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Research Article

Asymptotic Normality of the Estimators for Fractional Brownian Motions with Discrete Data

Lin Sun,[1] Xiaojian Yu,[2] Xuewei Guan,[3] and Qinghao Meng[3]

[1] *School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, China*
[2] *School of Economics and Commerce, South China University of Technology, Guangzhou 510006, China*
[3] *School of Business Administration, South China University of Technology, Guangzhou 510640, China*

Correspondence should be addressed to Xiaojian Yu; xjyuscut@gmail.com

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This paper deals with the problem of estimating the Hurst parameter in the fractional Brownian motion when the Hurst index is greater than one half. The estimation procedure is built upon the marriage of the autocorrelation approach and the maximum likelihood approach. The asymptotic properties of the estimators are presented. Using the Monte Carlo experiments, we compare the performance of our method to existing ones, namely, R/S method, variations estimators, and wavelet method. These comparative results demonstrate that the proposed approach is effective and efficient.

1. Introduction

It is well-known that many time series, in diverse fields of application, may exhibit the phenomenon of long-memory or long-range dependence. As a result, time series with long-memory are currently used as stochastic models in various applications including telecommunications, hydrodynamics, economics, and environment. Moreover, applications of long-memory are found in areas as diverse as energy market analysis [1], neurosciences and other biological applications [2], and more traditional financial analysis and statistical theory [3–6]. We refer to the monographs Beran [7] and Rao [8] for complete expositions on theoretical and practical aspects of long-memory processes.

In the literature, there exist many stochastic processes which exhibit long-memory property. The most popular self-similar stochastic process that exhibits long-range dependence is of course the fractional Brownian motion (fBm), which is a well-known Gaussian self-similar stochastic process with stationary increments. In fact, up to a multiplicative constant, the fBm is the only Gaussian process with these two properties. Obviously, if some phenomenon can be modeled by fBm, the estimation of the Hurst parameter in fBm is an important problem. In other words, to avoid using an arbitrary value of the unknown Hurst parameter of fBm, we should estimate the Hurst exponent. This leads to a demand for rigorous estimation procedures for fBm, which is the aim of this paper.

Actually, there are several estimation procedures for obtaining the Hurst parameter for fBm (see, e.g., [9]). One of the estimators worth mentioning is the celebrated rescaled range analysis (R/S analysis) since it is very popular among researchers until today. In fact, there are many other methods available in the literature for estimating the Hurst exponent. For example, variance-time analysis, Higuchi's approach, correlogram method, periodogram method, Whittle estimator, wavelet method, and Detrended fluctuation analysis method. Recently, Chronopoulou and Viens [10] proposed a new estimator of the Hurst parameter based on the discrete variations. Actually, some comparisons of these estimation methods have also been investigated in that paper. Taqqu et al. [11] stated that the Whittle method is better than the classical methods using the empirical study while Abry and Veitch [12] showed that the wavelet method is better than the Whittle method. For a general comparison of Fourier and wavelet approach, see Faÿ et al. [9].

As far as the estimation is concerned, there is not yet a perfect method that is agreed by all researchers. Each method has its own drawbacks and cannot be used as a sole estimator in all cases. For more detailed discussions, see the work by

Beran [7]. In this paper, for estimating the Hurst coefficient in fBm, we present a new estimation procedure, which is built upon the marriage of the autocorrelation approach and the maximum likelihood approach. Specifically, when $H \in (1/2, 3/4]$, we present the asymptotic normality of the estimator borrowing the idea of Hosking [13]. Moreover, when $H \in (3/4, 1)$, we propose the asymptotic normality of the maximum likelihood estimator under strictly weaker conditions than those employed by Lai [14]. We also describe the numerical implementation based on our method and compare the performance of our method with the other known approaches, namely, R/S method, variations estimators, and wavelet method.

The remainder of this paper is organized as follows. In Section 2, we give a brief description of the key result and present the asymptotic distributions of our estimators. In Section 3, the performance of the proposed estimator is illustrated by some numerical experiments. Finally, Section 4 draws the concluding remarks. All technical details are relegated to the appendix.

2. The Estimators Based on Discrete Observations

With the dramatic increase of the importance of application of long-memory in time series, the fBM model has been successfully applied in many fields of economics, finance, physics, chemistry, medicine, and environmental studies. Indeed, when fBm is used to describe some phenomena, a crucial problem is how to identify the Hurst parameter. Thus, the parameter estimation problem for the Hurst parameter was of great interest and became a challenging theoretical problem in the past decade. In what follows, we consider the problem of estimating the Hurst index in fBm.

Now, let us recall that a fractional Brownian motion B_t^H with the Hurst coefficient H, defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, is a centered Gaussian process. Its law is thus characterized by its covariance function, which is given by

$$\begin{aligned}\mathbb{E}\left[B_t^H B_s^H\right] &= R_H(t,s) \\ &= \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right), \quad t \geq 0,\ s \geq 0,\end{aligned} \tag{1}$$

where $\mathbb{E}[B_t^H] = 0$ and $B_0^H = 0$ and $H \in (0,1)$ denotes the long-memory parameter. Moreover, let $\mathbf{X} = (B_1, B_2, \ldots, B_N)^t$ be the column vector of this observed time series at instances $t_i = i$, where the superscript t denotes the transpose of a vector.

2.1. The Estimator When $1/2 < H \leq 3/4$. In this subsection, we develop the estimator of the Hurst parameter using the autocorrelation function.

From (1), we can see that the covariance can be written as $R_{1/2}(t,s) = \min(s,t)$ for $H = 1/2$, and the process $B_{1/2}$ is an ordinary Brownian motion. In this case the increments of the process in disjoint intervals are independent. However, for $H \neq 1/2$, the increments are not independent. Set $Y_k = B_k^H - B_{k-1}^H$, $k \geq 1$. Then $\{Y_k, k \geq 1\}$ is a Gaussian stationary sequence with unit variance and autocovariance function

$$\begin{aligned}\rho(k) &= \mathbb{E}\left[Y_{n+k} Y_n\right] \\ &= \frac{1}{2}\left(|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H}\right), \quad k \in \mathbb{Z}.\end{aligned} \tag{2}$$

Moreover, straightforward computations show that

$$\rho(k) \sim H(2H-1)|k|^{2H-1}, \tag{3}$$

as $k \to \infty$.

Therefore, the autocovariance function is nonsummable for $H > 1/2$. This phenomenon is called long-range dependence, indicating (relatively) a slow decay of the covariance function. Since Y_k is exactly second-order self-similar, we have from (2) that

$$\rho(1) = 2^{2H-1} - 1. \tag{4}$$

Thus, we can solve for H to get

$$\widehat{H} = \frac{1}{2}\left[1 + \log_2\left(1 + \rho(1)\right)\right]. \tag{5}$$

Now, given data $Y_1, \ldots, Y_N$, let

$$\begin{aligned}\widehat{\mu}_N &= \frac{1}{N}\sum_{i=1}^{N} Y_i, \\ \widehat{\gamma}_N(k) &= \frac{1}{N}\sum_{i=1}^{N-k}\left(Y_i - \widehat{\mu}_N\right)\left(Y_{i+k} - \widehat{\mu}_N\right), \\ \widehat{\sigma}_N^2 &= \widehat{\gamma}_N(0), \\ \widehat{\rho}_N(k) &= \frac{\widehat{\gamma}_N(k)}{\widehat{\sigma}_N^2}\end{aligned} \tag{6}$$

denote the sample mean, the sample covariance, the sample variance, and the sample autocorrelation, respectively. Based on (4), we can write the estimator for the Hurst parameter in fBm as

$$\widehat{H} = \frac{1}{2}\left[1 + \log_2\left(1 + \widehat{\rho}_N(1)\right)\right]. \tag{7}$$

To assess the performance of the proposed estimate, we appeal to the following result due to Hosking [13].

Theorem 1. *Let $Y_k = B_k^H - B_{k-1}^H$, $k \geq 1$ be an exactly second-order self-similar Gaussian process, that is, a fractional Gaussian noise. Assume in (2) that $0 < H \leq 3/4$. Then for a large sample size N, $\rho_N(1)$ is approximately $\mathcal{N}(\mu_N, \sigma_N^2)$, where*

$$\begin{aligned}\mu_N &= \rho(1) - \left(1 - \rho(1)\right)N^{2H-2}, \\ \sigma_N^2 &= \frac{1}{N}\left\{\left(1 + 3\rho^2(1)\right)\right. \\ &\quad + 2\sum_{k=1}^{\infty}\left[\left(1 + 2\rho^2(1)\right)\rho^2(k)\right. \\ &\quad + \rho(k-1)\rho(k+1) \\ &\quad \left.\left. - 4\rho(1)\rho(k-1)\rho(k)\right]\right\},\end{aligned} \tag{8}$$

when $H \in (0, 3/4)$ *and*

$$\sigma_N^2 = \frac{\log N}{N}\left[2H(2H-1)(1+\rho(1))\right]^2, \qquad (9)$$

when $H = 3/4$.

Proof. This is a special case of Hosking [13]. □

2.2. The Estimator When $3/4 < H < 1$. In what follows, we deal with the problem of estimating the Hurst index in fBm when $3/4 < H < 1$. The technique we employed here is the maximum likelihood method. The reason for choosing this approach is that this technique has been applied efficiently in a large set. Since the fBm is Gaussian the log-likelihood for the discrete observations $\mathbf{X} = (B_1, B_2, \ldots, B_N)^t$ may be explicitly computed. Borrowing the idea of Lai [14], we can obtain the exact maximum likelihood estimator of the Hurst parameter for the fBm. However, since our assumptions are slightly different from those in Lai [14], we present the detailed derivations and make some comparisons.

Note that for any i, B_{ih} is Gaussian. Thus the joint probability density function of $\mathbf{X}$ is

$$f_N(\mathbf{X}, H) = (2\pi)^{-N/2}\left|\Gamma_H\right|^{-1/2}\exp\left(-\frac{1}{2}\mathbf{X}^t\Gamma_H^{-1}\mathbf{X}\right), \qquad (10)$$

where

$$\begin{aligned}\Gamma_H &= \left[\operatorname{Cov}\left[B_i^H, B_j^H\right]\right]_{i,j=1,2,\ldots,N} \\ &= \frac{1}{2}\left(i^{2H} + j^{2H} - \left|i-j\right|^{2H}\right)_{i,j=1,2,\ldots,N}.\end{aligned} \qquad (11)$$

As a consequence, the log-likelihood function of $\mathbf{X}$ is given as

$$\begin{aligned}L_N(\mathbf{X}; H) &= \ln f_N(\mathbf{X}; H) \\ &= -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\left|\Gamma_H\right|\right) - \frac{1}{2}\mathbf{X}^t\Gamma_H^{-1}\mathbf{X}.\end{aligned} \qquad (12)$$

Moreover, let $\widehat{H}$ be the maximum likelihood estimator of H. Then, based on Taylor expansion we have

$$L'_N\left(\mathbf{X}; \widehat{H}\right) = 0 = L'_N(\mathbf{X}; H) + L''_N\left(\mathbf{X}; \widetilde{H}\right)\left(\widehat{H} - H\right), \qquad (13)$$

where $\widetilde{H}$ is a random point between $\widehat{H}$ and H; the single prime $'$ and the double prime $''$ denote the first and the second derivatives with respect to H, respectively. From Lai [14], we have the following result.

Lemma 2. *For* $H \in (0, 1)$, *the expectation of the first derivative of the log-likelihood function,* $L_N(X; H)$, *is zero and the variance is* $(1/2)\operatorname{Tr}([\Gamma_H^{-1}\Gamma'_H]^2)$.

Below we will establish the asymptotic distribution for H. First we list some technical conditions.

Assumption 3. Let $\lambda_{\max}(A)$ denote the largest eigenvalue of the matrix A. Then one assumes that $\lambda_{\max}(\Gamma_H) = o(N^{H+1/4}\sqrt{\ln N})$.

Actually, we should verify the feasibility of Assumption 3.

Proposition 4. *Assumption 3 is reasonable when* $H \in (3/4, 1)$

Proof. First, noting that the lower bound for the largest eigenvalue of Γ_H can be obtained from the result of Walker and Mieghem [15], we have

$$\lambda_{\max}\left(\Gamma_H\right) \geq \frac{\sum_{i,j=1}^{N} a_{ij}}{N} \geq NC_l, \qquad (14)$$

where a_{ij} are the coefficients and C_l is the minimum element of the matrix Γ_H.

On the other hand, we obtain the higher bound for the largest eigenvalue of Γ_H by the Gerschgorin Circle Theorem (see [16]: Theorem 8.1.3, P395)

$$\lambda_{\max}\left(\Gamma_H\right) \leq N^{2H+1}C_r, \qquad (15)$$

with C_r being a positive constant.

Consequently, we obtain

$$\lambda_{\max}\left(\Gamma_H\right) \in \left[NC_l, N^{2H+1}C_r\right]. \qquad (16)$$

From (16), it is obvious that Assumption 3 is practicable when $H \geq 3/4$. □

We are now ready to state the key result, whose proof is postponed to the appendix.

Theorem 5. *Suppose that Assumption 3 is satisfied and* $H \in (3/4, 1)$. *Then the maximum likelihood estimator* $\widehat{H}$ *of the parameter* H *is approximately normally distributed such that*

$$\sqrt{\frac{\operatorname{Tr}\left(\left[\Gamma_H^{-1}\Gamma'_H\right]^2\right)}{2}}\left(\widehat{H} - H\right) \xrightarrow{\mathscr{L}} \mathscr{N}(0, 1) \qquad (17)$$

as N *tends to infinity, where* $\xrightarrow{\mathscr{L}}$ *denotes the convergence in distribution.*

3. Simulation Studies

In this section, we conduct Monte Carlo simulations to study the performance of our estimator and compare our estimator with existing methods (the simulation in this paper has been implemented in the Matlab language. Readers should contact the authors if they are interested in obtaining the code of this study). Although there are many approaches available for estimating the Hurst parameter, here we compare some of these methods using the simulated time series. In particular, we consider the R/S, the variations estimators, the wavelet method, and our estimators. First, for a fixed time-step $h = 1/12$, we generate the fBm for different values of the parameters H and for different sample sizes: $N = 50$ and 100. Moreover, for each case we consider $s = 100$ independent realizations. Thus, for a given estimation method we obtain $s = 100$as the estimated values for H. We calculate

$$\widehat{H} = \frac{1}{s}\sum_{i=1}^{s} H_i, \qquad (18)$$

Table 1: The estimation results of comparative methods for sample size $N = 50$.

Estimation method	Statistic	True values of H			
		0.5600	0.6200	0.8500	0.9200
R/S method	Mean	0.50506	0.59232	0.82667	0.87692
	S.D.	0.62322	0.43601	0.48195	0.51479
	$\sqrt{\text{MSE}}$	0.62564	0.43689	0.48251	0.51659
	CPU time	13	15	18	20
Variations estimators	Mean	0.59451	0.65676	0.89267	0.92050
	S.D.	0.12365	0.23050	0.23739	0.15369
	$\sqrt{\text{MSE}}$	0.12838	0.23341	0.24119	0.15369
	CPU time	3	5	4	6
Wavelet method	Mean	0.58500	0.63520	0.87150	0.92963
	S.D.	0.04265	0.02518	0.06251	0.09654
	$\sqrt{\text{MSE}}$	0.04944	0.02941	0.06610	0.09702
	CPU time	22	27	28	28
Our method	Mean	0.56667	0.62905	0.85507	0.92102
	S.D.	0.03319	0.03092	0.01350	0.01325
	$\sqrt{\text{MSE}}$	0.03385	0.03222	0.01442	0.01329
	CPU time	3	3	5	5

Table 2: The estimation results of comparative methods for sample size $N = 100$.

Estimation method	Statistic	True values of H			
		0.5600	0.6200	0.8500	0.9200
R/S method	Mean	0.53625	0.60581	0.83751	0.89032
	S.D.	0.37218	0.32670	0.36258	0.36521
	$\sqrt{\text{MSE}}$	0.37294	0.32701	0.36280	0.36641
	CPU time	50	48	49	50
Variations estimators	Mean	0.58002	0.63370	0.86261	0.93143
	S.D.	0.01606	0.01952	0.01910	0.03162
	$\sqrt{\text{MSE}}$	0.02567	0.02385	0.02289	0.03362
	CPU time	21	18	19	22
Wavelet method	Mean	0.56497	0.62951	0.86232	0.91931
	S.D.	0.01268	0.01497	0.05436	0.07484
	$\sqrt{\text{MSE}}$	0.01362	0.01774	0.05574	0.07484
	CPU time	100	108	112	118
Our method	Mean	0.56003	0.62003	0.85003	0.92007
	S.D.	0.00214	0.00967	0.01470	0.00948
	$\sqrt{\text{MSE}}$	0.00215	0.00968	0.01472	0.00949
	CPU time	15	13	21	24

where H_i are the estimated values for a single realization. We calculate the standard deviation (S.D.)

$$\text{S.D.} = \sqrt{\frac{1}{s-1}\sum_{i=1}^{s}\left(H_i - \overline{H}\right)^2}, \tag{19}$$

where $\overline{H}$ is the mean over all the realizations and the mean-square error (MSE)

$$\text{MSE} = \frac{1}{s}\sum_{i=1}^{s}\left(H_i - H\right)^2, \tag{20}$$

where H is the value of the parameter that we have used to generate the model.

Using this way we can obtain approximate values for the standard deviation and the mean-squares error of the estimators. These results are important because they are necessary to investigate the performances of the estimators. Simulation results are summarized in Tables 1 and 2 with the mean, the standard deviation (S.D.), and the mean- square error (MSE) of these estimators. The CPU times (in seconds), which are the average, are also presented in these tables (all the procedures were programmed using a PC with 2.4 GHz Intel Duo Core CPU and 2-GB RAM).

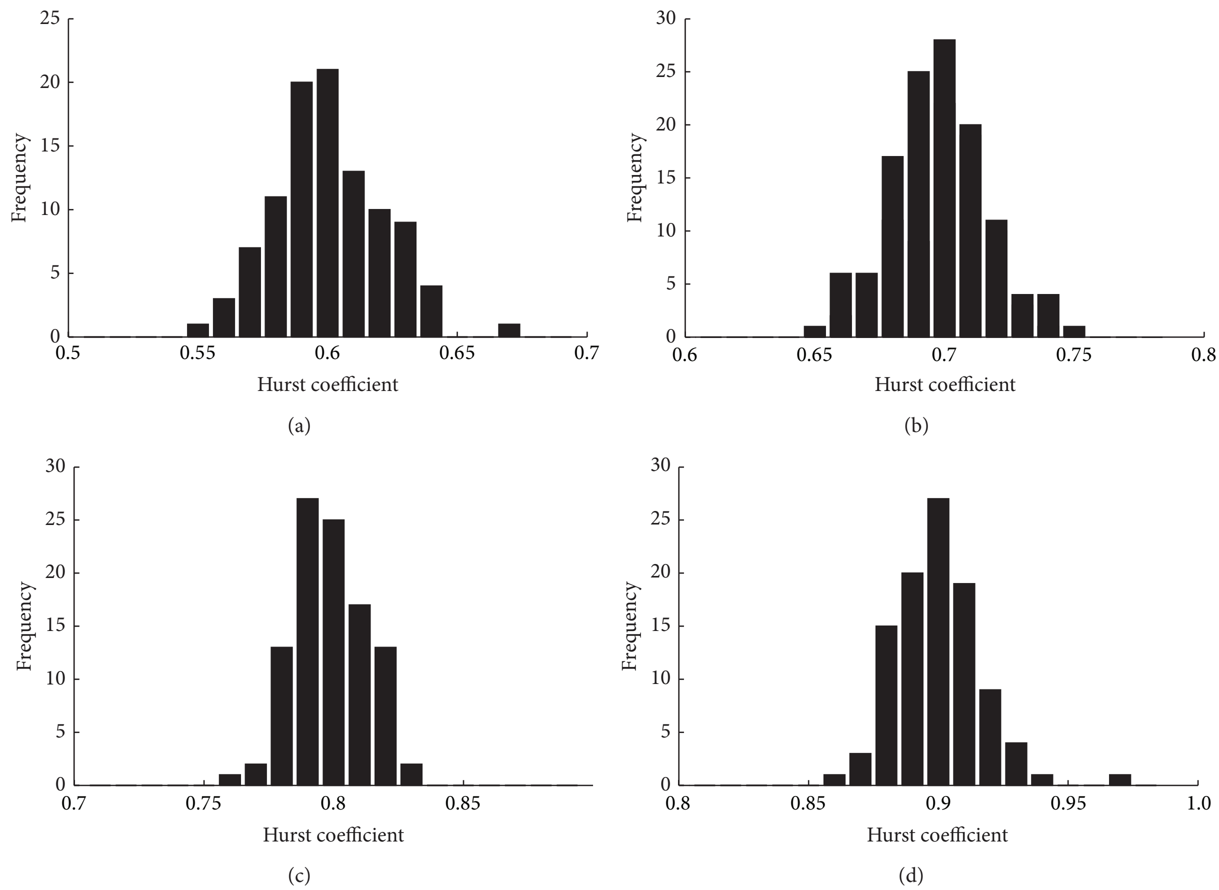

FIGURE 1: Histograms of the statistic $\widehat{H}$ for (a) $H = 0.60$, (b) $H = 0.70$, (c) $H = 0.80$, and (d) $H = 0.90$.

From these numerical computations, we can conclude that the simulated mean converges to the true value rapidly and the bias tends quickly to zero when the sample size increases. Indeed, by carefully observing Tables 1 and 2, we can see that the mean values obtained by our method and wavelet method are closer to the true values than those obtained based on R/S method and variation approach. Moreover, both the S.D. and the MSE obtained by our method and the wavelet method are smaller than those obtained based on R/S method and variation approach. Furthermore, it is interesting to see that our approach and wavelet method have the same accuracy and bias when the number of sample paths is large enough. Hence both methods are efficient. However, the most important finding is that the computation time costed by our approach is lower than that by the wavelet method. Therefore, our method is effective and efficient. In summary, the Monte Carlo simulations verify our theory and indicate that our estimators perform reasonably well in finite samples.

We next investigate the asymptotic distribution of $\widehat{H}$ and when N is not so large. Here, the chosen parameters are $H = 0.60, 0.70, 0.80, 0.90$ and we take $N = 120$ and $h = 1/12$ (i.e., $T = 10$). We just perform 100 Monte Carlo simulations of the sample paths generated by the process of fBm. The results are presented in Figure 1.

From Figure 1, we can see that the normal approximations of the distributions of the Hurst parameter H based on $\widehat{H}$ are reasonable even when N is not so large. This confirms our theoretical analysis: the convergence of the distributions of these two estimators is fast. In conclusion, our simulation results show that our estimators perform well since the estimating results match the chosen parameters exactly.

4. Conclusion

Research on estimating the Hurst parameters of fBm has been ongoing in the econometric and statistical literatures for more than two decades. But the subject has received its greatest attention in the last decade, as researchers in empirical finance have sought to use fBm to capture the long-range dependency of the prices of financial assets. This paper considered the inference problem for fBm based on the marriage of the autocorrelation approach and the maximum likelihood approach. The main contribution of this paper was in the establishment of asymptotic normality when the Hurst parameter satisfied $H \in (1/2, 1)$. This paper therefore extended the Lai [14] seminal contribution, which was done under strong conditions. Finally, we also showed the performance of our method to existing ones and illustrated that the proposed approach is effective and efficient. Certainly, for a future study, to improve the methodology it is required to use different schemes of estimation with a higher order of convergence. The field is therefore of growing importance for both theorists and practitioners.

Appendix

Proof of Theorem 5. Let $D_N^H = \sqrt{(1/2)\operatorname{Tr}([\Gamma_H^{-1}\Gamma_H']^2)}$. Then we obtain

$$\operatorname{Tr}\left(\left[\Gamma_H^{-1}\Gamma_H'\right]^2\right) \geq \frac{\operatorname{Tr}\left(\left(\Gamma_H'\right)^2\right)}{\lambda_{\max}^2\left(\Gamma_H\right)} = \frac{\left\|\Gamma_H'\right\|_E^2}{\lambda_{\max}^2\left(\Gamma_H\right)}, \tag{A.1}$$

where $\|\cdot\|_E$ denotes the Euclidean norm. The calculation of $\|\Gamma_H'\|_E^2$ is shown by the following computation:

$$\begin{aligned}
\left\|\Gamma_H'\right\|_E^2 &= \sum_{i,j=1}^{N}\left(a_{ij}'\right)^2 \geq \frac{1}{N^2}\left(\sum_{i,j=1}^{N} a_{ij}'\right)^2 \\
&= \frac{2}{N^2}\left[\sum_{\substack{i,j=1\\ i>j}}^{N} i^{2H}\ln i + j^{2H}\ln j \right.\\
&\qquad \left. - (i-j)^{2H}\ln(i-j) + \sum_{i=1}^{N} i^{2H}\ln i\right]^2 \\
&= \frac{2}{N^2}\left(\sum_{i=2}^{N}\left[(i-1)\, i^{2H}\ln i + i^{2H}\ln i\right]\right)^2 \\
&= \frac{2}{N^2}\left(\sum_{i=2}^{N} i^{2H+1}\ln i\right)^2 \\
&= \mathcal{O}\left(N^{4H+2}\ln^2 N\right),
\end{aligned} \tag{A.2}$$

where a_{ij}' are the coordinates of the matrix Γ_H'. The last asymptotic term follows since $\sum_{i=1}^{N} i^{2H+1}\cdot\ln i = \mathcal{O}(N^{2H+2}\cdot\ln N)$.

The Gerschgorin Circle Theorem (see [16]: Theorem 8.1.3, P395), combined with (A.2), implies that

$$\frac{\left\|\Gamma_H'\right\|_E^2}{\lambda_{\max}^2\left(\Gamma_H\right)} \geq C\frac{N^{4H+2}\ln^2 N}{\left(N^{2H+1}\right)^2} = C\ln^2 N, \tag{A.3}$$

where C is a positive constant. Consequently, we have $(D_N^H)^{-1} \to 0$ as $N \to \infty$.

Now let $W_N^H = (D_N^H)^{-2}(-L''(Y;H))$. Then a simple computation shows that $\mathbb{E}[W_N^H] = 1$. To compute the variance of W_N^H we first calculate the variance of $-L''(Y;H)$:

$$\begin{aligned}
&\operatorname{Var}\left[-L''(Y;H)\right] \\
&\quad = 2\operatorname{Tr}\left[\left(\frac{1}{2}\right)^2\left(\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right)^2\right] \\
&\quad = \frac{1}{2}\operatorname{Tr}\left(\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right)^2 \\
&\quad = \frac{1}{2}\left\|\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right\|_E^2 \\
&\quad = \frac{1}{2}\sum_{i,j=1}^{N} b_{ij}^2 = \frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\left(\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right) \\
&\quad \leq \frac{N}{2}\lambda_{\max}^2\left(\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right) \\
&\quad = \frac{N}{2}\left\|\Gamma_H^{1/2}\left(\Gamma_H^{-1}\right)''\Gamma_H^{1/2}\right\|_S \\
&\quad \leq \frac{N}{2}\left(2\left\|\Gamma_H^{-1/2}\Gamma_H'\Gamma_H^{-1}\Gamma_H'\Gamma_H^{-1/2}\right\|_S\right. \\
&\qquad \left. + \left\|\Gamma_H^{-1/2}\Gamma_H''\Gamma_H^{-1/2}\right\|_S\right) \\
&\quad = \frac{N}{2}\left(2\lambda_{\max}\left(\Gamma_H^{-1/2}\Gamma_H'\Gamma_H^{-1}\Gamma_H'\Gamma_H^{-1/2}\right)\right. \\
&\qquad \left. + \lambda_{\max}\left(\Gamma_H^{-1/2}\Gamma_H''\Gamma_H^{-1/2}\right)\right) \\
&\quad \leq \frac{N}{2}\left(2\frac{\lambda_{\max}^2\left(\Gamma_H'\right)}{\lambda_{\min}^2\left(\Gamma_H\right)} + \frac{\lambda_{\max}\left(\Gamma_H''\right)}{\lambda_{\min}\left(\Gamma_H\right)}\right) \\
&\quad = \mathcal{O}\left(N^{4H+3}\ln^2 N\right),
\end{aligned} \tag{A.4}$$

where b_{ij} are the coefficients, $\lambda_i(\cdot)$ is the eigenvalues, $\lambda_{\max}(\cdot)$ is the largest eigenvalue, and $\|\cdot\|_S$ is the spectral norm. The last term of the asymptotic relationship follows from the Gerschgorin Circle Theorem and the positive definiteness of the matrix Γ_H.

Indeed, by applying (A.2) and using (A.4), we immediately obtain that

$$\begin{aligned}
&\left(D_N^H\right)^{-4}\cdot\operatorname{Var}\left[-L''(Y;H)\right] \\
&\quad = \frac{4}{\left[\operatorname{Tr}\left(\left[\Gamma_H^{-1}\Gamma_H'\right]^2\right)\right]^2}\cdot\operatorname{Var}\left[-L''(Y;H)\right] \\
&\quad \leq \frac{4\lambda_{\max}^4\left(\Gamma_H\right)}{\left[\operatorname{Tr}\left(\left[\Gamma_H'\right]^2\right)\right]^2}\cdot\operatorname{Var}\left[-L''(Y;H)\right]
\end{aligned}$$

$$\le C^* \frac{\lambda_{\max}^4(\Gamma_H) \cdot N^{4H+3}}{N^{8H+4} \cdot \ln^2 N}$$
$$= C^* \frac{\lambda_{\max}^4(\Gamma_H)}{N^{4H+1} \cdot \ln^2 N}, \tag{A.5}$$

where C^* is a positive constant. Consequently, we obtain $(D_N^H)^{-4} \cdot \mathrm{Var}[-L''(Y;H)] \to 0$ as N goes to infinity when $\lambda_{\max}(\Gamma_H) = o(N^{H+1/4}\sqrt{\ln N})$. Now, we obtain $W_N^H \to 1$, which is the condition (C1) of Sweeting [17]. Moreover, the continuity condition (C2) in Sweeting [17] holds trivially since D_N^H and $L'(Y;H)$ are continuous functions of H.

From Corollary 1 in Sweeting [17], we obtain the desired result. □

Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work; there is no professional or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in or the review of this paper.

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