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Research Article

Analytical Solutions of Fractional Differential Equations Using the Convenient Adomian Series

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Due to the memory trait of the fractional calculus, numerical or analytical solution of higher order becomes very difficult even impossible to obtain in real engineering problems. Recently, a new and convenient way was suggested to calculate the Adomian series and the higher order approximation was realized. In this paper, the Adomian decomposition method is applied to nonlinear fractional differential equation and the error analysis is given which shows the convenience.

1. Introduction

The fractional calculus has frequently appeared in various applied areas and has become an increasing interesting topic in the past decades [1–7]. Many efforts to analytical and numerical methods have been made. The often used numerical methods are the fractional difference method [8–10] and the predictor corrector method [11–13] and others as well as the analytical methods such as the variational iteration method (VIM) [14–17] and the Adomian decomposition method (ADM) [17–21] and others.

In fact, these methods are developed from original versions for ordinary differential equation of the integer order equation. Compared with the ordinary calculus, the fractional calculus has the long interaction traits or the so-called memory effects; this characteristic can better depict various nonlinear dynamics in both theories and engineering mathematical modeling. However, it also results in finding the solution to the fractional models. The challenge is analyzed in the analytical methods in [17].

The ADM has been extensively applied to fractional differential equations due to its convenience. The Adomian series should be calculated in each iteration which greatly affects the efficiency and accuracy of the analytical approximation. In order to solve this problem, Duan very recently suggested a new way to calculate the Adomian series in [22–25] and successfully extended it to the fractional differential equation [26].

In this paper, we adopt Duan's way to calculate the Adomian series and apply it to FDEs for $1 < \alpha < 2$. We define the residual function and give the error analysis and investigate the validness of the iteration formulae.

2. Algorithm of the Fractional Differential Equations

Definition 1 (see [2]). The Caputo derivative is defined as

$$ {}_0^C D_t^\alpha u = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} \frac{d^m}{d\tau^m} u(\tau)\, d\tau, \quad 0 < t,\ 0 < \alpha,\ m = [\alpha] + 1, \tag{1} $$

where Γ is the Gamma function.

Definition 2 (see [2]). The R-L integration of α order is defined by

$$ {}_0I_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad 0 < t,\ 0 < \alpha. \tag{2} $$

Now we present our analytical schemes using the convenient Adomian series, Laplace transform, and Pade approximation. We adopt the steps in [27, 28]. Considering the following general fractional differential equations (FDEs),

$$ L[u] + N[u] = g(t), \tag{3} $$

where $L[u]$ is a linear operator with respect to u such as ${}_0^C D_t^{\alpha} u$, ${}_0^C D_t^{\alpha} u + u$, ${}_0^C D_t^{\alpha} u + {}_0^C D_t^{\beta} u, \ldots$, we show the following iteration schemes.

(a) Take Laplace transform $\widetilde{L}$ to both sides:

$$ \widetilde{L}[L[u]] + \widetilde{L}[N[u]] = \widetilde{L}[g(t)]. \tag{4} $$

We can have iteration formula (4) through inverse of Laplace transform $\widetilde{L}^{-1}$:

$$ u(t) = f(t) + \widetilde{L}^{-1}\left[\widetilde{\lambda}(s)\,\widetilde{L}[N[u]]\right], \tag{5} $$

where $\widetilde{\lambda}(s)$ and $f(t)$ can be determined by calculation of Laplace transform to $L[u]$, $g(t)$. This step makes (3) in time domain equivalently defined in the Laplace domain. This idea is illustrated in the solution of differential equations [27–29]. Here the $\widetilde{\lambda}(s)$ similarly plays a role as the Lagrange multipliers in the VIM [29].

(b) Through the Picard successive approximation, we can obtain the following iteration formula:

$$ u_{n+1} = f(t) + \widetilde{L}^{-1}\left[\widetilde{\lambda}(s)\,\widetilde{L}[N[u_n]]\right]. \tag{6} $$

(c) Let $u_n = \sum_{i=0}^{n} v_i$ and apply the Adomian series to expand the term $N[u]$ as $\sum_{i=0}^{\infty} A_i$. Then the iteration formula reads

$$ \begin{aligned} v_{n+1} &= \widetilde{L}^{-1}\left[\widetilde{\lambda}(s)\,\widetilde{L}[A_n]\right], \\ v_0 &= f(t) \end{aligned} \tag{7} $$

according to [24], where A_i is calculated by

$$ A_i = \frac{1}{i} \sum_{k=0}^{i-1} (k+1)\, v_{k+1} \frac{dA_{i-1-k}}{dv_0}. \tag{8} $$

(d) The nth-order approximation is explicitly given as

$$ u_n = \sum_{i=0}^{n} v_i. \tag{9} $$

The A_i depends on the $v_0, v_1, \ldots, v_i$ which is denoted as

$$ A_i = A_i(v_0, v_1, \ldots, v_i). \tag{10} $$

This characteristic allows us to obtain the approximate solutions from v_0 and $A_0 = N(v_0)$.

3. Numerical Example and Error Analysis

Example 1. Consider the following nonlinear FDE:

$$ {}_0^C D_t^{\alpha} u + 2u^2 = 0, \quad 1 < \alpha < 2, \tag{11} $$

with the initial conditions

$$ u(0) = 1, \qquad u'(0) = 0. \tag{12} $$

Following to the steps (a) to (d) in Section 2, we can have

$$ A_0 = 2v_0^2, \tag{13} $$

where v_0 can be identified using the Laplace as $v_0 = 1$.

The iteration formula is written as

$$ \begin{aligned} v_{n+1} &= -\widetilde{L}^{-1}\left[\frac{1}{s^{\alpha}}\widetilde{L}[A_n]\right], \quad n \geq 0, \\ A_0 &= 2v_0^2, \quad v_0 = 1. \end{aligned} \tag{14} $$

For $\alpha = 1.9$, the Adomian series and the approximate solutions can be calculated as

$$ \begin{aligned} &A_1 = -4v_0v_1, \\ &u_1 = v_0 + v_1 = 1 - 1.094478036t^{19/10}, \\ &A_2 = -2{v_1}^2 - 4v_2v_0, \\ &u_2 = 1 - 1.094478036t^{19/10} + 0.4484842412t^{19/5}, \\ &\vdots \end{aligned} \tag{15} $$

We find the new way to calculate the solution and the series is very convenient compared with the classical one. Set $n = 20$ and plot the defined residual function in Figure 1:

$$ g_n = \log_{10}\left|{}_0^C D_t^{\alpha} u_n + 2u_n^2\right|. \tag{16} $$

From Figure 2, we can conclude that iteration formula (14) and the approximate solutions are correct.

Example 2. The second example is given as

$$ {}_0^C D_t^{\alpha} u = u^3, \quad u(0) = 0,\ u'(0) = 1,\ 1 < \alpha < 2. \tag{17} $$

We obtain the following formulae based on the Adomian series:

$$ \begin{aligned} v_{n+1} &= \widetilde{L}^{-1}\left[\frac{1}{s^{\alpha}}\widetilde{L}[A_n]\right], \quad n \geq 0, \\ A_0 &= v_0^3, \quad v_0 = t. \end{aligned} \tag{18} $$

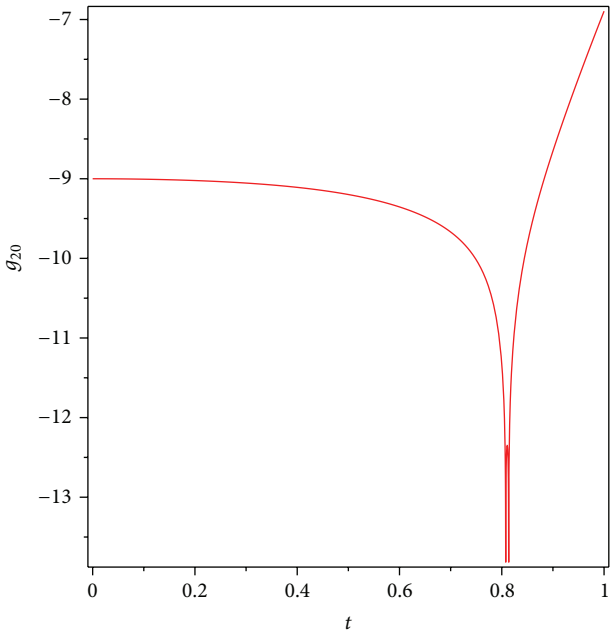

Figure 1: Residual function (16).

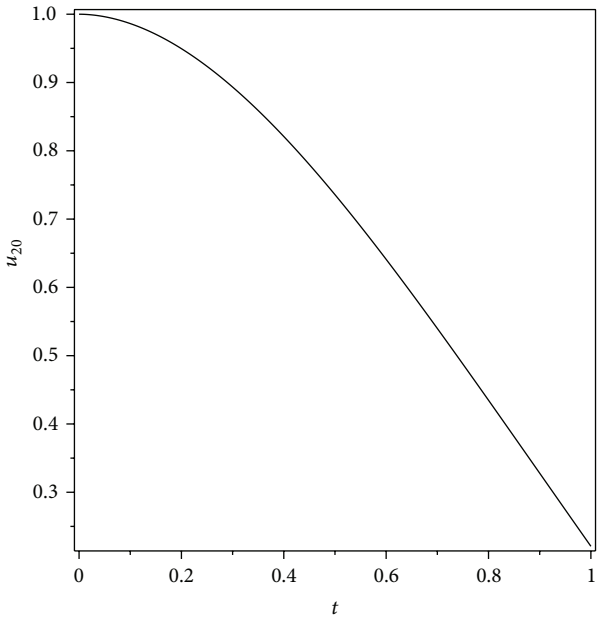

Figure 2: Approximate solution of (11).

For $\alpha = 1.5$, we can have the approximate solutions successively as

$$u_0 = t,$$

$$u_1 = t + 0.1146289948t^{9/2},$$

$$u_2 = t + 0.1146289948t^{9/2} + 0.01595982144t^{8}$$

$$u_3 = t + 0.1146289948t^{9/2} + 0.01595982144t^{8} + 0.002314983909t^{23/2}$$

$$\vdots \tag{19}$$

We plot the u_{100} in Figure 3.

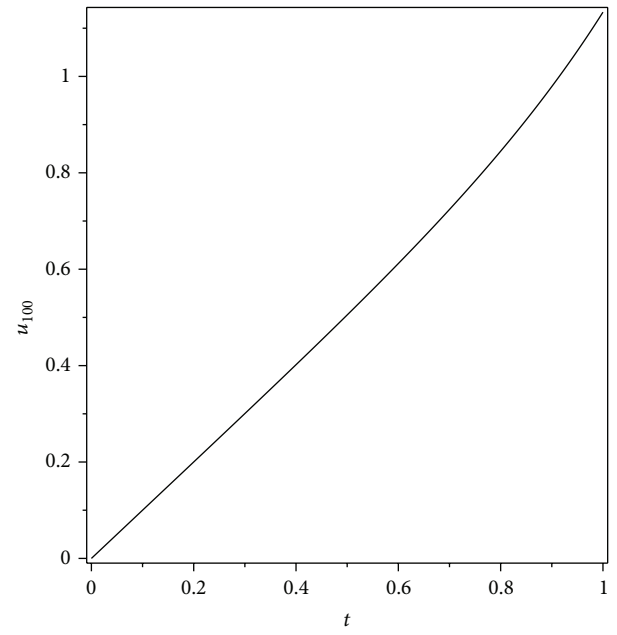

Figure 3: Approximate solution of (17).

4. Conclusions

This study applied the convenient Adomian series to fractional differential equations whose order is between one and two. Two nonlinear examples are used to illustrate the basics of the steps. We found the calculation of the solutions is more convenient and more rapid compared with the classical version. The approximate order here can be chosen as $n = 100$ while this choice is impossible for the classical definition. This merit is particularly nice for the fractional differential equations and the results show this purpose.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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