Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 283158, 12 pages
http://dx.doi.org/10.1155/2014/283158

Research Article

$\mathcal{JH}$-Operator Pairs with Application to Functional Equations Arising in Dynamic Programming

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Received 16 July 2013; Revised 21 December 2013; Accepted 21 December 2013; Published 13 February 2014

Academic Editor: Hichem Ben-El-Mechaiekh

Some common fixed point theorems for $\mathcal{JH}$-operator pairs are proved. As an application, the existence and uniqueness of the common solution for systems of functional equations arising in dynamic programming are discussed. Also, an example to validate all the conditions of the main result is presented.

1. Introduction and Preliminaries

Jungck [1] introduced compatible mappings as a generalization of weakly commuting mappings. Jungck and Pathak [2] defined the concept of the biased mappings in order to generalize the concept of compatible mappings. Also, several authors [3–6] studied various classes of compatible mappings and proved common fixed point theorems for these classes. Recently, Hussain et al. [7] introduced $\mathcal{JH}$-operator pairs as a new class of noncommuting self-mappings that contains the occasionally weakly compatible, and Sintunavarat and Kumam [8] introduced generalized $\mathcal{JH}$-operator pairs that contain $\mathcal{JH}$-operator pairs. On the other hand, fixed point theory has various applications in other fields, for instance, obtaining a solution of several classes of functional equations (or a system of functional equations) arising in dynamic programming (see [9–12]). Bellman and Lee [13], Zhang [14], and Chang and Ma [15] point out that the basic form of the functional equations of dynamic programming and the system of functional equations of dynamic programming are as follows:

$$f(x) = \sup_{y\in D} H(x, y, f(T(x, y))), \quad \forall x \in S,$$

$$f(x) = \sup_{y\in D} \{u(x, y) + G(x, y, g(T(x, y)))\}, \quad \forall x \in S,$$

$$g(x) = \sup_{y\in D} \{u(x, y) + F(x, y, f(T(x, y)))\}, \quad \forall x \in S. \tag{1}$$

In this presented work, $\mathcal{JH}$-operator pairs are compared with the various type of compatible mappings and it is shown that the $\mathcal{JH}$-operator pairs reduce to symmetric Banach operator pairs under relaxed conditions. We omit the completeness condition of the space. Then some common fixed point theorems are proved for $\mathcal{JH}$-operator pairs. Eventually, the results are used to show the existence and uniqueness of common solution for systems of functional equations without completeness of the space.

The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of S and T if $Sx = Tx (x = Sx = Tx)$. Let $C(S,T)$, $PC(S,T)$ denote the sets of all coincidence points and points of coincidence, respectively, of the pair (S,T). The pair (S,T) is called a Banach operator pair if the set $F(T)$ is S-invariant, namely, $S(F(T)) \subseteq F(T)$. If (S,T) is a Banach operator pair, then (T,S) need not be a Banach operator pair. Let (X,d) be a metric space and f, S self-mappings on X; the pair (f,S) is called as follows:

(0) symmetric Banach operator if both (f,S) and (S,f) are Banach operator pairs [16];

(1) compatible if $d(fSx_n, Sfx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that fx_n and $Sx_n \to t \in X$ [1];

(2) $\mathscr{P}$-operator pair if $d(x, Sx) \leq \operatorname{diam}(C(f,S))$, for some $x \in C(f,S)$ [17];

(3) $\mathscr{J}\mathscr{H}$-operator pair if there exists a point $w = fx = Sx$ in $PC(f,S)$ such that

$$d(w,x) \leq \operatorname{diam}(PC(f,S)); \tag{2}$$

see [7];

(4) compatible of type (A) if

$$d(fSx_n, SSx_n) \longrightarrow 0, \quad d(Sfx_n, ffx_n) \longrightarrow 0, \tag{3}$$

whenever $\{x_n\}$ is a sequence in X such that fx_n and $Sx_n \to t \in X$ [6];

(5) weakly S-biased of type (A) if $fp = Sp$ implies that

$$d(SSp, fp) \leq d(fSp, Sp); \tag{4}$$

see [18];

(6) compatible of type (B) if

$$\begin{aligned}
&\lim_{n\to\infty} d(Sfx_n, ffx_n) \\
&\quad \leq \frac{1}{2}\left[\lim_{n\to\infty} d(Sfx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right], \\
&\lim_{n\to\infty} d(fSx_n, SSx_n) \\
&\quad \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fSx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\right],
\end{aligned} \tag{5}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [3];

(7) compatible of type (P) if

$$\lim_{n\to\infty} d(ffx_n, SSx_n) = 0, \tag{6}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [4];

(8) compatible of type (C) if

$$\begin{aligned}
&\lim_{n\to\infty} d(Sfx_n, ffx_n) \\
&\quad \leq \frac{1}{3}\left[\lim_{n\to\infty} d(Sfx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\right. \\
&\qquad \left. + \lim_{n\to\infty} d(St, ffx_n)\right], \\
&\lim_{n\to\infty} d(fSx_n, SSx_n) \\
&\quad \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fSx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\right. \\
&\qquad \left. + \lim_{n\to\infty} d(ft, SSx_n)\right],
\end{aligned} \tag{7}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t \in X$ [5].

2. $\mathscr{J}\mathscr{H}$-Operator Pair

Proposition 1. *Let f and S be self-mappings of metric space (X,d), and $C(f,S) \neq \emptyset$. If f and S are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair.*

Proof. If f and S are one of the assumptions listed, then f and S are weakly compatible and, hence, they are occasionally weakly compatible; then (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair. □

Notation 1. The following example shows that the converse of Proposition 1 is not true, in general.

Example 2. Suppose that $(X = [0,1], d)$ is a metric space with $d(x,y) = |x - y|$ and f, S are defined by

$$\begin{aligned}
fx &= \begin{cases} x^2, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases} \\
Sx &= \begin{cases} \dfrac{x}{2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}
\end{aligned} \tag{8}$$

Then, $C(f,S) = \{0, 1/2\}$, $PC(f,S) = \{1, 1/4\}$. On the other hand, for $w = 1/4 \in PC(f,S)$ we have $f(1/2) = S(1/2) = 1/4$ and

$$d\left(\frac{1}{2}, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| \leq \operatorname{diam}(PC(f,S)) = \left|1 - \frac{1}{4}\right|. \tag{9}$$

Thus, (f,S) is a $\mathscr{J}\mathscr{H}$-operator pair.

Now, suppose that $\{x_n\}$ is a sequence in $[0,1]$ defined by $x_n = 1/2$. Then, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = t = 1/4$, $fx_n = Sx_n = 1/4$, and $fSx_n = 1/16$, $Sfx_n = 1/8$. Since

$$\lim_{n\to\infty} |fSx_n - Sfx_n| = \frac{1}{16} \neq 0, \tag{10}$$

so (f,S) is not compatible.

$SSx_n = 1/8$, $ffx_n = 1/16$. Since

$$\lim_{n\to\infty} |fSx_n - SSx_n| = \frac{1}{16} \neq 0, \tag{11}$$

thus (f,S) is not compatible of type (A).

Since

$$\begin{aligned}
&\lim_{n\to\infty} |fSx_n - SSx_n| \\
&\quad = \frac{1}{16} > \frac{1}{2}\left[\lim_{n\to\infty} |fSx_n - ft| + \lim_{n\to\infty} |ft - ffx_n|\right] = 0,
\end{aligned} \tag{12}$$

then (f,S) is not compatible of type (B).

Since

$$\lim_{n\to\infty} |SSx_n - ffx_n| = \frac{1}{16} \neq 0, \tag{13}$$

thus (f,S) is not compatible of type (P).

Since

$$\lim_{n\to\infty} |fSx_n - SSx_n| = \frac{1}{16} > \frac{1}{3}\left[\lim_{n\to\infty}|fSx_n - ft| + \lim_{n\to\infty}|ft - ffx_n| + \lim_{n\to\infty}|ft - SSx_n|\right] = \frac{1}{48}, \quad (14)$$

therefore, (f,S) is not compatible of type (C).

Proposition 3. *Let f and S be self-mappings of metric space (X,d). If (f,S) is a $\mathcal{JH}$-operator pair such that $PC(f,S)$ is singleton, then (f,S) is symmetric Banach operator pair.*

Proof. By hypothesis, there is a point $fx = Sx = w \in PC(f,S)$ such that $d(x,w) \le \mathrm{diam}(PC(f,S)) = 0$. Thus, $x = w = fx = Sx$ and x is a unique point of $C(f,S)$. Also, by Proposition 2.4 [19] (f,S) is weakly compatible and hence, by Lemma 2.1 [19], $w = x$ is a unique common fixed point of f and S. Now, since the sets $PC(f,S)$ and $C(f,S)$ are singleton, then $F(f) = F(S) = \{x\}$, $f(F(S)) \subseteq F(S)$ and $S(F(f)) \subseteq F(f)$; that is, (f,S) is symmetric Banach operator pair. □

Example 4. Suppose that $(X = [0,5], d)$ is a metric space with $d(x,y) = |x-y|$ and f, S are defined by

$$fx = \begin{cases} 0, & \text{if } x = 0, \\ x+4, & \text{if } x \in (0,1], \\ x-1, & \text{if } x \in (1,5], \end{cases} \qquad Sx = \begin{cases} 2, & \text{if } x \in (0,1], \\ 0, & \text{if } x \in \{0\} \cup (1,5]. \end{cases} \quad (15)$$

Then $C(f,S) = PC(f,S) = \{0\}$. Clearly (f,S) is $\mathcal{JH}$-operator pair and symmetric Banach operator pair.

Proposition 5. *Let f and S be self-mappings of metric space (X,d). If (f,S) is a $\mathcal{JH}$-operator pair and for all $x, y \in X$ we have*

$$d(fx,fy) \le \phi\left(\max\left\{d(Sx,Sy), d(Sx,fx), d(fy,Sy), \frac{1}{2}\left(d(Sx,fy) + d(Sy,fx)\right)\right\}\right), \quad (16)$$

where $\phi : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying the condition $\phi(t) < t$ for $t > 0$, then (f,S) is symmetric Banach operator pair.

Proof. Since (f,S) is a $\mathcal{JH}$-operator pair, there is a point $fx = Sx = w$ in $PC(f,S)$ such that $d(x,w) \le \mathrm{diam}(PC(f,S))$. Now, if there is another point $fy = Sy = z$ in $PC(f,S)$ and $z \ne w$, then, by (16),

$$d(w,z) = d(fx,fy) \le \phi\left(\max\left\{d(w,z), 0, 0, \frac{1}{2}\left(d(w,z) + d(w,z)\right)\right\}\right); \quad (17)$$

therefore, $d(w,z) \le \phi(d(w,z)) < d(w,z)$ which is a contradiction. Then $w = z$, that is, $PC(f,S)$ is singleton and, hence, by Proposition 3 (f,S) is symmetric Banach operator pair. □

Proposition 6. *Let f and S be self-mappings of metric space (X,d). If (f,S) is a $\mathcal{P}$-operator pair such that $C(f,S)$ is singleton, then (f,S) is symmetric Banach operator pair.*

Corollary 7. *Let (f,S) be an occasionally weakly compatible pair of self-mappings on X that $C(f,S)$ is singleton; then (f,S) is symmetric Banach operator pair.*

Proof. Clearly, occasionally weakly compatible mappings are $\mathcal{P}$-operators; then by Proposition 6 the result is obtained. □

3. Common Fixed Point

Definition 8 (see [20]). A function $\psi : [0,\infty] \to [0,\infty]$ is called an altering distance function if

(i) ψ is monotone increasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 9. *Suppose that S and T are self-mappings of metric space (X,d). The pair (S,T) is a $\mathcal{JH}$-operator pair and, for all $x, y \in X$,*

$$\psi(d(Sx,Ty)) \le \psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}\left[d(Tx,Ty) + d(Sy,Tx)\right]\right\}\right) - \phi\left(\max\left\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\right\}\right), \quad (18)$$

where ψ is an altering distance function and $\phi : [0,\infty] \to [0,\infty]$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then S and T have a unique common fixed point. Moreover, any fixed point of S is a fixed point of T and conversely.

Proof. By hypothesis, there exists a point $w \in X$ such that $w = Sx = Tx$ and

$$d(w,x) \le \mathrm{diam}(PC(S,T)). \quad (19)$$

Suppose that there exists another point $z \in X$ and $z \ne w$, for which $z = Sy = Ty$. Then, from (18), we get

$$\psi(d(w,z)) \le \psi\left(\max\left\{d(w,z), 0, 0, \frac{1}{2}\left[d(w,z) + d(z,w)\right]\right\}\right) - \phi\left(\max\left\{d(w,z), 0, 0\right\}\right); \quad (20)$$

accordingly, $\psi(d(w,z)) \le \psi(d(w,z)) - \phi(d(w,z))$, which is a contradiction with definition of ϕ. Therefore, $PC(S,T)$ is singleton so $\operatorname{diam}(PC(S,T)) = 0$. By using (19), $d(w,x) \le \operatorname{diam}(PC(S,T)) = 0$; thus, $w = x$; that is, x is a unique common fixed point of S and T.

Now, suppose that u is a fixed point of S but $u \neq Tu$, from (18),

$$\begin{aligned}\psi(d(u,Tu)) &= \psi(d(Su,Tu)) \\ &\le \psi\left(\max\left\{0, d(Tu,u), d(u,Tu), \frac{1}{2}[0 + d(u,Tu)]\right\}\right) \\ &\quad - \phi(\max\{0, d(u,Tu), d(u,Tu)\});\end{aligned} \tag{21}$$

thus, $\psi(d(u,Tu)) \le \psi(d(u,Tu)) - \phi(d(u,Tu))$, which is a contradiction with definition of ϕ. Hence, $u = Tu$. By using a similar argument, the conclusion will be obtained. □

Example 10. Suppose that $X = \{0, 2, 4, 6, \ldots\}$ and $d : X \times X \to \mathbb{R}$ is given by

$$d(x,y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{22}$$

Then (X,d) is a metric space.

Let $\psi : [0,\infty) \to [0,\infty)$ be defined as

$$\psi(t) = 2t^2, \quad \text{for } t \in [0,\infty). \tag{23}$$

Suppose that $\phi : [0,\infty) \to [0,\infty)$ is defined as

$$\phi(s) = \begin{cases} s, & \text{if } s \le 1, \\ 1, & \text{if } s > 1. \end{cases} \tag{24}$$

Then $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Let $S, T : X \to X$ be defined as

$$\begin{aligned} Sx &= \begin{cases} 2x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \\ Tx &= \begin{cases} 2x - 2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{aligned} \tag{25}$$

Now, we have the following cases for $x, y \in X$.

Case 1. $x \neq y$.

(i) If $y \neq 0$ and $x > y$, then

$$\begin{aligned} &\psi(d(Sx,Ty)) \\ &\quad = \psi(d(2x, 2y-2)) = \psi(2x + 2y - 2) \\ &\quad = 8(x+y-1)^2, \\ &\psi\Big(\max\Big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \\ &\qquad \frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\Big\}\Big) \\ &\quad = \psi(\max\{2x+2y, 4x-2, 4y-2, \\ &\qquad 2x+2y-3\}) \\ &\quad = \psi(4x-2) = 2(4x-2)^2, \\ &\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\ &\quad = \phi(\max\{2x+2y, 4x-2, 4y-2\}) \\ &\quad = \phi(4x-2) = 1. \end{aligned} \tag{26}$$

Since, $8(x+y-1)^2 \le 2(4x-2)^2 - 1$, then relation (18) is established.

(ii) If $x \neq 0$ and $y > x$, then

$$\begin{aligned} &\psi(d(Sx,Ty)) \\ &\quad = \psi(d(2x, 2y-2)) = \psi(2x+2y-2) = 8(x+y-1)^2, \\ &\psi\Big(\max\Big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \\ &\qquad \frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\Big\}\Big) \\ &\quad = \psi(\max\{2x+2y, 4x-2, 4y-2, 2x+2y-3\}) \\ &\quad = \psi(4y-2) = 2(4y-2)^2, \\ &\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\ &\quad = \phi(\max\{2x+2y, 4x-2, 4y-2\}) \\ &\quad = \phi(4y-2) = 1. \end{aligned} \tag{27}$$

Since, $8(x+y-1)^2 \le 2(4y-2)^2 - 1$, then relation (18) is established.

(iii) $y = 0$; then

$$\begin{aligned} &\psi(d(Sx,Ty)) = \psi(d(2x,0)) = \psi(2x) = 8x^2, \\ &\psi\Big(\max\Big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \\ &\qquad \frac{1}{2}[d(Tx,Ty) + d(Sy,Tx)]\Big\}\Big) \\ &\quad = \psi(\max\{2x, 4x-2, 0, 2x-2\}) \\ &\quad = \psi(4x-2) = 2(4x-2)^2, \\ &\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}) \\ &\quad = \phi(\max\{2x, 4x-2, 0\}) = \phi(4x-2) = 1. \end{aligned} \tag{28}$$

Since, $8x^2 \le 2(4x-2)^2 - 1$, then relation (18) is established.

(iv) $x = 0$; then

$$\begin{aligned} &\psi(d(Sx,Ty)) \\ &\quad = \psi(d(0, 2y-2)) = \psi(2y-2) = 2(2y-2)^2, \end{aligned}$$

$$\psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Tx,Ty)+d(Sy,Tx)]\right\}\right)$$
$$= \psi(\max\{2y, 0, 4y-2, 2y-1\})$$
$$= \psi(4y-2) = 2(4y-2)^2,$$
$$\phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\})$$
$$= \phi(\max\{2y, 0, 4y-2\}) = \phi(4y-2) = 1. \quad (29)$$

Since, $2(2y-2)^2 \leq 2(4y-2)^2 - 1$, then relation (18) is established.

Case 2. $x = y$.

In this case, it is easy to see that the relation (18) is hold. Therefore, for all $x, y \in X$,

$$\psi(d(Sx,Ty)) \leq \psi\left(\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}[d(Tx,Ty)+d(Sy,Tx)]\right\}\right) - \phi(\max\{d(Sx,Sy), d(Tx,Sx), d(Sy,Ty)\}). \quad (30)$$

Accordingly, the conditions of Theorem 9 are satisfied and 0 is the unique common fixed point of S and T.

Suppose that Φ is the collection of mappings $\phi : [0,\infty) \to [0,\infty)$ which are upper semicontinuous, nondecreasing in each coordinate variable and $\phi(t) < t$ for all $t > 0$ [21].

Lemma 11 (see [21]). *If $\phi_i \in \Phi$ and $i \in I$ where I is a finite index set, then there exists some $\phi \in \Phi$ such that $\max\{\phi_i(t) : i \in I\} \leq \phi(t)$ for all $t > 0$.*

Let f, g, S, and T be self-mappings of a metric space (X, d) such that

$$d(fx,gy) \leq a\phi_0(d(Sx,Ty)) + (1-a) \times \max\left\{\phi_1(d(Sx,Ty)), \phi_2\left(\frac{1}{2}[d(Sx,fx)+d(Ty,gy)]\right), \phi_3\left(\frac{1}{2}[d(Sx,gy)+d(Ty,fx)]\right), \phi_4\left(\frac{1}{2}[d(Sx,fx)+d(Ty,fx)]\right), \phi_5\left(\frac{1}{2}[d(Sx,gy)+d(Ty,gy)]\right)\right\}, \quad (31)$$

for all $x, y \in X$, where $\phi_i \in \Phi$, $i = 0,1,2,3,4,5$, $0 \leq a \leq 1$.

Theorem 12. *Let f, g, S, and T be self-mappings of a metric space (X, d) satisfying (31). If (f, S) and (g, T) are each $\mathcal{JH}$-operator pairs, then f, g, S, and T have a unique common fixed point.*

Proof. By hypothesis there exist points $x, y \in X$ such that $fx = Sx = w$ and $gy = Ty = z$. If $fx \neq gy$, then, from (31), we get

$$d(fx,gy) \leq a\phi_0(d(fx,gy)) + (1-a) \times \max\left\{\phi_1(d(fx,gy)), \phi_2(0), \phi_3\left(\frac{1}{2}[d(fx,gy)+d(gy,fx)]\right), \phi_4\left(\frac{1}{2}[d(gy,fx)]\right), \phi_5\left(\frac{1}{2}[d(fx,gy)]\right)\right\}, \quad (32)$$

which implies that $d(fx,gy) \leq a\phi(d(fx,gy)) + (1-a)\phi(d(fx,gy)) = \phi(d(fx,gy)) < d(fx,gy)$, a contradiction. Thus, $w = fx = gy = z$. Suppose that there exists another point u such that $fu = Su$. Then condition (31) implies that $fu = Su = gy = Ty = fx = Sx$. Hence, $w = fx = fu$. That is, $PC(f,S)$ is singleton. Since $d(x,w) \leq \operatorname{diam}(PC(f,S)) = 0$, so $d(x,w) = 0$ and $x = w$ is a unique common fixed point of f and S. Similarly, $y = z$ is a unique common fixed point of g and T. Therefore, $w = z$ is a unique common fixed point of f, g, S, and T. □

Corollary 13. *Let f, g, S, and T be self-mappings of a metric space (X, d) satisfying $d(fx,gy) \leq kd(Sx,Ty)$, for all $x, y \in X$ where $0 < k < 1$. If (f, S) and (g, T) are each $\mathcal{JH}$-operator pairs, then f, g, S, and T have a unique common fixed point.*

Proof. It is sufficient to set $a = 1$ and take $\phi_0(t) = kt \in \Phi$ in Theorem 12. □

Corollary 14. *Let f, S be self-mappings of a metric space (X, d) satisfying the following condition:*

$$d(fx,fy) \leq a\phi_0(d(Sx,Sy)) + (1-a) \times \max\left\{\phi_1(d(Sx,Sy)), \phi_2\left(\frac{1}{2}[d(Sx,fx)+d(Sy,fy)]\right),\right.$$

$$\phi_3\left(\frac{1}{2}\left[d(Sx,fy)+d(Sy,fx)\right]\right),$$
$$\phi_4\left(\frac{1}{2}\left[d(Sx,fx)+d(Sy,fx)\right]\right),$$
$$\phi_5\left(\frac{1}{2}\left[d(Sx,fy)+d(Sy,fy)\right]\right)\Bigg\}. \quad (33)$$

If (f,S) is $\mathcal{JH}$-operator pair, then f and S have a unique common fixed point.

Proof. Considering that $g := f$ and $T := S$ in Theorem 12, the result is obtained. □

Theorem 15. *Let f,S be self-mappings of a metric space (X,d) satisfying (33). Suppose that (f,S) is nontrivial Banach operator pair on X, then f and S have a unique common fixed point.*

Proof. By hypothesis $F(S)\neq\emptyset$ and $f(F(S))\subseteq F(S)$. From (33), for any $x,y\in F(S)$

$$\begin{aligned}
d(fx,fy) &\le a\phi_0(d(x,y))+(1-a)\\
&\times\max\Bigg\{\phi_1(d(x,y)),\\
&\phi_2\left(\frac{1}{2}\left[d(x,fx)+d(y,fy)\right]\right),\\
&\phi_3\left(\frac{1}{2}\left[d(x,fy)+d(y,fx)\right]\right),\\
&\phi_4\left(\frac{1}{2}\left[d(x,fx)+d(y,fx)\right]\right),\\
&\phi_5\left(\frac{1}{2}\left[d(x,fy)+d(y,fy)\right]\right)\Bigg\}.
\end{aligned} \quad (34)$$

By Corollary 14 (with S as identity map on X), f has a unique fixed point on $F(S)$ and hence f and S have a unique common fixed point. □

Corollary 16. *Let f,S be self-mappings of a metric space (X,d) satisfying $d(fx,fy)\le kd(Sx,Sy)$, for all $x,y\in X$ where $0<k<1$. If (f,S) is a nontrivial Banach operator pair, then f and S have a unique common fixed point.*

Proof. It is sufficient to set $a=1$ and take $\phi_0(t)=kt\in\Phi$ in Theorem 15. □

Example 17. Let $X=[0,1]$ be a metric space with the usual metric $d(x,y)=|x-y|$ for all $x,y\in X$. Define $fx=gx=-1+\sqrt{3}$, $Sx=1-(1/2)x^2$, and $Tx=x$ for all $x\in[0,1]$. Obviously, $|fx-gy|=0$ and $|fx-gy|\le k|Sx-Ty|$ for all $x,y\in X$ and $0<k<1$. Also, $C(f,S)=\{-1+\sqrt{3}\}$, $PC(f,S)=\{-1+\sqrt{3}\}$, $C(g,T)=\{-1+\sqrt{3}\}$, and $PC(g,T)=\{-1+\sqrt{3}\}$. So, clearly (f,S) and (g,T) are each $\mathcal{JH}$-operator pairs. Thus, all the conditions of Corollary 13 are satisfied and $-1+\sqrt{3}$ is the unique common fixed point of f,g,S, and T.

4. Applications

In this section, we utilize the common fixed point theorems and their results to deduce the existence and uniqueness of the common solution for the system of functional equations in dynamic programming.

Remark 18. Many authors (e.g., see [9, 11–15, 22], or [3–5, 8–12, 17, 22] in [22]) used the fixed point theory to solve functional equations arising in dynamic programming on complete metric spaces such as Banach spaces. But, in the final section, we omit the completeness of the space and we state the result in the normed vector spaces and metric spaces setting.

Let X,Y be normed vector spaces, $S\subseteq X$ the state space, and $D\subseteq Y$ the decision space. Denote by $B(S)$ the set of all bounded real-valued functions on S and $d(f,g)=\sup\{|f(x)-g(x)|:x\in S\}$. It is clear that $(B(S),d)$ is a metric space:

$$\begin{aligned}
f_i(x) &= \underset{y\in D}{\operatorname{opt}}\{u(x,y)+H_i(x,y,f_i(T(x,y)))\},\\
&\qquad \forall x\in S,\quad i=1,2,\\
g_i(x) &= \underset{y\in D}{\operatorname{opt}}\{u(x,y)+F_i(x,y,,g_i(T(x,y)))\},\\
&\qquad \forall x\in S,\quad i=1,2,
\end{aligned} \quad (35)$$

where opt stands for sup or inf, $u:S\times D\to\mathbb{R}$, $T:S\times D\to S$, and $H_i,F_i:S\times D\times\mathbb{R}\to\mathbb{R}$ for $i=1,2$. Suppose that the mappings A_i and T_i $(i=1,2)$ are defined:

$$\begin{aligned}
A_ih(x) &= \underset{y\in D}{\operatorname{opt}}\{u(x,y)+H_i(x,y,h(T(x,y)))\},\\
&\qquad \forall x\in S,\quad h\in B(S),\quad i=1,2,\\
T_ik(x) &= \underset{y\in D}{\operatorname{opt}}\{u(x,y)+F_i(x,y,k(T(x,y)))\},\\
&\qquad \forall x\in S,\quad k\in B(S),\quad i=1,2.
\end{aligned} \quad (36)$$

Theorem 19. *Suppose that the following conditions are satisfied:*

(i) for given $h\in B(S)$, there exist $r(h)>0$ such that

$$\begin{aligned}
&|u(x,y)|+\max\{|H_i(x,y,h(T(x,y)))|,\\
&\qquad |F_i(x,y,h(T(x,y)))|\ i=1,2\}\\
&\le r(h),\quad \forall(x,y)\in S\times D;
\end{aligned} \quad (37)$$

(ii)

$$\begin{aligned}
&|H_1(x,y,h(t))-H_2(x,y,k(t))|\\
&\le a\phi_0(d(T_1h(t),T_2k(t)))+(1-a)\\
&\times\max\Big\{\phi_1(d(T_1h(t),T_2k(t))),
\end{aligned}$$

$$\phi_2\left(\frac{1}{2}\left[d\left(T_1h\left(t\right),A_1h\left(t\right)\right)\right]\right.$$
$$\left.+\,d\left(T_2k\left(t\right),A_2k\left(t\right)\right)\right]\Big),$$
$$\phi_3\left(\frac{1}{2}\left[d\left(T_1h\left(t\right),A_2k\left(t\right)\right)\right.\right.$$
$$\left.\left.+d\left(T_2k\left(t\right),A_1h\left(t\right)\right)\right]\right),$$
$$\phi_4\left(\frac{1}{2}\left[d\left(T_1h\left(t\right),A_1h\left(t\right)\right)\right.\right.$$
$$\left.\left.+d\left(T_2k\left(t\right),A_1h\left(t\right)\right)\right]\right),$$
$$\phi_5\left(\frac{1}{2}\left[d\left(T_1h\left(t\right),A_2k\left(t\right)\right)\right.\right.$$
$$\left.\left.+d\left(T_2k\left(t\right),A_2k\left(t\right)\right)\right]\right)\Big\},\tag{38}$$

for all $(x,y)\in S\times D$, $h,k\in B(S)$, $t\in S$, *where* $\phi_i\in\Phi$, $i=0,1,2,3,4,5$, $0\le a\le 1$;

(iii) for $i=1,2$, $\emptyset\neq\Gamma_i=\{\tau_{p_i}:A_i\tau_{p_i}=T_i\tau_{p_i}=\Theta_{p_i}\}\subseteq B(S)$;

(iv) there exist $\tau_{p_i}\in\Gamma_i(i=1,2)$, *such that*

$$\left|H_i\left(x,y,\tau_{q_i}\left(t\right)\right)-F_i\left(x,y,\tau_{r_i}\left(t\right)\right)\right|\ge\left|\Theta_{p_i}-\tau_{p_i}\right|,\tag{39}$$

for some $\tau_{q_i},\tau_{r_i}\in\Gamma_i$ *and for all* $(x,y)\in S\times D$, $t\in S$.

Then the system of functional equations (35) *possesses a unique common solution in* $B(S)$.

Proof. Assume that $\mathrm{opt}_{y\in D}=\inf_{y\in D}$. By condition (i) and (36), A_i and T_i are self-mappings of $B(S)$. Using (i) and (36), one can deduce that there exist $y,z\in D$ such that

$$A_1h(x)>u(x,y)+H_1(x,y,h(T(x,y)))-\epsilon,\tag{40}$$

$$A_2k(x)>u(x,y)+H_2(x,z,k(T(x,z)))-\epsilon.\tag{41}$$

Note that

$$A_1h(x)\le u(x,z)+H_1(x,z,h(T(x,z))),\tag{42}$$

$$A_2k(x)\le u(x,y)+H_2(x,y,h(T(x,y))).\tag{43}$$

By virtue of (41) and (42),

$$\begin{aligned}A_1h(x)-A_2k(x)&<H_1(x,z,h(T(x,z)))-H_2(x,z,k(T(x,z)))+\epsilon\\&\le\left|H_1(x,z,h(T(x,z)))-H_2(x,z,k(T(x,z)))\right|+\epsilon.\end{aligned}\tag{44}$$

From (40) and (43), we conclude that

$$\begin{aligned}A_1h(x)-A_2k(x)&>H_1(x,y,h(T(x,y)))-H_2(x,y,k(T(x,y)))-\epsilon\\&\ge\left|H_1(x,y,h(T(x,y)))-H_2(x,y,k(T(x,y)))\right|-\epsilon.\end{aligned}\tag{45}$$

It follows from (44) and (45) that

$$\begin{aligned}&\left|A_1h(x)-A_2k(x)\right|\\&\le\max\{\left|H_1(x,y,h(T(x,y)))-H_2(x,y,k(T(x,y)))\right|,\\&\qquad\left|H_1(x,z,h(T(x,z)))-H_2(x,z,k(T(x,z)))\right|\}+\epsilon.\end{aligned}\tag{46}$$

Equation (46) and (ii) lead to

$$\begin{aligned}&\left|A_1h(x)-A_2k(x)\right|\\&\le\max\{\left|H_1(x,y,h(T(x,y)))-H_2(x,y,k(T(x,y)))\right|,\\&\qquad\left|H_1(x,z,h(T(x,z)))-H_2(x,z,k(T(x,z)))\right|\}+\epsilon\\&\le a\phi_0(d(T_1h(t),T_2k(t)))+(1-a)\\&\quad\times\max\Big\{\phi_1(d(T_1h(t),T_2k(t))),\\&\qquad\phi_2\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))+d(T_2k(t),A_2k(t))]\Big),\\&\qquad\phi_3\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))+d(T_2k(t),A_1h(t))]\Big),\\&\qquad\phi_4\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))+d(T_2k(t),A_1h(t))]\Big),\\&\qquad\phi_5\Big(\frac{1}{2}[d(T_1h(t),A_2k(t))+d(T_2k(t),A_2k(t))]\Big)\Big\}+\epsilon\end{aligned}\tag{47}$$

which yields that

$$\begin{aligned}&d(A_1h,A_2k)\\&=\sup_{x\in S}\left|A_1h(x)-A_2k(x)\right|\\&\le a\phi_0(d(T_1h(t),T_2k(t)))+(1-a)\\&\quad\times\max\Big\{\phi_1(d(T_1h(t),T_2k(t))),\\&\qquad\phi_2\Big(\frac{1}{2}[d(T_1h(t),A_1h(t))+d(T_2k(t),A_2k(t))]\Big),\end{aligned}$$

$$\begin{aligned}&\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t),A_2k(t)\right)\right.\right.\\&\qquad\left.\left.+d\left(T_2k(t),A_1h(t)\right)\right]\right),\\&\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t),A_1h(t)\right)\right.\right.\\&\qquad\left.\left.+d\left(T_2k(t),A_1h(t)\right)\right]\right),\\&\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t),A_2k(t)\right)\right.\right.\\&\qquad\left.\left.\left.+d\left(T_2k(t),A_2k(t)\right)\right]\right)\right\}+\epsilon.\end{aligned}\tag{48}$$

Let $\epsilon \to 0$ in (48); then

$$\begin{aligned}&d\left(A_1h,A_2k\right)\\&\quad=\sup_{x\in S}\left|A_1h(x)-A_2k(x)\right|\\&\quad\leq a\phi_0\left(d\left(T_1h(t),T_2k(t)\right)\right)+(1-a)\\&\qquad\times\max\left\{\phi_1\left(d\left(T_1h(t),T_2k(t)\right)\right),\right.\\&\qquad\phi_2\left(\frac{1}{2}\left[d\left(T_1h(t),A_1h(t)\right)\right.\right.\\&\qquad\qquad\left.\left.+d\left(T_2k(t),A_2k(t)\right)\right]\right),\\&\qquad\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t),A_2k(t)\right)\right.\right.\\&\qquad\qquad\left.\left.+d\left(T_2k(t),A_1h(t)\right)\right]\right),\\&\qquad\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t),A_1h(t)\right)\right.\right.\\&\qquad\qquad\left.\left.+d\left(T_2k(t),A_1h(t)\right)\right]\right),\\&\qquad\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t),A_2k(t)\right)\right.\right.\\&\qquad\qquad\left.\left.\left.+d\left(T_2k(t),A_2k(t)\right)\right]\right)\right\}.\end{aligned}\tag{49}$$

Now, we shall show that (A_1,T_1) and (A_2,T_2) are $\mathcal{JH}$-operator pairs. By (iii) there exists $\tau_{p_1}\in\Gamma_1$; thus, $A_1\tau_{p_1}=T_1\tau_{p_1}=\Theta_{p_1}$ and by (iv) for all $(x,y)\in S\times D$, $t\in S$, we have

$$\left|\Theta_{p_1}-\tau_{p_1}\right|\leq\left|H_1\left(x,y,\tau_{q_1}(t)\right)-F_1\left(x,y,\tau_{r_1}(t)\right)\right|,\tag{50}$$

for some $\tau_{q_1},\tau_{r_1}\in\Gamma_1$. Therefore, for all $x\in S$

$$\begin{aligned}&\left|\Theta_{p_1}(x)-\tau_{p_1}(x)\right|\\&\quad\leq\left|\inf_{y\in D}\left(H_1\left(x,y,\tau_{q_1}(T(x,y))\right)-F_1\left(x,y,\tau_{r_1}(T(x,y))\right)\right)\right|\\&\quad=\left|\inf_{y\in D}\left(u(x,y)+H_1\left(x,y,\tau_{q_1}(T(x,y))\right)-u(x,y)\right.\right.\\&\qquad\left.\left.-F_1\left(x,y,\tau_{r_1}(T(x,y))\right)\right)\right|\\&\quad=\left|A_1\tau_{q_1}-T_1\tau_{r_1}\right|\\&\quad=\left|\Theta_{q_1}-\Theta_{r_1}\right|\\&\quad\leq\sup_{x\in S}\left|\Theta_{q_1}(x)-\Theta_{r_1}(x)\right|\\&\quad=d\left(\Theta_{q_1},\Theta_{r_1}\right)\\&\quad\leq\operatorname{diam}\left(PC\left(A_1,T_1\right)\right).\end{aligned}\tag{51}$$

So

$$\sup_{x\in S}\left|\Theta_{p_1}(x)-\tau_{p_1}(x)\right|\leq\operatorname{diam}\left(PC\left(A_1,T_1\right)\right),\tag{52}$$

and, hence, $d(\Theta_{p_1},\tau_{p_1})\leq\operatorname{diam}(PC(A_1,T_1))$. That is, (A_1,T_1) is $\mathcal{JH}$-operator pair. Similarly, (A_2,T_2) is also $\mathcal{JH}$-operator pair. Clearly, all the above process also holds for $\operatorname{opt}_{y\in D}=\sup_{y\in D}$. Then all of the conditions of Theorem 15 are satisfied and $h\in B(S)$ is a unique common fixed point of A_1,T_1,A_2, and T_2; that is, $h(x)$ is a unique common solution of functional equations (35). □

Corollary 20. *Suppose that the conditions* (*i*), (*iii*)*, and* (*iv*) *of Theorem 19 are satisfied. Moreover, if the following condition also holds:*

$$\left|H_1\left(x,y,h(t)\right)-H_2\left(x,y,k(t)\right)\right|\leq\alpha d\left(T_1h(t),T_2k(t)\right),\tag{53}$$

for all $(x,y)\in S\times D$, $h,k\in B(S)$, $t\in S$, *where* $0<\alpha<1$, *then the system of functional equations* (35) *possesses a unique common solution in* $B(S)$.

Proof. It is sufficient to set $a=1$ and take $\phi_0(t)=\alpha t\in\Phi$ in Theorem 19. □

Example 21. Let $X=Y=\mathbb{R}$ be normed vector spaces endowed with the usual norm $\|\cdot\|$ defined by $\|x\|=|x|$ for all $x\in\mathbb{R}$. Let $S=[0,1]\subseteq X$ be the state space and $D=[1,\infty)\subseteq Y$ the decision space. Define $T:S\times D\to S$ and $u:S\times D\to\mathbb{R}$ by

$$T(x,y)=\frac{x+1}{y^2+2},\quad u(x,y)=0,\quad\forall x\in S,\ y\in D.\tag{54}$$

Define $f_i,g_i:S\to\mathbb{R}$ $(i=1,2)$ by

$$\begin{gathered}f_i(x)=\frac{1}{16}\left[x-x^2\right],\\g_1(x)=\frac{1}{2}\sqrt{x},\qquad g_2(x)=\frac{1}{2}x^3.\end{gathered}\tag{55}$$

Now, for all $h, k \in B(S)$; $x \in S$, we define mappings A_i and T_i $(i = 1, 2)$ by

$$\begin{aligned} A_1 h(x) &= \sup_{y\in D}\{u(x,y) + H_1(x,y,h(T(x,y)))\},\\ A_2 k(x) &= \sup_{y\in D}\{u(x,y) + H_2(x,y,k(T(x,y)))\},\\ T_1 h(x) &= \sup_{y\in D}\{u(x,y) + F_1(x,y,h(T(x,y)))\},\\ T_2 k(x) &= \sup_{y\in D}\{u(x,y) + F_2(x,y,k(T(x,y)))\}, \end{aligned} \tag{56}$$

for which $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are defined as follows:

$$\begin{aligned} H_i(x,y,t) &= \frac{1}{16}\left[\left(x - x^2\right)\cos\left(t\cdot\left(1 - \frac{1}{y+2}\right)\right)\right],\\ F_1(x,y,t) &= \frac{1}{2}\left[\sqrt{x}\sin\left(t\cdot\left(1 - \frac{1}{y+2}\right)\right)\right],\\ F_2(x,y,t) &= \frac{1}{2}\left[x^3\sin\left(t\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]. \end{aligned} \tag{57}$$

So,

$$\begin{aligned} A_1 h(x) &= \sup_{y\in D} H_1\left(x, y, h\left(\frac{x+1}{y^2+1}\right)\right)\\ &= \sup_{y\in D}\frac{1}{16}\left[\left(x - x^2\right)\cos\left(h\left(\frac{x+1}{y^2+2}\right)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]\\ &= \frac{1}{16}\left[x - x^2\right] = f_1(x),\\ A_2 k(x) &= \sup_{y\in D} H_2\left(x, y, k\left(\frac{x+1}{y^2+2}\right)\right)\\ &= \sup_{y\in D}\frac{1}{16}\left[\left(x - x^2\right)\cos\left(k\left(\frac{x+1}{y^2+2}\right)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]\\ &= \frac{1}{16}\left[x - x^2\right] = f_2(x),\\ T_1 h(x) &= \sup_{y\in D} F_1\left(x, y, h\left(\frac{x+1}{y^2+1}\right)\right)\\ &= \sup_{y\in D}\frac{1}{2}\left[\sqrt{x}\sin\left(h\left(\frac{x+1}{y^2+2}\right)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]\\ &= \frac{1}{2}\sqrt{x} = g_1(x),\\ T_2 k(x) &= \sup_{y\in D} F_2\left(x, y, k\left(\frac{x+1}{y^2+2}\right)\right)\\ &= \sup_{y\in D}\frac{1}{2}\left[x^3\sin\left(k\left(\frac{x+1}{y^2+2}\right)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]\\ &= \frac{1}{2}x^3 = g_2(x), \end{aligned} \tag{58}$$

for all $x \in S$, $h, k \in B(S)$. Also, $\|H_i\| \le 1/16$ and $\|F_i\| \le 1/2$, $(i = 1, 2)$; then easily we have the following:

(i) for given $h \in B(S)$, there exist $r(h) > 0$ such that

$$\begin{aligned} &|u(x,y)| + \max\{|H_i(x,y,h(T(x,y)))|,\\ &\qquad |F_i(x,y,h(T(x,y)))| \ i = 1, 2\}\\ &\quad \le r(h), \quad \forall (x,y) \in S \times D, \end{aligned} \tag{59}$$

if choose $a = 1$, and $\phi_0(n) = (1/4)n$, for $n \in [0, \infty)$ we have (ii)

$$\begin{aligned} &|H_1(x,y,h(t)) - H_2(x,y,k(t))|\\ &\quad = \left|\frac{1}{16}\left[\left(x - x^2\right)\cos\left(h(t)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right]\right.\\ &\qquad \left. - \frac{1}{16}\left[\left(x - x^2\right)\cos\left(k(t).\left(1 - \frac{1}{y+2}\right)\right)\right]\right|\\ &\quad = \frac{1}{16}\left|x - x^2\right|\left|\cos\left(h(t)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right.\\ &\qquad \left. - \cos\left(k(t)\cdot\left(1 - \frac{1}{y+2}\right)\right)\right|\\ &\quad = \frac{1}{16}\left|x - x^2\right|\\ &\qquad \cdot 2\left|\sin\left(\left(h(t)\cdot\left(1 - \frac{1}{y+2}\right) - k(t)\right.\right.\right.\\ &\qquad \left.\left.\left.\cdot\left(1 - \frac{1}{y+2}\right)\right)\times(2)^{-1}\right)\right|\\ &\qquad \times\left|\sin\left(\left(h(t)\cdot\left(1 - \frac{1}{y+2}\right) + k(t)\right.\right.\right.\\ &\qquad \left.\left.\left.\cdot\left(1 - \frac{1}{y+2}\right)\right)\times(2)^{-1}\right)\right|\\ &\quad \le \frac{1}{8}\left|x - x^2\right| \le \frac{1}{8}\left[\sqrt{x} - x^3\right] = \frac{1}{4}\left[\frac{\sqrt{x}}{2} - \frac{x^3}{2}\right]\\ &\quad = a\phi_0\left(\left|\frac{\sqrt{x}}{2} - \frac{x^3}{2}\right|\right)\\ &\quad = a\phi_0\left(d\left(T_1 h(x), T_2 k(x)\right)\right). \end{aligned} \tag{60}$$

Therefore, $|H_1(x,y,h(t)) - H_2(x,y,k(t))| \le a\phi_0(d(T_1 h(t), T_2 k(t)))$, for all $(x,y) \in S \times D$, $h, k \in B(S)$, $t \in S$.

(iii) $\Gamma_1 = \{\tau_{p_1} : A_1\tau_{p_1} = T_1\tau_{p_1} = \Theta_{p_1}\} = B(\{0\}) \neq \emptyset$ and $\Gamma_2 = \{\tau_{p_2} : A_2\tau_{p_2} = T_2\tau_{p_2} = \Theta_{p_2}\} = B(\{0\}) \neq \emptyset$.

(iv) Clearly, there exist $\tau_{p_i} \in \Gamma_i (i = 1, 2)$, such that

$$\left|H_i\left(x, y, \tau_{q_i}(t)\right) - F_i\left(x, y, \tau_{r_i}(t)\right)\right| \geq \left|\Theta_{p_i} - \tau_{p_i}\right|, \quad (61)$$

for some $\tau_{q_i}, \tau_{r_i} \in \Gamma_i$ and for all $(x, y) \in S \times D$.

Thus, all the assumptions of Theorem 19 are satisfied. So, the system of (35) has a unique common solution in $B(S)$.

Let

$$f(x) = \operatorname*{opt}_{y\in D}\{u(x, y) + H(x, y, f(T(x, y)))\}, \quad \forall x \in S,$$
$$g(x) = \operatorname*{opt}_{y\in D}\{u(x, y) + F(x, y, g(T(x, y)))\}, \quad \forall x \in S, \quad (62)$$

where $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$ and $H, F : S\times D\times\mathbb{R} \to \mathbb{R}$. Suppose that the mappings A_1 and T_1 are defined:

$$\begin{aligned} A_1h(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + H(x, y, h(T(x, y)))\}, \\ &\qquad \forall x \in S, \quad h \in B(S), \\ T_1k(x) &= \operatorname*{opt}_{y\in D}\{u(x, y) + F(x, y, k(T(x, y)))\}, \\ &\qquad \forall x \in S, \quad k \in B(S), \end{aligned} \quad (63)$$

Theorem 22. *Suppose that the following conditions are satisfied:*

(i) for given $h \in B(S)$, there exist $r(h) > 0$ such that

$$\begin{aligned} |u(x, y)| &+ \max\{|H(x, y, h(T(x, y)))|, \\ &\qquad |F(x, y, h(T(x, y)))|\} \\ &\leq r(h), \quad \forall (x, y) \in S \times D; \end{aligned} \quad (64)$$

(ii)

$$\begin{aligned} &|H(x, y, h(t)) - H(x, y, k(t))| \\ &\quad \leq a\phi_0(d(T_1h(t), T_1k(t))) + (1 - a) \\ &\quad \times \max\Big\{\phi_1(d(T_1h(t), T_1k(t))), \\ &\quad \phi_2\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) + d(T_1k(t), A_1k(t))]\Big), \\ &\quad \phi_3\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) + d(T_1k(t), A_1h(t))]\Big), \\ &\quad \phi_4\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) + d(T_1k(t), A_1h(t))]\Big), \\ &\quad \phi_5\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) + d(T_1k(t), A_1k(t))]\Big)\Big\}, \end{aligned} \quad (65)$$

for all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$, where $\phi_i \in \Phi$, $i = 0, 1, 2, 3, 4, 5$, $0 \leq a \leq 1$;

(iii) $\emptyset \neq \Gamma = \{\tau_p : A_1\tau_p = T_i\tau_p = \Theta_p\} \subseteq B(S)$;

(iv) there exist $\tau_p \in \Gamma$, such that

$$\left|H\left(x, y, \tau_q(t)\right) - F\left(x, y, \tau_r(t)\right)\right| \geq \left|\Theta_p - \tau_p\right|, \quad (66)$$

for some $\tau_q, \tau_r \in \Gamma$ and for all $(x, y) \in S \times D$, $t \in S$.

Then the functional equations (62) have a unique common solution in $B(S)$.

Proof. Assume that $\text{opt}_{y\in D} = \sup_{y\in D}$. By conditions (i) and (63), A_1 and T_1 are self-mappings of $B(S)$. Let h, k be any two points of $B(S)$, $x \in S$, and $\epsilon > 0$ any positive number; using (i) and (63), we deduce that there exist $y, z \in D$ such that

$$A_1h(x) < u(x, y) + H(x, y, h(T(x, y))) + \epsilon, \quad (67)$$
$$A_1k(x) < u(x, y) + H(x, z, k(T(x, z))) + \epsilon, \quad (68)$$
$$A_1h(x) \geq u(x, y) + H(x, z, h(T(x, z))), \quad (69)$$
$$A_1k(x) \geq u(x, y) + H(x, y, k(T(x, y))). \quad (70)$$

Subtracting (70) from (67) and using (ii),

$$\begin{aligned} &A_1h(x) - A_1k(x) \\ &\quad < H(x, y, h(T(x, y))) - H(x, y, k(T(x, y))) + \epsilon \\ &\quad \leq |H(x, y, h(T(x, y))) - H(x, y, k(T(x, y)))| + \epsilon \\ &\quad \leq a\phi_0(d(T_1h(t), T_1k(t))) + (1 - a) \\ &\quad \times \max\Big\{\phi_1(d(T_1h(t), T_1k(t))), \\ &\quad \phi_2\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) + d(T_1k(t), A_1k(t))]\Big), \\ &\quad \phi_3\Big(\frac{1}{2}[d(T_1h(t), A_1k(t)) + d(T_1k(t), A_1h(t))]\Big), \\ &\quad \phi_4\Big(\frac{1}{2}[d(T_1h(t), A_1h(t)) + d(T_1k(t), A_1h(t))]\Big), \end{aligned}$$

$$\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\bigg\} + \epsilon. \tag{71}$$

From (68) and (69), we get

$$\begin{aligned}
A_1h(x) - A_1k(x) &> H(x,z,h(T(x,z))) - H(x,z,k(T(x,z))) - \epsilon \\
&\geq |H(x,z,h(T(x,z))) - H(x,z,k(T(x,z)))| - \epsilon \\
&\geq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\
&\quad\times \max\bigg\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\
&\quad\phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\
&\quad\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\
&\quad\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right)\right]\right) + d\left(T_1k(t), A_1h(t)\right)\Big]\Big), \\
&\quad\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\bigg\} - \epsilon.
\end{aligned} \tag{72}$$

Hence, from (71) and (72)

$$\begin{aligned}
|A_1h(x) - A_1k(x)| &\leq |H(x,y,h(T(x,y))) - H(x,y,k(T(x,y)))| + \epsilon \\
&\leq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\
&\quad\times \max\bigg\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\
&\quad\phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\
&\quad\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\
&\quad\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\
&\quad\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\bigg\} + \epsilon.
\end{aligned} \tag{73}$$

So

$$\begin{aligned}
d\left(A_1h, A_1k\right) &= \sup_{x\in S}|A_1h(x) - A_1k(x)| \\
&\leq |H(x,y,h(T(x,y))) - H(x,y,k(T(x,y)))| + \epsilon \\
&\leq a\phi_0\left(d\left(T_1h(t), T_1k(t)\right)\right) + (1-a) \\
&\quad\times \max\bigg\{\phi_1\left(d\left(T_1h(t), T_1k(t)\right)\right), \\
&\quad\phi_2\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right), \\
&\quad\phi_3\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\
&\quad\phi_4\left(\frac{1}{2}\left[d\left(T_1h(t), A_1h(t)\right) + d\left(T_1k(t), A_1h(t)\right)\right]\right), \\
&\quad\phi_5\left(\frac{1}{2}\left[d\left(T_1h(t), A_1k(t)\right) + d\left(T_1k(t), A_1k(t)\right)\right]\right)\bigg\} + \epsilon.
\end{aligned} \tag{74}$$

Also from (iii) and (iv) and similar to Theorem 19, it is easy to prove that the pair (A_1, T_1) is $\mathcal{JH}$-operator pair. Therefore, by Corollary 14, A_1 and T_1 have a unique common fixed point in $B(S)$ and hence the functional equations (62) have a unique common solution in $B(S)$. □

Corollary 23. *Suppose that the conditions (i), (iii), and (iv) of Theorem 22 are satisfied. Moreover, if the following condition also holds:*

$$|H(x,y,h(t)) - H(x,y,k(t))| \leq \alpha d\left(T_1h(t), T_1k(t)\right), \tag{75}$$

for all $(x,y) \in S \times D$, $h,k \in B(S)$, $t \in S$, where $0 < \alpha < 1$, then the functional equations (62) have a unique common solution in $B(S)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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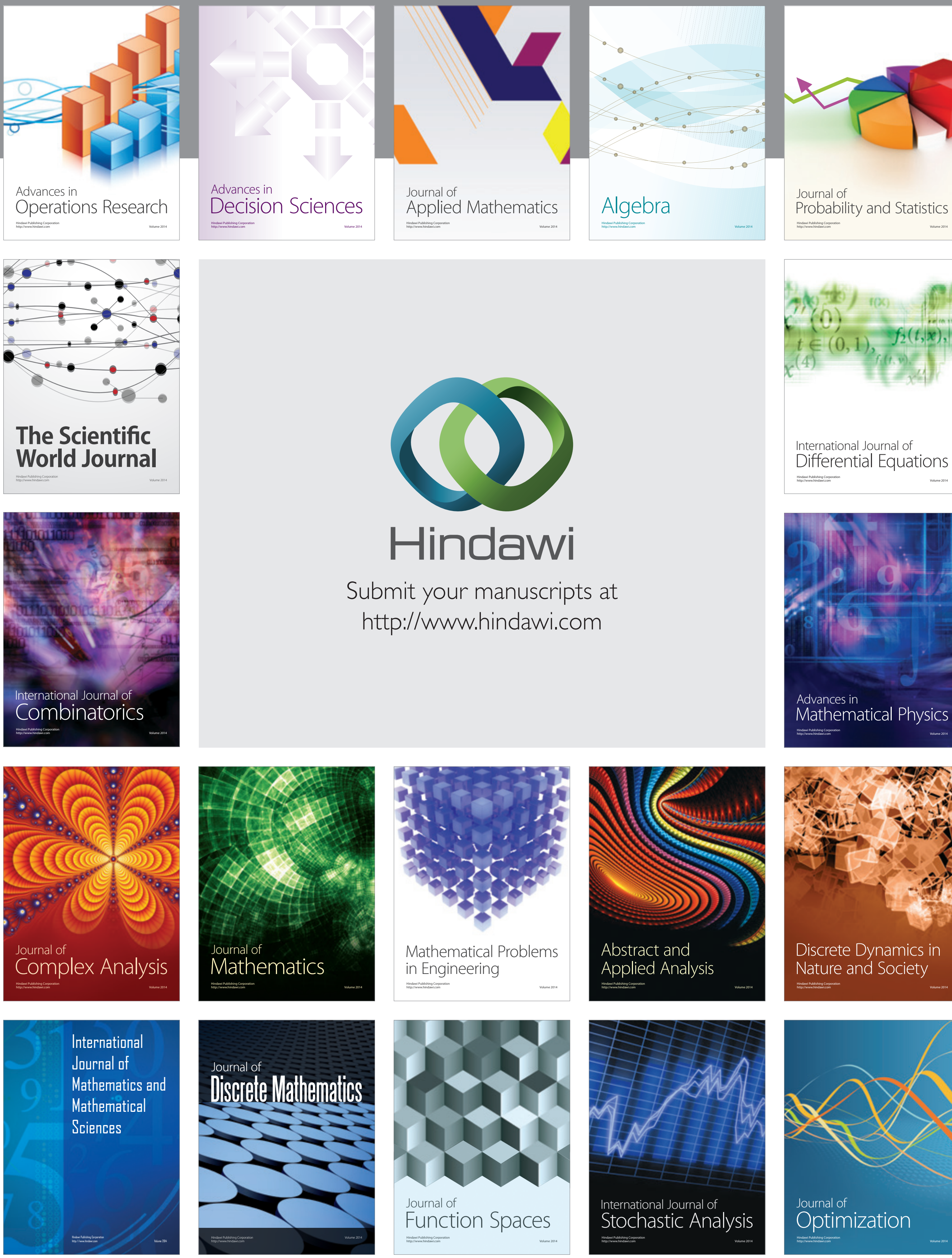
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