)\right)\\&=S(r,f)+O\left(\sum_{j=1}^s m\left(r,a_j\right)\right),\end{aligned}\tag{6}$$

for all r outside an exceptional set of finite logarithmic measures.

Lemma 15 (see [3]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, and let $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions that satisfy the following:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

(ii) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not a constant;*

(iii) *when* $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \longrightarrow \infty,\ r \notin E), \tag{7}$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Lemma 16. *Let f be a transcendental entire function of finite order and let a_i, c_i be constants satisfying $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$. Then $H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f(z + c_i)$ $(n \geq 1)$ is transcendental.*

Proof. If $H_n(z) \equiv 0$, then $\sum_{i=1}^{k} a_i f(z + c_i) \equiv 0$ which contradicts our condition $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$. Now we suppose that

$$H_n(z) = f(z)^n \sum_{i=1}^{k} a_i f(z + c_i) = P(z), \tag{8}$$

where $P(z) \not\equiv 0$ is a polynomial. Applying Lemma 14 to (8), we obtain that

$$T\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) = m\left(r, \sum_{i=1}^{k} a_i f(z + c_i)\right) = S(r, f). \tag{9}$$

Thus by (8), (9), and the first fundamental theorem of Nevanlinna theory, we obtain that

$$T(r, f(z)^n) = T\left(r, \frac{P(z)}{\sum_{i=1}^{k} a_i f(z + c_i)}\right) = S(r, f). \tag{10}$$

Since $n \geq 1$, this is a contradiction. Hence $H_n(z)$ is a transcendental entire function. □

Lemma 17 (see [17]). *Let $f(z)$ be a nonconstant finite-order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f). \tag{11}$$

3. Proof of Theorems 5 and 10

Proof of Theorem 5. (i) If $f(z)$ has infinitely many zeros, then $H_n(z)$ has infinitely many zeros since $\sum_{i=1}^{k} a_i f(z + c_i)$ is an entire function and $\sum_{i=1}^{k} a_i f(z + c_i) \not\equiv 0$.

Now we suppose that $f(z)$ has only finitely many zeros and $\sigma(f) \neq 1$. Thus since f is transcendental, $f(z)$ can be written as follows:

$$f(z) = g(z) e^{h(z)}, \tag{12}$$

where $g(z)(\not\equiv 0)$, $h(z)$ are polynomials, $\deg h(z) \geq 2$. Thus

$$f(z + c_i) = g(z + c_i) e^{h(z + c_i)}. \tag{13}$$

Now we suppose that $H_n(z)$ has only finitely many zeros. By Lemma 16, we see that $H_n(z)$ is transcendental. So $H_n(z)$ can be written as

$$\begin{aligned} H_n(z) &= g(z)^n \sum_{i=1}^{k} a_i g(z + c_i) e^{nh(z) + h(z + c_i)} \\ &= g_1(z) e^{h_1(z)}, \end{aligned} \tag{14}$$

where $g_1(z)(\not\equiv 0)$, $h_1(z)$ are polynomials, $\deg h_1(z) \geq 1$. Set

$$h(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0, \quad b_m \neq 0, \tag{15}$$

where $b_m, \ldots, b_0$ are constants and $m \geq 2$. Thus

$$\begin{aligned} h(z + c_i) &= b_m z^m + (b_m m c_i + b_{m-1}) z^{m-1} \\ &\quad + b'_{m-2} z^{m-2} + \cdots + b'_0, \end{aligned} \tag{16}$$

where $b'_{m-2}, \ldots, b'_0$ are constants. Since $m \geq 2$ and

$$h(z + c_i) - h(z + c_j) = b_m m (c_i - c_j) z^{m-1} + \cdots \quad (i \neq j), \tag{17}$$

we see that $nh(z) + h(z + c_i) - (nh(z) + h(z + c_j))$ $(i \neq j)$ are not constants.

Case 1. If for any i, $nh(z) + h(z + c_i) - h_1(z)$ are not constants, then by Lemma 15 and (14), we see that

$$a_i g(z)^n g(z + c_i) \equiv 0, \qquad g_1(z) \equiv 0, \tag{18}$$

which is a contradiction.

Case 2. If there exists a j satisfying $nh(z) + h(z + c_j) - h_1(z) = \delta$ where δ is a constant, then by (14), we have

$$\begin{aligned} &\left(g(z)^n a_j g(z + c_j) - e^{-\delta} g_1(z)\right) e^{nh(z) + h(z + c_j)} \\ &\quad + g(z)^n \sum_{i \neq j} a_i g(z + c_i) e^{nh(z) + h(z + c_i)} = 0. \end{aligned} \tag{19}$$

By (19), Lemma 15, and $k \geq 2$, we obtain that

$$\begin{aligned} &a_i g(z)^n g(z + c_i) \equiv 0 \quad (i \neq j), \\ &g(z)^n a_j g(z + c_j) - e^{-\delta} g_1(z) \equiv 0, \end{aligned} \tag{20}$$

which is also a contradiction. Hence, $H_n(z)$ has infinitely many zeros.

(ii) Suppose that $f(z)$ has only finitely many zeros and $\sigma(f) = 1$. Then $f(z)$ can be written as

$$f(z) = g_2(z) e^{bz + d}, \tag{21}$$

where $g_2(z)(\not\equiv 0)$ is a polynomial and $b(\neq 0), d$ are constants. Thus

$$f(z+c_i) = g_2(z+c_i)e^{bc_i}e^{bz+d},$$
$$H_n(z) = \sum_{i=1}^{k} a_i g_2(z)^n g_2(z+c_i)e^{bc_i}e^{(n+1)(bz+d)}. \tag{22}$$

By the condition $\sum_{i=1}^{k} a_i f(z+c_i) \not\equiv 0$, we see that $\sum_{i=1}^{k} a_i g_2(z+c_i)e^{bc_i} \not\equiv 0$.

Hence $H_n(z)$ has only finitely many zeros.

(iii) Case 1. $\sigma(f) = 0$. From $0 \leq \lambda(H_n(z)-\alpha(z)) \leq \sigma(H_n(z)-\alpha(z)) \leq \sigma(f) = 0$, we get $\lambda(H_n(z)-\alpha(z)) = \sigma(H_n(z)-\alpha(z)) = \sigma(f) = 0$. If $H_n(z)-\alpha(z)$ has only finitely zeros, then $H_n(z)-\alpha(z)$ can be written as

$$H_n(z)-\alpha(z) = p(z), \quad \text{i.e.,} \quad H_n(z) = p(z)+\alpha(z), \tag{23}$$

where $p(z)$ is a polynomial. By using a similar method as in the proof of Lemma 16, we get a contradiction. Thus $H_n(z)-\alpha(z)$ has infinitely many zeros.

Case 2. $\sigma(f) > 0$. Suppose on contrary to the assertion that $\lambda(H_n(z)-\alpha(z)) < \sigma(f)$. If $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)-\alpha(z) \equiv 0$, that is, $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) \equiv \alpha(z)$. By using a similar method as in the proof of Lemma 16, we get a contradiction. So we have $f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)-\alpha(z) \not\equiv 0$. Thus, by Hadamard's theorem, $H_n(z)-\alpha(z)$ can be written as

$$\begin{aligned} H_n(z)-\alpha(z) &= f(z)^n \sum_{i=1}^{k} a_i f(z+c_i)-\alpha(z) \\ &= \frac{P(z)}{Q(z)}e^{h(z)}, \end{aligned} \tag{24}$$

where $h(z)$ is a polynomial and $P(z)(\not\equiv 0), Q(z)(\not\equiv 0)$ are the canonical products formed by zeros and poles of $H_n(z)-\alpha(z)$, respectively, such that

$$\lambda(P(z)) = \sigma(P(z)) = \lambda(H_n(z)-\alpha(z)) < \sigma(f) = \sigma. \tag{25}$$

Since $T(r,\alpha(z)) = S(r,f)$, we get that

$$\lambda(Q(z)) = \sigma(Q(z)) = \lambda\left(\frac{1}{\alpha(z)}\right) < \sigma(f) = \sigma. \tag{26}$$

We set $g(z) = P(z)/Q(z)$; then from (25) and (26), we get

$$\sigma(g) = \max\{\sigma(P(z)), \sigma(Q(z))\} < \sigma(f) = \sigma. \tag{27}$$

Differentiating (24) and eliminating $e^{h(z)}$, we get

$$f(z)^{n-1}F(z,f) = \alpha'(z)g(z)-\alpha(z)\left(g(z)h'(z)+g'(z)\right), \tag{28}$$

where

$$\begin{aligned} F(z,f) &= nf'(z)g(z)\sum_{i=1}^{k} a_i f(z+c_i) \\ &\quad + f(z)g(z)\sum_{i=1}^{k} a_i f'(z+c_i) \\ &\quad - \left(g(z)h'(z)+g'(z)\right)f(z)\sum_{i=1}^{k} a_i f(z+c_i). \end{aligned} \tag{29}$$

Case 2.1. $F(z,f) \equiv 0$. Then from (28), we have

$$\alpha'(z)g(z)-\alpha(z)\left(g(z)h'(z)+g'(z)\right) \equiv 0. \tag{30}$$

By integrating, we have

$$\alpha(z) = cg(z)e^{h(z)}, \tag{31}$$

where c is a nonzero constant. From (24) and (31), we have

$$f(z)^n \sum_{i=1}^{k} a_i f(z+c_i) = \left(1+\frac{1}{c}\right)\alpha(z). \tag{32}$$

By using a similar method as in the proof of Lemma 16, we get a contradiction.

Case 2.2. $F(z,f) \not\equiv 0$. Let

$$\begin{aligned} F^*(z,f) = \frac{F(z)}{f(z)^2} &= n\frac{f'(z)}{f(z)}g(z)\sum_{i=1}^{k} a_i \frac{f(z+c_i)}{f(z)} \\ &\quad + g(z)\sum_{i=1}^{k} a_i \frac{f'(z+c_i)}{f(z+c_i)} \cdot \frac{f(z+c_i)}{f(z)} \\ &\quad - \left(g(z)h'(z)+g'(z)\right)\sum_{i=1}^{k} a_i \frac{f(z+c_i)}{f(z)}. \end{aligned} \tag{33}$$

Then from (28), we have

$$f(z)^{n+1}F^*(z,f) = \alpha'(z)g(z)-\alpha(z)\left(g(z)h'(z)+g'(z)\right). \tag{34}$$

From Lemma 13 and Lemma 14, we have

$$\begin{aligned} m\left(r, f(z)^k F^*(z,f)\right) &\leq S(r,f) + O(m(r,g)) \\ &\quad + O\left(\sum_{i=1}^{k} m\left(r, \frac{f'(z+c_i)}{f(z+c_i)}\right)\right), \quad k = 1,2. \end{aligned} \tag{35}$$

Now for any given ε $(0 < \varepsilon < 1)$, we obtain from Lemma 17 and (27) that

$$\begin{aligned} m\left(r, \frac{f'(z+c_i)}{f(z+c_i)}\right) &= S(r, f(z+c_i)) \\ &= S(r, f(z)), T(r,g) = O(r^{\sigma-\varepsilon}). \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$m\left(r, f(z) F^{*}(z, f)\right)=O\left(r^{\sigma-\varepsilon}\right)+S(r, f), \quad (37)$$

$$m\left(r, f(z)^{2} F^{*}(z, f)\right)=O\left(r^{\sigma-\varepsilon}\right)+S(r, f). \quad (38)$$

We obtain from the definition of $F(z, f)$ that

$$N(r, F(z, f))=O(N(r, g(z)))=O\left(r^{\sigma-\varepsilon}\right). \quad (39)$$

Thus from (38) and (39), we have

$$\begin{aligned} T\left(r, f(z)^{2} F^{*}(z, f)\right) &= T(r, F(z, f)) \\ &= O\left(r^{\sigma-\varepsilon}\right)+S(r, f). \end{aligned} \quad (40)$$

Note that a zero of $f(z)$ which is not a pole of $g(z)$ is a pole of $f(z) F^{*}(z, f)$ with the multiplicity at most 1, so from (34) and (27) we get that, for ε (>0) sufficiently small,

$$\begin{aligned} &(n-1) N\left(r, \frac{1}{f(z)}\right) \\ &\leq N\left(r, \frac{1}{\alpha'(z) g(z)-\alpha(z)\left(g(z) h'(z)+g'(z)\right)}\right) \\ &\quad+O(N(r, g(z)))=O\left(r^{\sigma-\varepsilon}\right)+S(r, f). \end{aligned} \quad (41)$$

Hence from (33) and the above formula, we have

$$\begin{aligned} N\left(r, f(z) F^{*}(z, f)\right) &= O\left(N\left(r, \frac{1}{f(z)}\right)+N(r, g(z))\right) \\ &= O\left(r^{\sigma-\varepsilon}\right)+S(r, f). \end{aligned} \quad (42)$$

It follows from (37) and (42) that

$$T\left(r, f(z) F^{*}(z, f)\right)=O\left(r^{\sigma-\varepsilon}\right)+S(r, f). \quad (43)$$

Therefore, from (40) and (43), we have

$$T(r, f(z))=O\left(r^{\sigma-\varepsilon}\right)+S(r, f), \quad (44)$$

which contradicts the assumption that $f(z)$ is a transcendental entire function of finite order σ. This completes the proof of Theorem 5.

By using the same methods as in the proof of Theorem 5 (i) and (ii), we complete the proof of Theorem 10. □

4. Proof of Theorem 8

Proof. Firstly, we prove (ii) and (iii). (ii) Suppose that $d(\neq 0)$ is the Borel exceptional value of $f(z)$. Then $f(z)$ can be written as follows:

$$f(z)=d+p(z) e^{\alpha z^{k}}, \quad (45)$$

where k is a positive integer, α $(\neq 0)$ is a constant, and $p(z)(\not\equiv 0)$ is an entire function satisfying

$$\sigma(p)<\sigma(f)=k. \quad (46)$$

Thus

$$f\left(z+c_{i}\right)=d+p\left(z+c_{i}\right) p_{i}(z) e^{\alpha z^{k}}, \quad (47)$$

where $p_{i}(\not\equiv 0)$ is an entire function satisfying $\sigma(p_{i})=k-1$. So by using $\sum_{i=1}^{k} a_{i}=0$, we have

$$\begin{aligned} H(z) &= \sum_{i=1}^{k} a_{i}\left(d+p(z) e^{\alpha z^{k}}\right)\left(d+p\left(z+c_{i}\right) p_{i}(z) e^{\alpha z^{k}}\right) \\ &= \sum_{i=1}^{k} d a_{i} p\left(z+c_{i}\right) p_{i}(z) e^{\alpha z^{k}} \\ &\quad+\sum_{i=1}^{k} a_{i} p(z) p\left(z+c_{i}\right) p_{i}(z) e^{2 \alpha z^{k}}. \end{aligned} \quad (48)$$

Since $\sum_{i=1}^{k} a_{i} f\left(z+c_{i}\right) \not\equiv 0$, we see that

$$\sum_{i=1}^{k} a_{i} p\left(z+c_{i}\right) p_{i}(z) \not\equiv 0. \quad (49)$$

By (48) and (49), we see that

$$\sigma(H)=\sigma(f)=k. \quad (50)$$

If $H(z)$ has the Borel exceptional value d^{*}, then

$$H(z)=d^{*}+p^{*}(z) e^{\beta z^{k}}, \quad (51)$$

where $\beta(\neq 0)$ is a constant and $p^{*}(z)(\not\equiv 0)$ is an entire function satisfying

$$\sigma\left(p^{*}(z)\right)<\sigma(H)=k. \quad (52)$$

By (48) and (51), we have

$$\begin{aligned} &\sum_{i=1}^{k} d a_{i} p\left(z+c_{i}\right) p_{i}(z) e^{\alpha z^{k}}+\sum_{i=1}^{k} a_{i} p(z) p\left(z+c_{i}\right) p_{i}(z) e^{2 \alpha z^{k}} \\ &\quad-p^{*}(z) e^{\beta z^{k}}-d^{*}=0. \end{aligned} \quad (53)$$

Case 1. If $\beta \neq 2\alpha$ and $\beta \neq \alpha$, then by Lemma 15 and (53), we can obtain that

$$\sum_{i=1}^{k} d a_{i} p\left(z+c_{i}\right) p_{i}(z) \equiv 0. \quad (54)$$

This contradicts with (49).

Case 2. If $\beta=2\alpha$ or $\beta=\alpha$, then using the same method as above, we can also obtain a contradiction. Hence $H(z)$ has no Borel exceptional value.

(iii) Suppose that $d=0$ is the Borel exceptional value of $f(z)$. Using the same method as above, we obtain

$$H(z)=\sum_{i=1}^{k} a_{i} p(z) p\left(z+c_{i}\right) p_{i}(z) e^{2 \alpha z^{k}}. \quad (55)$$

From (49) and

$$\sigma\left(\sum_{i=1}^{k} a_i p(z)\, p(z+c_i)\, p_i(z)\right) < k, \tag{56}$$

we see that 0 is the finite Borel exceptional value of $H(z)$. Thus, $H(z)$ has no nonzero finite Borel exceptional value.

Finally, we prove (i). By the assertion of (ii) and (iii), we see that if $f(z)$ has the finite Borel exceptional value, then any nonzero finite value a must not be the Borel exceptional value of $H(z)$. Hence $H(z)$ takes the value a infinitely often. By (50), we obtain $\lambda(H-a) = \sigma(H) = \sigma(f)$. □

5. Proof of Theorem 9

Proof. Clearly, if $a = 0$, then $H(z)$ has infinitely many zeros since $\sum_{i=1}^{k} a_i f(z+c_i)$ ($\not\equiv 0$) is an entire function and $f(z)$ has infinitely many zeros.

Now we suppose that $a \neq 0$. Suppose that $H(z)-a$ has only finitely many zeros. Then $H(z)-a$ can be written as follows:

$$H(z) - a = \sum_{i=1}^{k} a_i f(z)\, f(z+c_i) - a = p(z)\, e^{q(z)}, \tag{57}$$

where $p(z)$, $q(z)$ are polynomials. By Lemma 16, we see that $p(z) \not\equiv 0$, $\deg q(z) \geq 1$. Differentiating (57) and eliminating $e^{q(z)}$, we obtain

$$\frac{\left(f(z)\sum_{i=1}^{k} a_i f(z+c_i)\right)'}{f(z)\sum_{i=1}^{k} a_i f(z+c_i)} = \frac{p'(z) + p(z)\, q'(z)}{p(z)} \times \left(1 - \frac{a}{f(z)\sum_{i=1}^{k} a_i f(z+c_i)}\right). \tag{58}$$

Since there exists an infinite sequence $\{z_n\}$ satisfying $f(z_n) = \sum_{i=1}^{k} a_i f(z_n + c_i) = 0$, we see that there is a sufficiently large point z_0 such that $f(z_0) = \sum_{i=1}^{k} a_i f(z_0 + c_i) = 0$ and $p'(z_0) + p(z_0) q'(z_0) \neq 0$, $p(z_0) \neq 0$ at the same time.

From observation, we have the following: $(f(z)\sum_{i=1}^{k} a_i f(z+c_i))' / f(z)\sum_{i=1}^{k} a_i f(z+c_i)$ has a simple pole at z_0 and $a/f(z)\sum_{i=1}^{k} a_i f(z+c_i)$ has pole at z_0 of multiplicity at least 2. This shows that (58) is a contradiction. Hence $H(z)$ takes every value a infinitely often. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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